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Research Article

The Obstacle Problem for the A-Harmonic Equation

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Firstly, we define an order for differential forms. Secondly, we also define the supersolution and subsolution of the A-harmonic equation and the obstacle problems for differential forms which satisfy the A-harmonic equation, and we obtain the relations between the solutions to A-harmonic equation and the solution to the obstacle problem of the A-harmonic equation. Finally, as an application of the obstacle problem, we prove the existence and uniqueness of the solution to the A-harmonic equation on a bounded domain Ω with a smooth boundary $\partial\Omega$, where the A-harmonic equation satisfies $d^\star A(x,du)=0, x\in\Omega$; $u=\rho, x\in\partial\Omega$, where ρ is any given differential form which belongs to $W^{1,p}(\Omega,\Lambda^{l-1})$.

1. Introduction

Recently, a large amount of work about the A-harmonic equation for the differential forms has been done. In 1999 Nolder gave some properties for the solution to the A-harmonic equation in [1], and different versions of these properties had been established in [2–4]. The properties of the nonhomogeneous A-harmonic equation have been discussed in [5–10]. In the above papers, we can think that the boundary values were zero. In this paper, we mainly discuss the existence and uniqueness of the solution to A-harmonic equation with boundary values on a bounded domain Ω.

Now let us see some notions and definitions about the A-harmonic equation $d^\star A(x,du)=0$.

Let $e_1, e_2, \dots, e_n$ denote the standard orthogonal basis of $\mathbb{R}^n$. For $l=0,1,\dots,n$, we denote by $\Lambda^l=\Lambda^l(\mathbb{R}^n)$ the linear space of all l-vectors, spanned by the exterior product $e_I = e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_l}$ corresponding to all ordered l-tuples $I=(i_1,i_2,\dots,i_l)$, $1\le i_1<i_2<\cdots<i_l\le n$. The Grassmann algebra $\Lambda=\oplus\Lambda^l$ is a graded algebra with respect to the exterior products of $\alpha=\sum\alpha_I e_I\in\Lambda$ and $\beta=\sum\beta_I e_I\in\Lambda$, then its inner product is obtained by

$$\langle\alpha,\beta\rangle=\sum\alpha_I\beta_I \tag{1.1}$$

with the summation over all $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. And the norm of $\alpha = \sum \alpha_I e_I \in \Lambda$ is given by $|\alpha| = \langle \alpha, \alpha \rangle^{1/2}$.

The Hodge star operator $\star : \Lambda^l \to \Lambda^{n-l}$ is defined by the rule if $\omega = \omega_I dx_I = \omega_{i_1, i_2, \ldots, i_l} dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_l}$, then

$$\star\omega = (-1)^{\Sigma(I)} \omega_I dx_J, \tag{1.2}$$

where $\sum(I) = l(l+1)/2 + \sum_{k=1}^{l} i_k$ and $J = 1, 2, \ldots, n - I$. So we have $\star\star\omega = (-1)^{l(n-l)}\omega$.

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is an open subset, for any constant $\sigma > 1$, Q denotes a cube such that $Q \subset \sigma Q \subset \Omega$, where σQ denotes the cube whose center is as same as Q and $\operatorname{diam}(\sigma Q) = \sigma \operatorname{diam} Q$. We say that $\alpha = \sum \alpha_I e_I \in \Lambda$ is a differential l-form on Ω if every coefficient α_I of α is Schwartz distribution on Ω. The space spanned by differential l-form on Ω is denoted by $D'(\Omega, \Lambda^l)$. We write $L^p(\Omega, \Lambda^l)$ for the l-form $\alpha = \sum \alpha_I dx_I$ on Ω with $\alpha_I \in L^p(\Omega)$ for all ordered l-tuple I. Thus $L^p(\Omega, \Lambda^l)$ is a Banach space with the norm

$$\|\alpha\|_{p,\Omega} = \left(\int_\Omega |\alpha|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum_I |\alpha_I|^2 \right)^{p/2} dx \right)^{1/p}. \tag{1.3}$$

Similarly $W^{k,p}(\Omega, \Lambda^l)$ denotes those l-forms on Ω with all coefficients in $W^{k,p}(\Omega)$. We denote the exterior derivative by

$$d : D'\left(\Omega, \Lambda^l\right) \longrightarrow D'\left(\Omega, \Lambda^{l+1}\right) \quad \text{for } l = 0, 1, 2, \ldots, n \tag{1.4}$$

and its formal adjoint operator (the Hodge codifferential operator)

$$d^\star : D'\left(\Omega, \Lambda^l\right) \longrightarrow D'\left(\Omega, \Lambda^{l-1}\right). \tag{1.5}$$

