

Research Article

On Hilbert-Pachpatte Multiple Integral Inequalities

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We establish some multiple integral Hilbert-Pachpatte-type inequalities. As applications, we get some inverse forms of Pachpatte's inequalities which were established in 1998.

1. Introduction

In 1934, Hilbert [1] established the following well-known integral inequality.

If $f \in L^p(0, \infty)$, $g \in L^q(0, \infty)$, $f, g \geq 0$, $p > 1$ and $1/p + 1/q = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(x) dx \right)^{1/q}, \quad (1.1)$$

where $\pi / \sin(\pi/p)$ is the best value.

In recent years, considerable attention has been given to various extensions and improvements of the Hilbert inequality from different viewpoints [2–10]. In particular, Pachpatte [11] proved some inequalities similar to Hilbert's integral inequalities in 1998. In this paper, we establish some new multiple integral Hilbert-Pachpatte-type inequalities.

2. Main Results

Theorem 2.1. Let $h_i \geq 1$, let $f_i(\sigma_i) \in C^1[(x_i, 0), (0, \infty)]$, $i = 1, \dots, n$, where x_i are positive real numbers, and define $F_i(s_i) = \int_{s_i}^0 f_i(\sigma_i) d\sigma_i$, for $s_i \in (x_i, 0)$. Then for $1/\alpha_i + 1/\beta_i = 1$, $0 < \beta_i < 1$ and $\sum_{i=1}^n (1/\alpha_i) = 1/\alpha$,

$$\int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i))^{1/\alpha}} ds_1 \cdots ds_n \geq \prod_{i=1}^n (-x_i)^{1/\alpha_i} h_i \left(\int_{x_i}^0 (s_i - x_i) (F_i^{h_i-1}(s_i) f_i(s_i))^{\beta_i} ds_i \right)^{1/\beta_i}. \quad (2.1)$$

Proof. From the hypotheses and in view of inverse Hölder integral inequality (see [12]), it is easy to observe that

$$\begin{aligned} \prod_{i=1}^n F_i^{h_i}(s_i) &= \prod_{i=1}^n h_i \int_{s_i}^0 F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) d\sigma_i \\ &\geq \prod_{i=1}^n h_i (-s_i)^{1/\alpha_i} \left(\int_{s_i}^0 (F_i^{h_i-1}(\sigma_i) f_i(\sigma_i))^{\beta_i} d\sigma_i \right)^{1/\beta_i}, \quad s_i \in (x_i, 0), \quad i = 1, \dots, n. \end{aligned} \quad (2.2)$$

Let us note the following means inequality:

$$\prod_{i=1}^n m_i^{1/\alpha_i} \geq \left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} m_i \right)^{1/\alpha}, \quad m > 0. \quad (2.3)$$

We obtain that

$$\frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i))^{1/\alpha}} \geq \prod_{i=1}^n h_i \left(\int_{s_i}^0 (F_i^{h_i-1}(\sigma_i) f_i(\sigma_i))^{\beta_i} d\sigma_i \right)^{1/\beta_i}. \quad (2.4)$$

Integrating both sides of (2.4) over s_i from x_i ($i = 1, 2, \dots, n$) to 0 and using the special case of inverse Hölder integral inequality, we observe that

$$\begin{aligned} &\int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i))^{1/\alpha}} ds_1 \cdots ds_n \\ &\geq \prod_{i=1}^n h_i \int_{x_i}^0 \left(\int_{s_i}^0 (F_i^{h_i-1}(\sigma_i) f_i(\sigma_i))^{\beta_i} d\sigma_i \right)^{1/\beta_i} ds_i \\ &\geq \prod_{i=1}^n h_i (-x_i)^{1/\alpha_i} \left(\int_{x_i}^0 \left(\int_{s_i}^0 (F_i^{h_i-1}(\sigma_i) f_i(\sigma_i))^{\beta_i} d\sigma_i \right) ds_i \right)^{1/\beta_i} \\ &= \prod_{i=1}^n (-x_i)^{1/\alpha_i} h_i \left(\int_{x_i}^0 (s_i - x_i) (F_i^{h_i-1}(s_i) f_i(s_i))^{\beta_i} ds_i \right)^{1/\beta_i}. \end{aligned} \quad (2.5)$$

