Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 768469, 16 pages
doi:10.1155/2010/768469

Research Article

Controllability for Variational Inequalities of Parabolic Type with Nonlinear Perturbation

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Received 21 October 2009; Accepted 1 February 2010

Academic Editor: Jong Kim

We deal with the approximate controllability for the nonlinear functional differential equation governed by the variational inequality in Hilbert spaces and present a general theorems under which previous results easily follow. The common research direction is to find conditions on the nonlinear term such that controllability is preserved under perturbation.

1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. If H is identified with its dual space, we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H, and V^* will be denoted by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For $l \in V^*$, we denote (l, v) by the value $l(v)$ of l at $v \in V$. We assume that V has a stronger topology than H and, for the brevity, we may regard that

$$\|u\|_* \leq |u| \leq \|u\|, \quad \forall u \in V. \tag{1.1}$$

Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality, and let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function, and $h : \mathbb{R}^+ \times V \times U \to H$ is a nonlinear mapping. Let U be some Hilbert space and the controller operator B a bounded linear operator from U to H. Then we study the following variational inequality problem with nonlinear term:

$$\begin{aligned} &\left(x'(t) + Ax(t), x(t) - z\right) + \phi(x(t)) - \phi(z) \\ &\quad \le \left(\int_0^t k(t-s)h(s,x(s),u(s))ds + Bu(t), x(t) - z\right), \quad \text{a.e., } \forall z \in V, \\ &\qquad x(0) = x_0. \end{aligned} \tag{NDE}$$

Noting that the subdifferential operator $\partial\phi$ is defined by

$$\partial\phi(x) = \left\{x^* \in V^*; \phi(x) \le \phi(y) + \left(x^*, x - y\right),\ y \in V\right\}, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the duality pairing between V^* and V, the problem (NDE) is represented by the following nonlinear functional differential problem:

$$\begin{aligned} x'(t) + Ax(t) + \partial\phi(x(t)) &\ni \int_0^t k(t-s)h(s,x(s),u(s))ds + Bu(t), \quad 0 < t, \\ x(0) &= x_0. \end{aligned} \tag{NCE}$$

The existence and regularity for the parabolic variational inequality in the linear case ($h \equiv 0$), which was first investigated by Brézis [1, 2], have been developed as seen in Barbu [4, Section 4.3.2] (also see [4, Section 4.3.1]). The regularity for the nonlinear variational inequalities of semilinear parabolic type was studied in [5].

The solution (NCE) is denoted by $x(T;\phi,h,u)$ corresponding to the nonlinear term h and the control u. The system (NCE) is said to be approximately controllable in the time interval $[0,T]$, if for every given final state $x_1 \in H$, $T > 0$, and $\epsilon > 0$, there is a control function $u \in L^2(0,T;U)$ such that $|x(T;\phi,h,u) - x_1| < \epsilon$. Investigations of controllability of semilinear systems found in [6, 7] have been studied by many [6–10], which is shown the relation between the reachable set of the semilinear system and that of its corresponding.

In [7, 11], they dealt with the approximate controllability of a semilinear control system as a particular case of sufficient conditions for the approximate solvability of semilinear equations by assuming that

(1) $S(t)$ is compact operator, or the embedding $D(A) \subset V$ is compact;

(2) $h(\cdot,x,u)$ is (locally) Lipschitz continuous (or the sublinear growth condition and $\lim_{n\to\infty}(|h(\cdot,x,u)|/\|(x,u)\|) = 0$);

(3) the corresponding linear system (NCE) in case where $h \equiv 0$ and $\phi \equiv 0$ is approximately controllable.

Yamamoto and Park [12] studied the controllability for parabolic equations with uniformly bounded nonlinear terms instead of assumptions mentioned above. As for the some considerations on the trajectory set of (NCE) and that of its corresponding linear system (in case $h \equiv 0$) as matters connected with (3), we refer to Naito [10] and Sukavanam and Tomar [13], and references therein. In [13] and Zhou [14], they studied the control problems of the semilinear equations by assuming (1), (3), a Lipschitz continuity of G, and a range condition of the controller B with an inequality constraint.

In this paper, we no longer require the compact property in (1), the uniform boundedness in (2), and the inequality constraint on the range condition of the controller B, but instead we need the regularity and a variation of solutions of the given equations. For the basis of our study, we construct the fundamental solution and establish variations of constant formula of solutions for the linear systems.

This paper is composed of four sections. Section 2 gives assumptions and notations. In Section 3, we introduce the single valued smoothing system corresponding to (NCE). Then in Section 4, the relations between the reachable set of systems consisting of linear parts and possibly nonlinear perturbations are addressed. From these results, we can obtain the approximate controllability for (NCE), which is the extended result of [10, 13, 14] to (NCE).