The operators d and $d^\star$ are given by the formulas

$$d\alpha = \sum_I d\alpha_I \wedge dx_I, \qquad d^\star = (-1)^{nl+1} \star d \star. \tag{1.6}$$

2. The Obstacle Problem

In this section, we introduce the main work of this paper, which defining the supersolution and subsolution of the A-harmonic equation and the obstacle problems for differential forms which satisfy the A-harmonic equation, and the proof for the uniqueness of the solution to the obstacle problem of the A-harmonic equations for differential forms. We can see this work about functions in [11, Chapter 3 and Appendix I] in detail. We use the similar methods in [11] to do the main work for differential forms.

We firstly give the comparison about differential forms according to the comparison's definition about functions in $\mathbb{R}$.

Definition 2.1. Suppose that $\alpha = \sum_I \alpha_I(x)dx_I$ and $\beta = \sum_I \beta_I(x)dx_I$ belong to Λ^l, we say that $\alpha \geq \beta$ if for any given x, we have $\alpha_I(x) \geq \beta_I(x)$ for all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$.

Remark 2.2. The above definition involves the order for differential forms which we have been trying to avoid giving. We know that many differential forms can not be compared based on the above definition since there are so many inequalities to be satisfied. However, at the moment, we can not replace this definition by another one and we are working on it now. We just started our research on the obstacle problem for differential forms satisfying the A-harmonic equation and we hope that our work will stimulate further research in this direction.

By the some definitions as the solution, supersolution (or subsolution) to quasilinear elliptic equation, we can give the definitions of the solution, supersolution (or subsolution) to A-harmonic equation

$$d^\star A(x, du) = 0. \tag{2.1}$$

Definition 2.3. If a differential form $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \Lambda^{l-1})$ satisfies

$$\int_\Omega \langle A(x, du), d\varphi \rangle dx = 0, \tag{2.2}$$

for any $\varphi \in W^{1,p}_0(\Omega, \Lambda^{l-1})$, then we say that u is a solution to (2.1). If for any $0 \leq \varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \Lambda^{l-1})$, we have

$$\int_\Omega \langle A(x, du), d\varphi \rangle dx \geq 0 (\leq 0), \tag{2.3}$$

then we say that u is a supersolution (subsolution) to (2.1).

We can see that if u is a subsolution to (2.1), then for $0 \geq \varphi \in W^{1,p}_0(\Omega, \Lambda^{l-1})$, we have

$$\int_\Omega \langle A(x, du), d\varphi \rangle dx \geq 0. \tag{2.4}$$

According to the above definition, we can get the following theorem.

Theorem 2.4. *A differential form $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \Lambda^{l-1})$ is a solution to* (2.1) *if and only if u is both supersolution and subsolution to* (2.1).

Proof. The sufficiency is obvious, we only prove the necessity. For any $\varphi \in W^{1,p}_0(\Omega, \Lambda^{l-1})$, we suppose that $\varphi = \sum_I \varphi_I dx_I$,

$$\varphi_1 = \sum_I \varphi_I^+ dx_I \geq 0, \qquad \varphi_2 = \sum_I \varphi_I^- dx_I \leq 0; \tag{2.5}$$

by Definition 2.3, it holds that

$$\int_\Omega \langle A(x,du), d\varphi_1\rangle dx \geq 0, \qquad \int_\Omega \langle A(x,du), d\varphi_2\rangle dx \geq 0. \tag{2.6}$$

So

$$\begin{aligned} 0 &\leq \int_\Omega \langle A(x,du), d\varphi_1\rangle dx + \int_\Omega \langle A(x,du), d\varphi_2\rangle dx \\ &= \int_\Omega \langle A(x,du), d\varphi_1 + d\varphi_2\rangle dx = \int_\Omega \langle A(x,du), d\varphi\rangle dx. \end{aligned} \tag{2.7}$$

Using $-\varphi$ in place of φ, we also can get

$$\int_\Omega \langle A(x,du), d\varphi\rangle dx \leq 0. \tag{2.8}$$

Thus

$$\int_\Omega \langle A(x,du), d\varphi\rangle dx = 0. \tag{2.9}$$

Therefore u is a solution to (2.1). □

Next we will introduce the obstacle problem to A-harmonic equation, whose definition is according to the same definition as the obstacle problem of quasilinear elliptic equation. For the obstacle problem of quasilinear elliptic equation we can see [11] for details.