The proof is complete. \square

Remark 2.2. Taking $n = 2$, $\beta_i = 1/2$ to (2.1), (2.1) changes to

$$\begin{aligned} & \int_{x_1}^0 \int_{x_2}^0 \frac{F_1^{h_1}(s_1) F_2^{h_2}(s_2)}{(s_1 + s_2)^{-2}} ds_1 ds_2 \\ & \geq 4h_1 h_2 (x_1 x_2)^{-1} \left(\int_{x_1}^0 (s_1 - x_1) \left(F_1^{h_1-1}(s_1) f_1(s_1) \right)^{1/2} ds_1 \right)^2 \\ & \quad \times \left(\int_{x_2}^0 (s_2 - x_2) \left(F_2^{h_2-1}(s_2) f_2(s_2) \right)^{1/2} ds_2 \right)^2. \end{aligned} \quad (2.6)$$

This is just an inverse inequality similar to the following inequality which was proved by Pachpatte [11]:

$$\begin{aligned} & \int_0^x \int_0^y \frac{F^h(s) G^l(t)}{s+t} ds dt \leq \frac{1}{2} hl(xy)^{1/2} \left(\int_0^x (x-s) \left(F^{h-1}(s) f(s) \right)^2 ds \right)^{1/2} \\ & \quad \times \left(\int_0^y (y-t) \left(G^{l-1}(t) g(t) \right)^2 dt \right)^{1/2}. \end{aligned} \quad (2.7)$$

Theorem 2.3. Let $f_i(\sigma_i)$, $F_i(s_i)$, α_i , and β_i be as in Theorem 2.1. Let $p_i(\sigma_i)$ be n positive functions defined for $\sigma_i \in (x_i, 0)$ ($i = 1, 2, \dots, n$), and define $P_i(s_i) = \int_{s_i}^0 p_i(\sigma_i) d\sigma_i$, where x_i are positive real numbers. Let ϕ_i ($i = 1, 2, \dots, n$) be n real-valued nonnegative, concave, and super-multiplicative functions defined on \mathbb{R}_+ . Then

$$\begin{aligned} & \int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{\left(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i) \right)^{1/\alpha}} ds_1 \cdots ds_n \\ & \geq L(x_1, \dots, x_n) \prod_{i=1}^n \left(\int_{x_i}^0 (s_i - x_i) \left(p_i(s_i) \phi_i \left(\frac{f_i(s_i)}{p_i(s_i)} \right) \right)^{\beta_i} ds_i \right)^{1/\beta_i}, \end{aligned} \quad (2.8)$$

where

$$L(x_1, \dots, x_n) = \prod_{i=1}^n \left(\int_{x_i}^0 \left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)} \right)^{\alpha_i} ds_i \right)^{1/\alpha_i}. \quad (2.9)$$

Proof. By using Jensen integral inequality (see [11]) and inverse Hölder integral inequality (see [12]) and noticing that ϕ_i ($i = 1, 2, \dots, n$) are n real-valued super-multiplicative functions, it is easy to observe that

$$\begin{aligned}
 \phi_i(F_i(s_i)) &= \phi_i\left(\frac{P_i(s_i) \int_{s_i}^0 p_i(\sigma_i) (f_i(\sigma_i)/p_i(\sigma_i)) d\sigma_i}{\int_{s_i}^0 p_i(\sigma_i) d\sigma_i}\right) \\
 &\geq \phi_i(P_i(s_i)) \phi_i\left(\frac{\int_{s_i}^0 p_i(\sigma_i) (f_i(\sigma_i)/p_i(\sigma_i)) d\sigma_i}{\int_{s_i}^0 p_i(\sigma_i) d\sigma_i}\right) \\
 &\geq \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \int_{s_i}^0 p_i(\sigma_i) \phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right) d\sigma_i \\
 &\geq \left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right) (-s_i)^{1/\alpha_i} \left(\int_{s_i}^0 \left(p_i(\sigma_i) \phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i}.
 \end{aligned} \tag{2.10}$$