2. Solvability of the Nonlinear Variational Inequality Problems

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality:

$$\operatorname{Re}\ a(u,u) \geq \omega_1 \|u\|^2 - \omega_2 |u|^2, \tag{2.1}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u,v), \qquad u, v \in V. \tag{2.2}$$

Then A is a bounded linear operator from V to V^* by the Lax-Milgram theorem. The realization for the operator A in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\} \tag{2.3}$$

is also denoted by A. We also assume that there exists a constant C_0 such that

$$\|u\| \leq C_0 \|u\|_{D(A)}^{1/2} |u|^{1/2} \tag{2.4}$$

for every $u \in D(A)$, where

$$\|u\|_{D(A)} = \left(|Au|^2 + |u|^2\right)^{1/2} \tag{2.5}$$

is the graph norm of $D(A)$. Thus, in terms of the intermediate theory, we may assume that

$$(D(A), H)_{1/2,2} = V, \tag{2.6}$$

where $(D(A), H)_{1/2,2}$ denotes the real interpolation space between $D(A)$ and H.

Lemma 2.1. *Let $T > 0$. Then*

$$H = \left\{ x \in V^* : \int_0^T \left\| Ae^{tA}x \right\|_*^2 dt < \infty \right\}. \tag{2.7}$$

Proof. Put $u(t) = e^{tA}x$ for $x \in H$. Then,

$$u'(t) = Au(t), \quad u(0) = x. \tag{2.8}$$

As in [15, Theorem 4.1, Chapter 4], the solution u belongs to $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$; hence we obtain that

$$\int_0^T \left\| Ae^{tA}x \right\|_*^2 dt = \int_0^T \| u'(s) \|_*^2 ds < \infty. \tag{2.9}$$

Conversely, suppose that $x \in V^*$ and $\int_0^T \| Ae^{tA}x \|_*^2 dt < \infty$. Put $u(t) = e^{tA}x$. Then since A is an isomorphism operator from V to V^*, there exists a constant $c > 0$ such that

$$\int_0^T \| u(t) \|^2 dt \leq c \int_0^T \| Au(t) \|_*^2 dt = c \int_0^T \left\| Ae^{tA}x \right\|_*^2 dt. \tag{2.10}$$

From the assumptions and $\dot{u}(t) = Ae^{tA}x$, it follows that

$$u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H). \tag{2.11}$$

Therefore, $x = u(0) \in H$. □

By Lemma 2.1, from Butzer and Berens [16, Theorem 3.5.3], we can see that

$$(V, V^*)_{1/2,2} = H. \tag{2.12}$$

It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. The following Lemma is from [17, Lemma 3.6.2].

Lemma 2.2. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$:*

$$|S(t)x| \leq M|x|, \quad \|S(t)x\| \leq Mt^{-1/2}|x|. \tag{2.13}$$

Lemma 2.3. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$\|x\|_{L^2(0,T;D(A))} \le C_1\|k\|_{L^2(0,T;H)}, \tag{2.14}$$

$$\|x\|_{L^2(0,T;H)} \le C_2T\|k\|_{L^2(0,T;H)}, \tag{2.15}$$

$$\|x\|_{L^2(0,T;V)} \le C_2\sqrt{T}\|k\|_{L^2(0,T;H)}. \tag{2.16}$$

Proof. The assertion (2.14) is immediately obtained by virtue of [8, Theorem 3.3] (or [7, Theorem 3.1]). Since

$$\begin{aligned}\|x\|^2_{L^2(0,T;H)} &= \int_0^T\left|\int_0^t S(t-s)k(s)ds\right|^2 dt \le M\int_0^T\left(\int_0^t|k(s)|ds\right)^2 dt\\ &\le M\int_0^T t\int_0^t|k(s)|^2ds\,dt \le M\frac{T^2}{2}\int_0^T|k(s)|^2ds,\end{aligned} \tag{2.17}$$

it follows that

$$\|x\|_{L^2(0,T;H)} \le T\sqrt{\frac{M}{2}}\|k\|_{L^2(0,T;H)}. \tag{2.18}$$

From (2.4), (2.14), and (2.15), it holds that

$$\|x\|_{L^2(0,T;V)} \le C_0\sqrt{C_1T}\left(\frac{M}{2}\right)^{1/4}\|k\|_{L^2(0,T;H)}. \tag{2.19}$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\left\{\sqrt{\frac{M}{2}}, C_0\sqrt{C_1}\left(\frac{M}{2}\right)^{1/4}\right\}, \tag{2.20}$$

the proof is complete. □

Let $h : \mathbb{R}^+ \times V \times U \to H$ be a nonlinear mapping satisfying the following:

(G1) for any $x \in V$, $u \in U$, the mapping $h(\cdot, x, u)$ is strongly measurable;

(G2) there exist positive constants L_0, L_1, L_2 such that

(i) $|h(t,x,u) - h(t,\hat{x},\hat{u})| \le L_1\|x-\hat{x}\| + L_2\|u-\hat{u}\|_U$,

(ii) $|h(t,0,0)| \le L_0$ for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $u, \hat{u} \in U$.