Suppose that Ω is a bounded domain. that $\psi = \sum_I \psi_I dx_I$ is any differential form in Ω which satisfies any ψ_I that is function in Ω with values in the extended reals $[-\infty, \infty]$, and $\rho \in W^{1,p}(\Omega, \Lambda^{l-1})$. Let

$$\mathcal{K}_{\psi,\rho}\left(\Omega, \Lambda^{l-1}\right) = \left\{v \in W^{1,p}\left(\Omega, \Lambda^{l-1}\right) : v \geq \psi \text{ a.e.}, v - \rho \in W_0^{1,p}\left(\Omega, \Lambda^{l-1}\right)\right\}. \tag{2.10}$$

The problem is to find a differential form in $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$ such that for any $v \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$, we have

$$\int_\Omega \langle A(x,du)d(v-u)\rangle \geq 0. \tag{2.11}$$

Definition 2.5. A differential form $u \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$ is called a solution to the obstacle problem of A-harmonic equation (2.1) with obstacle ψ and boundary values ρ or a solution to the obstacle problem of A-harmonic equation (2.1) in $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$ if u satisfies (2.11) for any $v \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$.

If $\psi = \rho$, then we denote that $\mathcal{K}_{\psi,\psi}(\Omega, \Lambda^{l-1}) = \mathcal{K}_{\psi}(\Omega, \Lambda^{l-1})$. We have some relations between the solution to quasilinear elliptic equation and the solution to obstacle problem in PDE. As to differential forms, we also have some relations between the solution to A-harmonic equation and the solution to obstacle problem of A-harmonic equation. We have the following two theorems.

Theorem 2.6. *If a differential form u is a supersolution to* (2.1)*, then u is a solution to the obstacle problem of* (2.1) *in $\mathcal{K}_{\psi,u}(\Omega, \Lambda^{l-1})$. For any $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$, if u is a solution to the obstacle problem of* (2.1) *in $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$, then u is a supersolution to* (2.1) *in Ω.*

Proof. If u is a solution to the obstacle problem of (2.1) in $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$, then for any $0 \leq \varphi \in W_0^{1,p}(\Omega, \Lambda^{l-1})$, we have $v = u + \varphi \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$, so it holds that

$$\int_\Omega \langle A(x, du), d\varphi \rangle dx = \int_\Omega \langle A(x, du), dv - du \rangle dx \geq 0. \tag{2.12}$$

Thus u is a supersolution to (2.1) in Ω. Conversely, if u is a supersolution to (2.1) in Ω, then for any $v \in \mathcal{K}_u(\Omega, \Lambda^{l-1})$, we have

$$v - u \geq 0, \qquad v - u \in W_0^{1,p}\left(\Omega, \Lambda^{l-1}\right). \tag{2.13}$$

Thus let $\varphi = v - u$, then we have

$$0 \leq \int_\Omega \langle A(x, du), d\varphi \rangle dx = \int_\Omega \langle A(x, du), dv - du \rangle dx. \tag{2.14}$$

So u is a solution to the obstacle problem of (2.1) in $\mathcal{K}_{\psi,u}(\Omega, \Lambda^{l-1})$. □

Theorem 2.7. *A differential form u is a solution to* (2.1) *if and only if u is a solution to the obstacle problem of* (2.1) *in $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$ with ρ satisfying $u - \rho \in W_0^{1,p}(\Omega, \Lambda^{l-1})$.*

Proof. If is a solution to the obstacle problem of (2.1) in $\mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$, then for any $\varphi \in W_0^{1,p}(\Omega, \Lambda^{l-1})$, we have $v = u + \varphi = u - \rho + \rho + \varphi \in \mathcal{K}_{-\infty,\rho}(\Omega, \Lambda^{l-1})$. So we can obtain

$$\int_\Omega \langle A(x, du), d\varphi \rangle dx = \int_\Omega \langle A(x, du), dv - du \rangle dx \geq 0. \tag{2.15}$$

By using $-\varphi$ in place of φ, we have

$$\int_\Omega \langle A(x, du), d(-\varphi) \rangle dx = \int_\Omega \langle A(x, du), dv - du \rangle dx \geq 0. \tag{2.16}$$

So

$$\int_\Omega \langle A(x, du), d\varphi \rangle dx = 0. \tag{2.17}$$

Thus u is a solution to (2.1) in Ω.