In view of the means inequality and integrating two sides of (2.10) over s_i from x_i ($i = 1, 2, \dots, n$) to 0 and noticing Hölder integral inequality, we observe that

$$\begin{aligned}
 &\int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{(\alpha \sum_{i=1}^n (1/\alpha_i) (-s_i))^{1/\alpha}} ds_1 \cdots ds_n \\
 &\geq \prod_{i=1}^n \int_{x_i}^0 \left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right) \left(\int_{s_i}^0 \left(p_i(\sigma_i) \phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i} ds_i \\
 &\geq \prod_{i=1}^n \left(\int_{x_i}^0 \left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)^{\alpha_i} ds_i\right)^{1/\alpha_i} \left(\int_{x_i}^0 \int_{s_i}^0 \left(p_i(\sigma_i) \phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i} d\sigma_i ds_i\right)^{1/\beta_i} \\
 &= L(x_1, \dots, x_n) \prod_{i=1}^n \left(\int_{x_i}^0 (s_i - x_i) \left(p_i(s_i) \phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right)^{\beta_i} ds_i\right)^{1/\beta_i}.
 \end{aligned} \tag{2.11}$$

This completes the proof of Theorem 2.3. \square

Remark 2.4. Taking $n = 2$, $\beta_i = 1/2$ to (2.8), (2.8) changes to

$$\begin{aligned}
 &\int_{x_1}^0 \int_{x_2}^0 \frac{\phi_1(F_1(s_1)) \phi_2(F_2(s_2))}{(s_1 + s_2)^{-2}} ds_1 ds_2 \\
 &\geq L(x_1, x_2) \left(\int_{x_1}^0 (s_1 - x_1) \left(p_1(s_1) \phi_1\left(\frac{f_1(s_1)}{p_1(s_1)}\right)\right)^{1/2} ds_1\right)^2 \\
 &\quad \times \left(\int_{x_2}^0 (s_2 - x_2) \left(p_2(s_2) \phi_2\left(\frac{f_2(s_2)}{p_2(s_2)}\right)\right)^{1/2} ds_2\right)^2,
 \end{aligned} \tag{2.12}$$

where

$$L(x_1, x_2) = 4 \left(\int_{x_1}^0 \left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)} \right)^{-1} ds_1 \right)^{-1} \left(\int_{x_2}^0 \left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} ds_2 \right)^{-1}. \quad (2.13)$$

This is just an inverse inequality similar to the following inequality which was proved by Pachpatte [11]:

$$\begin{aligned} \int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t} ds dt &\leq L(x, y) \left(\int_0^x (x-s) \left(p(s) \phi \left(\frac{f(s)}{p(s)} \right) \right)^2 ds \right)^{1/2} \\ &\times \left(\int_0^y (y-t) \left(q(t) \psi \left(\frac{g(t)}{q(t)} \right) \right)^2 dt \right)^{1/2}, \end{aligned} \quad (2.14)$$

where

$$L(x, y) = \frac{1}{2} \left(\int_0^x \left(\frac{\phi(P(s))}{P(s)} \right)^2 ds \right)^{1/2} \left(\int_0^y \left(\frac{\psi(Q(t))}{Q(t)} \right)^2 dt \right)^{1/2}. \quad (2.15)$$

Theorem 2.5. Let $f_i(\sigma_i)$, $p_i(\sigma_i)$, $P_i(\sigma_i)$, α_i , and β_i be as Theorem 2.3, and define $F_i(s_i) = (1/P_i(s_i)) \int_{s_i}^0 p_i(\sigma_i) f_i(\sigma_i) d\sigma_i$ for $\sigma_i, s_i \in (x_i, 0)$, where x_i are positive real numbers. Let ϕ_i ($i = 1, 2, \dots, n$) be n real-valued, nonnegative, and concave functions on \mathbb{R}_+ . Then

$$\begin{aligned} &\int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n P_i(s_i) \phi_i(F_i(s_i))}{(\alpha \sum_{i=1}^n (1/\alpha_i) (-s_i))^{1/\alpha}} ds_1 \cdots ds_n \\ &\geq \prod_{i=1}^n x_i^{1/\alpha_i} \left(\int_{x_i}^0 (s_i - x_i) (p_i(s_i) \phi_i(f_i(s_i)))^{\beta_i} ds_i \right)^{1/\beta_i}. \end{aligned} \quad (2.16)$$