For $x \in L^2(0,T;V)$, we set

$$G(t,x,u) = \int_0^t k(t-s)h(s,x(s),u(s))ds, \tag{2.21}$$

where k belongs to $L^2(0,T)$.

Lemma 2.4. *Let $x \in L^2(0,T;V)$ and $u \in L^2(0,T;U)$ for any $T > 0$. Then $G(\cdot,x,u) \in L^2(0,T;H)$ and*

$$\|G(\cdot,x,u)\|_{L^2(0,T;H)} \le \frac{L_0\|k\|_{L^2(0,T)}T}{\sqrt{2}} + \|k\|_{L^2(0,T)}\sqrt{T}\Big(L_1\|x\|_{L^2(0,T;V)} + L_2\|u\|_{L^2(0,T;U)}\Big). \tag{2.22}$$

Moreover, if $x, \hat{x} \in L^2(0,T;V)$, then

$$\|G(\cdot,x,u) - G(\cdot,\hat{x},\hat{u})\|_{L^2(0,T;H)} \le \|k\|_{L^2(0,T)}\sqrt{T}\Big(L_1\|x-\hat{x}\|_{L^2(0,T;V)} + L_2\|u-\hat{u}\|_{L^2(0,T;U)}\Big). \tag{2.23}$$

Proof. From (G1), (G2), and using the Hölder inequality, it is easily seen that

$$\begin{aligned}
\|G(\cdot,x,u)\|_{L^2(0,T;H)} &\le \|G(\cdot,0,0)\| + \|G(\cdot,x,u) - G(\cdot,0,0)\| \\
&\le \left(\int_0^T\left|\int_0^t k(t-s)h(s,0,0)ds\right|^2 dt\right)^{1/2} \\
&\quad + \left(\int_0^T\left|\int_0^t k(t-s)\{h(s,x(s),u(s)) - h(s,0,0)\}ds\right|^2 dt\right)^{1/2} \\
&\le \frac{L_0\|k\|_{L^2(0,T)}T}{\sqrt{2}} + \|k\|_{L^2(0,T)}\sqrt{T}\|h(\cdot,x,u) - h(\cdot,0,0)\|_{L^2(0,T;H)} \\
&\le \frac{L_0\|k\|_{L^2(0,T)}T}{\sqrt{2}} + \|k\|_{L^2(0,T)}\sqrt{T}\Big(L_1\|x\|_{L^2(0,T;V)} + L_2\|u\|_{L^2(0,T;U)}\Big).
\end{aligned} \tag{2.24}$$

The proof of (2.23) is similar. □

By virtue of [5, Theorems 3.1 and 3.2], we have the following result on the solvability of (NDE) (see [3, 15] in case of corresponding to equations with $h \equiv 0$).

Proposition 2.5. *Let the assumptions (G1) and (G2) be satisfied. Assume that $(x_0,u) \in \overline{D(\phi)} \times L^2(0,T;U)$, where $\overline{D(\phi)}$ stands for the closure in H of the set $D(\phi) = \{u \in V : \phi(u) < \infty\}$. Then,* (NDE) *has a unique solution*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H), \tag{2.25}$$

and there exists a constant C_3 *depending on* T *such that*

$$\|x\|_{L^2\cap W^{1,2}\cap C} \le C_3\Big(1 + |x_0| + \|u\|_{L^2(0,T;U)}\Big). \tag{2.26}$$

3. Smoothing System Corresponding to (NDE)

For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = \inf\left\{ \frac{\|x-y\|_*^2}{2\epsilon} + \phi(y) : y \in H \right\}. \tag{3.1}$$

Then the function ϕ_ϵ is Fréchet differentiable on H, and its Fréhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1}, where $\partial\phi_\epsilon = \epsilon^{-1}(I-(I+\epsilon\partial\phi)^{-1})$ as is seen in [4, Corollary 2.2, Chapter II]. It is also well known results that $\lim_{\epsilon\to 0}\phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0}\partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$, where $(\partial\phi)^0 : H \to H$ is the minimum element of $\partial\phi$.