Conversely, if u is a solution to (2.1) in Ω, then for any $v \in \mathcal{K}_{-\infty,\rho}(\Omega,\Lambda^{l-1})$, we have $v-u \in W_0^{1,p}(\Omega,\Lambda^{l-1})$. Now let $\varphi = v-u$, then we have

$$0=\int_\Omega \langle A(x,du), d\varphi\rangle dx = \int_\Omega \langle A(x,du), dv-du\rangle dx. \tag{2.18}$$

Thus

$$0 \le \int_\Omega \langle A(x,du), dv-du\rangle dx. \tag{2.19}$$

So the theorem is proved. □

The following we will discuss the existence and uniqueness of the solution to the obstacle problem of (2.1) in $\mathcal{K}_{\psi,\rho}(\Omega,\Lambda^{l-1})$ and the solution to (2.1). First we introduce a definition and two lemmas.

Definition 2.8 (see [11]). Suppose that X is a reflexive Banach space in Ω with dual space X', and let $(\cdot,\cdot)$ denote a pairing between X' and X. If $\mathbf{K} \subset X$ is a closed convex set, then a mapping $\pounds : \mathbf{K} \to X'$ is called monotone if

$$(\pounds u - \pounds v, u-v) \ge 0, \tag{2.20}$$

for all uv in $\mathbf{K}$. Further, $\pounds$ is called coercive on $\mathbf{K}$ if there exists $\varphi \in \mathbf{K}$ such that

$$\frac{(\pounds u_j - \pounds\varphi, u_j-\varphi)}{\|u_j-\varphi\|} \longrightarrow \infty, \tag{2.21}$$

whenever u_j is a sequence in $\mathbf{K}$ with $\|u_j\| \to \infty$.

By the definition of ∇u in [12], we can easily get the following lemma.

Lemma 2.9. *For any $u \in W^{1,p}(\Omega,\Lambda^l)$, we have $|du| \le |\nabla u|$ and $|\nabla|u|| \le |\nabla u|$.*

Lemma 2.10 (see [11]). *Let $\mathbf{K}$ be a nonempty closed convex subset of X and let $\pounds : \mathbf{K} \to X'$ be monotone, coercive, and weakly continuous on $\mathbf{K}$. Then there exists an element u in $\mathbf{K}$ such that*

$$(\pounds u, u-v) \ge 0, \tag{2.22}$$

whenever $v \in \mathbf{K}$.

Using the same methods in [11, Appendix I], we can prove the existence and uniqueness of the solution to the obstacle problem of (2.1).

Theorem 2.11. *If $\mathcal{K}_{\psi,\rho}(\Omega,\Lambda^{l-1})$ is nonempty, then there exists a unique solution to the obstacle problem of* (2.1) *in $\mathcal{K}_{\psi,\rho}(\Omega,\Lambda^{l-1})$.*

Proof. Let $X = L^p(\Omega, \Lambda^l)$, then $X' = L^{p/(p-1)}(\Omega, \Lambda^l)$. Let

$$(f, g) = \int_\Omega \langle f, g \rangle dx, \tag{2.23}$$

where $f \in L^p(\Omega, \Lambda^l)$ and $g \in L^{p/(p-1)}(\Omega, \Lambda^l)$. Denote that

$$\mathbf{K} = \left\{ dv : v \in \mathcal{K}_{\psi,\rho}\left(\Omega, \Lambda^{l-1}\right) \right\}. \tag{2.24}$$

We define a mapping $£ : \mathbf{K} \to X'$ such that for any $v \in \mathbf{K}$, we have $£v = A(x, v)$. So for any $u \in L^p(\Omega, \Lambda^l)$, we have

$$(£v, u) = \int_\Omega \langle A(x, v), u \rangle dx. \tag{2.25}$$

Then we only prove that $\mathbf{K}$ is a closed convex subset of X and $£ : \mathbf{K} \to X'$ is monotone, coercive, and weakly continuous on $\mathbf{K}$.

(1) $\mathbf{K}$ is convex. For any $x_1, x_2 \in \mathbf{K}$, we have $v_1, v_2 \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$ such that

$$x_1 = dv_1, \qquad x_2 = dv_2. \tag{2.26}$$

So for any $t \in (0, 1)$, we have

$$tx_1 + (1 - t)x_2 = tdv_1 + (1 - t)dv_2 = d(tv_1 + (1 - t)v_2). \tag{2.27}$$

Since

$$tv_1 + (1 - t)v_2 - \rho = t\left(v_1 - \rho\right) + (1 - t)\left(v_2 - \rho\right) \in \mathcal{K}_{\psi,\rho}\left(\Omega, \Lambda^{l-1}\right), \tag{2.28}$$

thus

$$tx_1 + (1 - t)x_2 \in \mathbf{K}. \tag{2.29}$$

So $\mathbf{K}$ is convex.