Proof. From the hypotheses and by using Jensen integral inequality and the inverse Hölder integral inequality, we have

$$\begin{aligned} \phi_i(F_i(s_i)) &= \phi_i \left(\frac{1}{P_i(s_i)} \int_{s_i}^0 p_i(\sigma_i) f_i(\sigma_i) d\sigma_i \right) \\ &\geq \frac{1}{P_i(s_i)} \int_{s_i}^0 p_i(\sigma_i) \phi_i(f_i(\sigma_i)) d\sigma_i \\ &\geq \frac{1}{P_i(s_i)} (-s_i)^{1/\alpha_i} \left(\int_{s_i}^0 (p_i(\sigma_i) \phi_i(f_i(\sigma_i)))^{\beta_i} d\sigma_i \right)^{1/\beta_i}. \end{aligned} \quad (2.17)$$

Hence

$$\begin{aligned}
 & \int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n P_i(s_i) \phi_i(F_i(s_i))}{(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i))^{1/\alpha}} ds_1 \cdots ds_n \\
 & \geq \prod_{i=1}^n \int_{x_i}^0 \left(\int_{s_i}^0 (p_i(\sigma_i) \phi_i(f_i(\sigma_i)))^{\beta_i} d\sigma_i \right)^{1/\beta_i} ds_i \\
 & \geq \prod_{i=1}^n x_i^{1/\alpha_i} \left(\int_{x_i}^0 \int_{s_i}^0 (p_i(\sigma_i) \phi_i(f_i(\sigma_i)))^{\beta_i} d\sigma_i ds_i \right)^{1/\beta_i} \\
 & = \prod_{i=1}^n (-x_i)^{1/\alpha_i} \left(\int_{x_i}^0 (s_i - x_i) (p_i(s_i) \phi_i(f_i(s_i)))^{\beta_i} ds_i \right)^{1/\beta_i}.
 \end{aligned} \tag{2.18}$$

□

Remark 2.6. Taking $n = 2$, $\beta_i = 1/2$ to (2.16), (2.16) changes to

$$\begin{aligned}
 & \int_{x_1}^0 \int_{x_2}^0 \frac{P_1(s_1) P_2(s_2) \phi_1(F_1(s_1)) \phi_2(F_2(s_2))}{(s_1 + s_2)^{-2}} ds_1 ds_2 \\
 & \geq 4(x_1 x_2)^{-1} \left(\int_{x_1}^0 (s_1 - x_1) (p_1(s_1) \phi_1(f_1(s_1)))^{1/2} ds_1 \right)^2 \\
 & \quad \times \left(\int_{x_2}^0 (s_2 - x_2) (p_2(s_2) \phi_2(f_2(s_2)))^{1/2} ds_2 \right)^2.
 \end{aligned} \tag{2.19}$$

This is just an inverse inequality similar to the following inequality which was proved by Pachpatte [11]:

$$\begin{aligned}
 & \int_0^x \int_0^y \frac{P(s) Q(t) \phi(F(s)) \psi(G(t))}{s + t} ds dt \\
 & \leq \frac{1}{2} (xy)^{1/2} \left(\int_0^x (x - s) (p(s) \phi(f(s)))^2 ds \right)^{1/2} \left(\int_0^y (y - t) (q(t) \psi(g(t)))^2 dt \right)^{1/2}.
 \end{aligned} \tag{2.20}$$

Remark 2.7. In (2.20), if $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore (2.20) changes to

$$\begin{aligned}
 & \int_{x_1}^0 \int_{x_2}^0 \frac{\phi_1(F_1(s_1)) \phi_2(F_2(s_2))}{(s_1 + s_2)^{-2}} ds_1 ds_2 \\
 & \geq 4(x_1 x_2)^{-1} \left(\int_{x_1}^0 (s_1 - x_1) (\phi_1(f_1(s_1)))^{1/2} ds_1 \right)^2 \left(\int_{x_2}^0 (s_2 - x_2) (\phi_2(f_2(s_2)))^{1/2} ds_2 \right)^2.
 \end{aligned} \tag{2.21}$$

This is just an inverse inequality similar to the following Inequality which was proved by Pachpatte [11]:

$$\int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{(st)^{-1}(s+t)} ds dt \leq \frac{1}{2}(xy)^{1/2} \left(\int_0^x (x-s)(\phi(f(s)))^2 ds \right)^{1/2} \left(\int_0^y (y-t)(\psi(g(t)))^2 dt \right)^{1/2}. \quad (2.22)$$

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