Now, we introduce the smoothing system corresponding to (NCE) as follows.

$$\begin{gathered} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = G(t,x,u) + Bu(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{gathered} \tag{SCE}$$

Since A generates a semigroup $S(t)$ on H, the mild solution of (SCE) can be represented by

$$x_\epsilon(t) = S(t)x_0 + \int_0^t S(t-s)\{G(s,x_\epsilon,u) + Bu(s) - \partial\phi_\epsilon(x_\epsilon(s))\}ds. \tag{3.2}$$

In virtue of Proposition 2.5, we know that if the assumptions (G1-G2) are satisfied then for every $x_0 \in H$ and every $u \in L^2(0,T;U)$, (SCE) has a unique solution

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H), \tag{3.3}$$

and there exists a constant C_4 depending on T such that

$$\|x\|_{L^2\cap W^{1,2}\cap C} \le C_4\Big(1 + |x_0| + \|u\|_{L^2(0,T;U)}\Big). \tag{3.4}$$

Now, we assume the hypothesis that $V \subset D(\partial\phi)$ and $(\partial\phi)^0$ is uniformly bounded, that is,

$$\left|(\partial\phi)^0 x\right| \le M_1, \quad x \in H. \tag{A}$$

Lemma 3.1. *Let* x_ϵ *and* x_λ *be the solutions of* (SCE) *with same control* u*. Then there exists a constant* C *independent of* ϵ *and* λ *such that*

$$\|x_\epsilon - x_\lambda\|_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon+\lambda), \quad 0 < T. \tag{3.5}$$

Proof. For given $\epsilon, \lambda > 0$, let x_ϵ and x_λ be the solutions of (SCE) corresponding to ϵ and λ, respectively. Then from (SCE), we have

$$x'_\epsilon(t) - x'_\lambda(t) + A(x_\epsilon(t) - x_\lambda(t)) + \partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)) = G(t, x_\epsilon, u) - G(t, x_\lambda, u), \tag{3.6}$$

and hence, from (2.13) and multiplying by $x_\epsilon(t) - x_\lambda(t)$, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_\epsilon(t) - x_\lambda(t)|^2 + \omega_1\|x_\epsilon(t) - x_\lambda(t)\|^2 + \big(\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)\big) \\ &\quad\le (G(t, x_\epsilon, u) - G(t, x_\lambda, u), x_\epsilon(t) - x_\lambda(t)) + \omega_2|x_\epsilon(t) - x_\lambda(t)|^2. \end{aligned} \tag{3.7}$$

Let us choose a constant $c > 0$ such that $2\omega_1 - cL_1^2\|k\|^2_{L^2(0,T)} > 0$. Then by (G1), we have

$$\begin{aligned} &(G(t, x_\epsilon, u) - G(t, x_\lambda, u), x_\epsilon(t) - x_\lambda(t)) \\ &\quad\le |G(t, x_\epsilon, u) - G(t, x_\lambda, u)| \cdot |x_\epsilon(t) - x_\lambda(t)| \\ &\quad\le \frac{cL_1^2\|k\|^2_{L^2(0,T)}}{2}\int_0^T \|x_\epsilon(t) - x_\lambda(t)\|^2 dt + \frac{1}{2c}|x_\epsilon(t) - x_\lambda(t)|^2. \end{aligned} \tag{3.8}$$

Integrating (3.7) over $[0, T]$ and using the monotonicity of $\partial\phi$, we have

$$\begin{aligned} &\frac{1}{2}|x_\epsilon(t) - x_\lambda(t)|^2 + \left(\omega_1 - \frac{cL_1^2\|k\|^2_{L^2(0,T)}}{2}\right)\int_0^T \|x_\epsilon(t) - x_\lambda(t)\|^2 dt \\ &\quad\le \int_0^T \big(\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), \lambda\partial\phi_\lambda\big(x_\lambda(t) - \epsilon\partial\phi_\epsilon(x_\epsilon(t))\big)\big) dt \\ &\qquad + \left(\frac{1}{2c} + \omega_2\right)\int_0^T |x_\epsilon(t) - x_\lambda(t)|^2 dt. \end{aligned} \tag{3.9}$$

Here, we used

$$\partial\phi_\epsilon(x_\epsilon(t)) = \epsilon^{-1}\Big(x_\epsilon(t) - \big(I + \epsilon\partial\phi\big)^{-1}x_\epsilon(t)\Big). \tag{3.10}$$