(2) $\mathbf{K}$ is closed in X. Suppose that $dv_i \in \mathbf{K}$ is a sequence converging to $\tilde{v}$ in X. Then by the real functions' Poincaré inequality and Lemma 2.9, we have

$$\begin{aligned} \int_\Omega \left|v_i - \rho\right|^p dx &\le c(\operatorname{diam}\Omega)^p \int_\Omega \left|\nabla \left|v_i - \rho\right|\right|^p dx \\ &\le c(\operatorname{diam}\Omega)^p \int_\Omega \left|\nabla v_i - \nabla \rho\right|^p dx \le M < \infty. \end{aligned} \tag{2.30}$$

Thus v_i is a bounded sequence in $W^{1,p}(\Omega,\Lambda^{l-1})$. Because $\mathcal{K}_{\varphi,\rho}(\Omega,\Lambda^{l-1})$ is a closed and convex subset of $W^{1,p}(\Omega,\Lambda^{l-1})$, we denote that $v_i=\sum_I v_i^I dx_I$ and $\rho=\sum_I \rho^I dx_I$. Then for any I in $l-1$ tuples, according to Theorems 1.30 and 1.31 in [11], we have a function v^I such that

$$v_i^I \longrightarrow v^I \quad \text{weakly}, \quad v^I-\rho^I \in W_0^{1,p}(\Omega), \quad \nabla v_i^I \longrightarrow \nabla v^I = \left(\frac{\partial v^I}{\partial x_1},\dots,\frac{\partial v^I}{\partial x_n}\right) \quad \text{weakly}. \tag{2.31}$$

According to **Lemma 2.9** and the uniqueness of a limit of a convergence sequence, we only let

$$\widetilde{v}=\sum_I\sum_{i=1}^{n}\frac{\partial v^I}{\partial x_i}dx_i\wedge dx_I. \tag{2.32}$$

Thus $\widetilde{v}\in\mathbf{K}$, so $\mathbf{K}$ is closed in X.

(3) £ is monotone. Since operator A satisfies

$$\langle A(x,\xi_1)-A(x,\xi_2),\xi_1-\xi_2\rangle \geq 0, \tag{2.33}$$

so for all $u,v\in\mathbf{K}$, it holds that

$$(£u-£v,u-v)=\int_\Omega\langle A(x,u)-A(x,v),u-v\rangle dx\geq 0. \tag{2.34}$$

Thus £ is monotone.

(4) £ is coercive on $\mathbf{K}$. For any fixed $\varphi\in\mathbf{K}$, we have

$$\begin{aligned}(£u-£\varphi,u-\varphi)&=\int_\Omega\langle A(x,u)-A(x,\varphi),u-\varphi\rangle dx\\
&=\int_\Omega\langle A(x,u),u\rangle dx+\int_\Omega\langle A(x,\varphi),\varphi\rangle dx-\int_\Omega\langle A(x,u),\varphi\rangle dx\\
&\quad-\int_\Omega\langle A(x,\varphi),u\rangle dx\\
&\geq K^{-1}\int_\Omega|u|^p dx+K^{-1}\int_\Omega|\varphi|^p dx-K\int_\Omega|u|^{p-1}|\varphi|dx-\int_\Omega|\varphi|^{p-1}|u|dx\\
&\geq K^{-1}\left(\|u\|^p+\|\varphi\|^p\right)-K\left(\|u\|^{p-1}\|\varphi\|+\|u\|\|\varphi\|^{p-1}\right)\\
&\geq K^{-1}2^{-p}\|u-\varphi\|\|u-\varphi\|^{p-1}-K2^{p-1}\|\varphi\|\left(\|\varphi\|^{p-1}+\|u-\varphi\|^{p-1}\right)\\
&\quad-K\|\varphi\|^{p-1}\left(\|\varphi\|+\|u-\varphi\|\right).\end{aligned} \tag{2.35}$$

So

$$\frac{(\pounds u_j - \pounds\varphi, u_j - \varphi)}{\|u_j - \varphi\|} \geq K^{-1}2^{-p}\|u_j - \varphi\|^{p-1} - K2^{p-1}\|\varphi\|\left(\frac{\|\varphi\|^{p-1}}{\|u_j - \varphi\|} + \|u_j - \varphi\|^{p-2}\right)$$
$$- K\|\varphi\|^{p-1}\left(\frac{\|\varphi\|}{\|u_j - \varphi\|} + 1\right). \tag{2.36}$$

When $\|u_j\| \to \infty$ and $\|u_j - \varphi\| \to \infty$, we can obtain

$$\frac{(\pounds u_j - \pounds\varphi, u_j - \varphi)}{\|u_j - \varphi\|} \longrightarrow \infty. \tag{2.37}$$

Therefore £ is coercive on **K**.

(5) £ is weakly continuous on **K**. Suppose that $u_i \in \mathbf{K}$ is a sequence that converge to $u \in \mathbf{K}$ on X. Pick a subsequence u_{i_j} such that $u_{i_j} \to u$ a.e. in Ω. Since the mapping $\xi \to A(x,\xi)$ is continuous for a.e. x, we have

$$A\left(x, u_{i_j}\right) \longrightarrow A(x,u), \tag{2.38}$$

a.e. $x \in \Omega$. Because $L^{p/(p-1)}(\Omega, \Lambda^l)$-norms of $A(x, u_{i_j})$ are uniformly bounded, we have that

$$A\left(x, u_{i_j}\right) \longrightarrow A(x,u) \tag{2.39}$$

weakly in $L^{p/(p-1)}(\Omega, \Lambda^l)$. Because the weak limit is independent of the choice of the subsequence, it follows that

$$A(x,u_i) \longrightarrow A(x,u) \tag{2.40}$$

weakly in $L^{p/(p-1)}(\Omega, \Lambda^l)$. Thus for any $v \in L^p(\Omega, \Lambda^l)$, we have

$$(\pounds u_i, v) = \int_\Omega \langle \pounds u_i, v\rangle dx \longrightarrow \int_\Omega \langle \pounds u, v\rangle dx = (\pounds u, v). \tag{2.41}$$

Thus £ is weakly continuous on **K**.

By Lemma 2.10, we can find an element $\tilde{u}$ in **K** such that

$$(\pounds\tilde{u}, \tilde{v} - \tilde{u}) \geq 0, \tag{2.42}$$

for any $\tilde{v} \in \mathbf{K}$, that is to say, there exists $u \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$ such that $du = \tilde{u}$ and

$$\int_\Omega \langle A(x, du), dv - du\rangle dx = (\pounds du, dv - du) \geq 0, \tag{2.43}$$

for any $v \in \mathcal{K}_{\psi,\rho}(\Omega, \Lambda^{l-1})$. Then the theorem is proved. □

By Theorem 2.7, we can see that the solution u to the obstacle problem of (2.1) in $\mathcal{K}_{-\infty,\rho}(\Omega,\Lambda^{l-1})$ is a solution of (2.1) in Ω. Then by theorem, we can get the existence and uniqueness of the solution to A-harmonic equation.

Corollary 2.12. *Suppose that Ω is a bounded domain with a smooth boundary $\partial\Omega$ and $\rho \in W^{1,p}(\Omega,\Lambda^{l-1})$. There is a differential form $u \in W^{1,p}(\Omega,\Lambda^{l-1})$ such that*

$$\begin{aligned} d^{\star}A(x,du) &= 0, \quad x \in \Omega, \\ u &= \rho, \quad x \in \partial\Omega \end{aligned} \tag{2.44}$$

weakly in Ω, that is to say,

$$\int_{\Omega} \langle A(x,du), d\varphi \rangle dx = 0, \tag{2.45}$$

for any $\varphi \in W_0^{1,p}(\Omega,\Lambda^{l-1})$.

Proof. Let $\psi = -\infty$ and u be a solution to the obstacle problem of (2.1) in $\mathcal{K}_{\psi,\rho}(\Omega,\Lambda^{l-1})$. For any $\varphi \in W_0^{1,p}(\Omega,\Lambda^{l-1})$, we have both $u+\varphi$ and $u-\varphi$ belong to $\mathcal{K}_{\psi,\rho}(\Omega,\Lambda^{l-1})$. Then

$$\int_{\Omega} \langle A(x,du), d\varphi \rangle dx \geq 0, \qquad -\int_{\Omega} \langle A(x,du), d\varphi \rangle dx \geq 0. \tag{2.46}$$

Thus

$$\int_{\Omega} \langle A(x,du), d\varphi \rangle dx = 0. \tag{2.47}$$

So u is solution to A-harmonic equation $d^{\star}A(x,du) = 0$ in Ω with a boundary value ρ. □

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