Since $|\partial\phi_\epsilon(x)| \le |(\partial\phi)^0 x|$ for every $x \in D(\partial\phi)$, it follows from (A) and using Gronwall's inequality that

$$\|x_\epsilon - x_\lambda\|_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon + \lambda), \quad 0 < T. \tag{3.11}$$

□

Theorem 3.2. *Let the assumptions (G1-G2) and* (A) *be satisfied. Then* $x = \lim_{\epsilon\to} x_\epsilon$ *in* $L^2(0, T; V) \cap C([0, T]; H)$ *is a solution of* (NCE)*, where* x_ϵ *is the solution of* (SCE)*.*

Proof. In virtue of Lemma 3.1, there exists $x(\cdot) \in L^2(0,T;V)$ such that

$$x_\epsilon(\cdot) \longrightarrow x(\cdot) \quad \text{in } L^2(0,T;V) \cap C([0,T];H). \tag{3.12}$$

From (G1-G2), it follows that

$$\begin{aligned} G(\cdot, x_\epsilon, \cdot) &\longrightarrow G(\cdot, x, \cdot), \quad \text{strongly in } L^2(0,T;H), \\ Ax_n &\longrightarrow Ax, \quad \text{strongly in } L^2(0,T;V^*). \end{aligned} \tag{3.13}$$

Since $\partial\phi_\epsilon(x_\epsilon)$ are uniformly bounded by assumption (A), from (3.13) we have that

$$\frac{d}{dt}x_\epsilon \longrightarrow \frac{d}{dt}x, \quad \text{weakly in } L^2(0,T;V^*), \tag{3.14}$$

therefore,

$$\partial\phi_\epsilon(x_\epsilon) \longrightarrow G(\cdot, x, \cdot) + k - x' - Ax, \quad \text{weakly in } L^2(0,T;V^*). \tag{3.15}$$

Note that $\partial\phi_\epsilon(x_\epsilon) = \epsilon^{-1}(I - (I + \epsilon\partial\phi)^{-1})(x_\epsilon)$. Since $(I + \epsilon\partial\phi)^{-1}\ x_\epsilon \ \rightarrow\ x$ strongly and $\partial\phi$ is demiclosed, we have that

$$G(\cdot, x, \cdot) + k - x' - Ax \in \partial\phi(x), \quad \text{in } L^2(0,T;V^*). \tag{3.16}$$

Thus we have proved that $x(t)$ satisfies a.e. on $(0,T)$ (NCE). □

4. Controllability of the Nonlinear Variational Inequality Problems

Let $x(T;\phi,g,u)$ be a state value of the system (SCE) at time T corresponding to the function ϕ, the nonlinear term g, and the control u. We define the reachable sets for the system (SCE) as follows:

$$\begin{aligned} R_T(h) &= \left\{ x(T;\phi,h,u) : u \in L^2(0,T;U) \right\}, \\ R_T(0) &= \left\{ x(T;\phi,0,u) : u \in L^2(0,T;U) \right\}, \\ L_T(0) &= \left\{ x(T;0,0,u) : u \in L^2(0,T;U) \right\}. \end{aligned} \tag{4.1}$$

Definition 4.1. The system (NCE) is said to be approximately controllable in the time interval $[0,T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$, there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T;\phi,h,u)$ of (NCE) satisfies $|x(T;\phi,h,u) - x_1| < \epsilon$, that is, if $\overline{R_T(h)} = H$, where $\overline{R_T(h)}$ is the closure of $R_T(h)$ in H, then the system (NCE) is called approximately controllable at time T.

We need the following hypothesis:
for any $\varepsilon > 0$ and $p \in L^2(0,T;H)$, there exists a $u \in L^2(0,T;U)$ such that

$$\left|\widehat{S}p - \widehat{S}Bu\right| < \varepsilon,$$
$$\|Bu\|_{L^2(0,t;H)} \le q_1 \|p\|_{L^2(0,t;H)}, \quad 0 \le t \le T, \tag{B}$$

where q_1 is a constant independent of p.
As seen in [18], we obtain the following results.

Proposition 4.2. *Under the assumptions (G1-G2),* (A)*, and* (B)*, the following system*

$$y'(t) + Ay(t) + \partial\phi_\varepsilon(y(t)) = Bu(t), \quad 0 < t \le T,$$
$$y(0) = x_0. \tag{4.2}$$

is approximately controllable on $[0,T]$*, that is,* $\overline{R_T(0)} = H$.

Let $u \in L^1(0,T;U)$. Then it is well-known that

$$\lim_{h\to 0} h^{-1}\int_0^h \|u(t+s) - u(t)\|_U ds = 0 \tag{4.3}$$

for almost all point of $t \in (0,T)$.

Definition 4.3. The point t which permits (4.3) to hold is called the Lebesgue point of u.

Let $x_\epsilon(T;\phi,h,u)$ be a solution of (SCE) such that $x(T;\phi,h,u) = \lim_{\epsilon\to x_\epsilon}(T;\phi,h,u)$ in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$ is a solution of (NCE). First we consider the approximate controllability of the system (SCE) in case where the controller B is the identity operator on H under the Lipschitz conditions (G1-G2) on the nonlinear operator h in Proposition 4.2. So, $H = U$ obviously.

Proposition 4.4. *Let* $y(t)$ *be solution of* (4.2) *corresponding to a control* u*. Then there exists a* $v \in L^2(0,T;H)$ *such that*

$$v(t) = u(t) - G(t,y,v), \quad 0 < t \le T$$
$$v(0) = u(0). \tag{4.4}$$

Proof. Let T_0 be a Lebesgue point of u, v so that

$$L_2\sqrt{T_0}\|k\|_{L^2(0,T_0)} < 1. \tag{4.5}$$

For a given $u \in L^2(0,T;H)$, we define a mapping

$$\Upsilon : L^2(0,T;H) \longrightarrow L^2(0,T;H) \tag{4.6}$$

by

$$(\Upsilon v)(t) = u(t) - G\big(t, y(t), v(t)\big), \quad 0 < t \le T_0. \tag{4.7}$$

It follows readily from definition of W and Lemma 2.4 that

$$\begin{aligned} \|\Upsilon v_1 - \Upsilon v_2\|_{L^2(0,T_0;H)} &= \big\|G(\cdot, y, v_2) - G(\cdot, y, v_1)\big\|_{L^2(0,T_0;H)} \\ &\le L_2\sqrt{T_0}\|k\|_{L^2(0,T_0)}\|v_2 - v_1\|_{L^2(0,T_0;H)}. \end{aligned} \tag{4.8}$$

By a well-known contraction mapping principle, Υ has a unique fixed point v in $L^2(0,T_0;H)$ if the condition (4.5) is satisfied. Let

$$v(t) = u(t) - G\big(t, y(t), v(t)\big). \tag{4.9}$$

Then from (G1-G2), Lemma 2.4, and Proposition 2.5, it follows that

$$\begin{aligned} \|v\|_{L^2(0,T_0;H)} &\le \big\|G(\cdot, y, v) + u\big\|_{L^2(0,T_0;H)} \\ &\le \sqrt{T_0}\|k\|_{L^2(0,T_0)}\Big(L_1\big\|y\big\|_{L^2(0,T_0;V)} + L_2\|v\|_{L^2(0,T_0;H)}\Big) \\ &\quad + \|G(\cdot,0,0) + u\|_{L^2(0,T_0;H)} \\ &\le \sqrt{T_0}\|k\|_{L^2(0,T_0)}\Big\{L_1C_3\Big(|x_0| + \|u\|_{L^2(0,T_0;U)}\Big) \\ &\quad + L_2\|v\|_{L^2(0,T_0;H)}\Big\} + \|G(\cdot,0,0) + u\|_{L^2(0,T_0;H)}. \end{aligned} \tag{4.10}$$

Thus, from which, we have

$$\begin{aligned} \|v\|_{L^2(0,T_0;H)} \le \Big(1 - L_2\sqrt{T_0}\|k\|_{L^2(0,T_0)}\Big)^{-1}\Big\{&\sqrt{T_0}\|k\|_{L^2(0,T_0)}L_1C_3\Big(|x_0| + \|u\|_{L^2(0,T;U)}\Big) \\ &+ \|G(\cdot,0,0) + u\|_{L^2(0,T_0;H)}\Big\}. \end{aligned} \tag{4.11}$$

And we obtain

$$
\begin{aligned}
|v(T_0)| &= \left|G(T_0, y(T_0), v(T_0)) - u(T_0)\right| \\
&\le \left|\int_0^{T_0} k(T_0 - s)\{h(s, y(s), v(s) - h(s,0,0))\}ds\right| \\
&\quad + \left|\int_0^{T_0} k(T_0 - s)h(s,0,0)ds + u(T_0)\right| \\
&\le \|k\|_{L^2(0,T_0)} \|h(\cdot, y, v) - h(\cdot,0,0)\|_{L^2(0,T_0;H)} + L_0\|k\|_{L^2(0,T_0)}\sqrt{T_0} + |u(T_0)| \\
&\le \|k\|_{L^2(0,T_0)}\left(L_1\|y\|_{L^2(0,T_0;V)} + L_2\|v\|_{L^2(0,T_0;H)} + L_0\sqrt{T_0}\right) + |u(T_0)|.
\end{aligned}
\tag{4.12}
$$

If $2T_0$ is a Lebesgue point of u, v, then we can solve the equation in $[T_0, 2T_0]$ with the initial value $v(T_0)$ and obtain an analogous estimate to (4.10) and (4.12). If not, we can choose $T_1 \in [T_0, 2T_0]$ to be a Lebesgue point of u, v. Since the condition (4.5) is independent of initial values, the solution can be extended to the interval $[T_1, T_1 + T_0]$, and so we have showed that there exists a $v \in L^2(0, T; H)$ such that $v(t) = u(t) - G(t, y(t), v(t))$. □

Now, we consider the approximate controllability for the following semilinear control system in case where B is the identity operator,

$$
\begin{gathered}
z'(t) + Az(t) + \partial\phi_\epsilon(z(t)) = G(t, z, v) + v(t), \quad 0 < t \le T, \\
z(0) = x_0.
\end{gathered}
\tag{4.13}
$$

Let us define the reachable sets for the system (4.13) as follows:

$$
\begin{gathered}
r_T(h) = \left\{z(T;\phi,h,u) : u \in L^2(0,T;U)\right\}, \\
r_T(0) = \left\{z(T;\phi,0,u) : u \in L^2(0,T;U)\right\}.
\end{gathered}
\tag{4.14}
$$

Theorem 4.5. *Under the assumptions (G1-G2),* (A)*, and* (B)*, we have*

$$
r_T(0) \subset \overline{r_T(h)}. \tag{4.15}
$$

Therefore, if the system (4.2) *with* $h = 0$ *is approximately controllable, then so is the semilinear system* (4.13).

Proof. Let $v(t) = u(t) - G(t, y(t), v(t))$ and let $y = z(T;\phi,0,u)$ be a solution of (4.2) corresponding to a control u. Consider the following semilinear system:

$$
\begin{gathered}
z'(t) + Az(t) + \partial\phi_\epsilon(z(t)) = G(t, z(t), v(t)) + u(t) - G(t, y(t), v(t)), \quad 0 < t \le T, \\
z(0) = x_0.
\end{gathered}
\tag{4.16}
$$

The solution of (4.2) and (4.16), respectively, can be written as

$$y(t) = S(t)x_0 + \int_0^t S(t-s)\{u(s) - \partial\phi_\epsilon(z(s))\}ds,$$
$$z(t) = S(t)x_0 + \int_0^t S(t-s)\{u(s) - \partial\phi_\epsilon(z(s))\}ds$$
$$+ \int_0^t S(t-s)\{G(s,z(s),v(s)) - G(s,y(s),v(s))\}ds. \tag{4.17}$$

Then from Proposition 2.5, it is easily seen that $z(\cdot) \in C([0,T];H)$, that is, $z(s) \to z(t)$ as $s \to t$ in H. Let $\delta > 0$ be given. For $\delta \le t$, set

$$z^\delta(t) = S(t)x_0 + \int_0^{t-\delta} S(t-s)\left\{u(s) - \partial\phi_\epsilon\left(z^\delta(s)\right)\right\}ds$$
$$+ \int_0^{t-\delta} S(t-s)\left\{G\left(s, z^\delta(s), v(s)\right) - G(s,y(s),v(s))\right\}ds. \tag{4.18}$$

Then we have

$$z(t) - z^\delta(t) = \int_{t-\delta}^t S(t-s)\{u(s) - \partial\phi_\epsilon(z(s))\}ds - \int_{t-\delta}^t S(t-s)G(s,y(s),v(s))ds$$
$$+ \int_{t-\delta}^t S(t-s)G(s,z(s),v(s))ds$$
$$+ \int_0^{t-\delta} S(t-s)\left\{\partial\phi_\epsilon(z(s)) - \partial\phi_\epsilon\left(z^\delta(s)\right)\right\}ds$$
$$+ \int_0^{t-\delta} S(t-s)\left\{G(s,z(s),v(s)) - G\left(s,z^\delta(s),v(s)\right)\right\}ds. \tag{4.19}$$

So, for fixing $\epsilon > 0$, we choose some constant $T_1 > 0$ satisfying

$$C_2\sqrt{T_1}\left(L_1\|k\|_{L^2(0,T)} + \epsilon^{-1}\right) < 1, \tag{4.20}$$

and from (2.13), or (2.16) it follows that

$$\left\|z - z^\delta\right\|_{L^2(0,T_1;V)} \le C_2\sqrt{\delta}\left(M_1 + \|u\|_{L^2(0,T_1;H)}\right) + C_2L_1\sqrt{\delta}\|k\|_{L^2(0,T)}\|z-y\|_{L^2(0,T_1;V)}$$
$$+ C_2\sqrt{T_1}\left(L_1\|k\|_{L^2(0,T)} + \epsilon^{-1}\right)\left\|z - z^\delta\right\|_{L^2(0,T_1;V)}. \tag{4.21}$$

Thus, we know that $z^\delta \to z$ as $\delta \to 0$ in $L^2(0, T_1; V)$ for $\delta < t < T_1$. Noting that

$$\begin{aligned} z^\delta(t) - y(t) = &- \int_{t-\delta}^{t} S(t-s)\{u - \partial\phi_\epsilon(z(s))\}ds \\ &+ \int_{t-\delta}^{t} S(t-s)\left\{\partial\phi_\epsilon(z(s)) - \partial\phi_\epsilon\left(z^\delta(s)\right)\right\}ds \\ &+ \int_{0}^{t-\delta} S(t-s)\left\{G\left(s, z^\delta(s), v(s)\right) - G(s, y(s), v(s))\right\}ds, \end{aligned} \tag{4.22}$$

from (2.13), or (2.16), it follows that

$$\begin{aligned} \left\|z^\delta - y\right\|_{L^2(0,T_1;V)} = &\, C_2\sqrt{\delta}\|u - \partial\phi_\epsilon(z)\|_{L^2(0,T_1;H)} + C_2\sqrt{\delta}\epsilon^{-1}\left\|z - z^\delta\right\|_{L^2(0,T_1;V)} \\ &+ C_2\sqrt{T_1}L_1\|k\|_{L^2(0,T)}\left\|z^\delta - y\right\|_{L^2(0,T_1;V)}. \end{aligned} \tag{4.23}$$

Since the condition (4.20) is independent of δ, by the step by stem method, we get $z^\delta \to y$ as $\delta \to 0$ in $L^2(0, T; V)$, for all $\delta < t < T$. Therefore, noting that $z(\cdot)$, $y(\cdot) \in C([0, T; H])$, every solution of the linear system with control u is also a solution of the semilinear system with control v, that is, we have that $r_T(0) \subset \overline{r_T(h)}$ in case where $B = I$. □

From now on, we consider the initial value problem for the semilinear parabolic equation (SCE). Let U be some Banach space and let the controller operator $B \neq I$ be a bounded linear operator from U to H.

Theorem 4.6. *Let us assume that there exists a constant $\beta > 0$ such that*

$$\|Bu\| \geq \beta\|u\| \quad \forall u \in L^2(0, T; U), \quad R(G) \subset R(B). \tag{B1}$$

Assume that assumptions (G1-G2), (A)*, and* (B) *are satisfied. Then we have*

$$R_T(0) \subset \overline{R_T(h)}, \tag{4.24}$$

that is, the system (SCE) *is approximately controllable on* $[0, T]$.

Proof. Let x be a solution of the smoothing system (SCE) corresponding to (NCE). Set $v(t) = u(t) - B^{-1}G(t, y, v)$ where y is a solution of (4.2) corresponding to a control u. Then as seen in Theorem 4.5, we know that $v \in L^2(0, T; U)$. Consider the following semilinear system:

$$\begin{aligned} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) &= G(t, x, v) + Bv(t) \\ &= G(t, x, v) + Bu(t) - G(t, y, v), \quad 0 < t \leq T, \\ z(0) &= x_0. \end{aligned} \tag{4.25}$$

If we define x^δ as in proof of Theorem 3.2, then we get

$$\begin{aligned} x^\delta(t) - y(t) = &-\int_{t-\delta}^{t} S(t-s)\{u - \partial\phi_\epsilon(x(s))\}ds \\ &+ \int_{t-\delta}^{t} S(t-s)\Big\{\partial\phi_\epsilon(x(s)) - \partial\phi_\epsilon\Big(x^\delta(s)\Big)\Big\}ds \\ &+ \int_{0}^{t-\delta} S(t-s)\Big\{G\Big(s, x^\delta, v(s)\Big) - G(s, y, v(s))\Big\}ds. \end{aligned} \tag{4.26}$$

So, as similar to the proof of Theorem 3.2, we obtain that $R_T(0) \subset \overline{R_T(h)}$. □

From Theorems 3.2 and 4.6, we obtain the following results.

Theorem 4.7. *Under the assumptions (G1-G2),* (A)*,* (B)*, and* (B1)*, the system* (NCE) *is approximately controllable on* $[0, T]$.

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (no. 2009-0071344).

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