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Research Article

Generalized Ulam-Hyers Stability of Jensen Functional Equation in Šerstnev PN Spaces

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We establish a generalized Ulam-Hyers stability theorem in a Šerstnev probabilistic normed space (briefly, Šerstnev PN-space) endowed with Π_M. In particular, we introduce the notion of approximate Jensen mapping in PN-spaces and prove that if an approximate Jensen mapping in a Šerstnev PN-space is continuous at a point then we can approximate it by an everywhere continuous Jensen mapping. As a version of a theorem of Schwaiger, we also show that if every approximate Jensen type mapping from the natural numbers into a Šerstnev PN-space can be approximated by an additive mapping, then the norm of Šerstnev PN-space is complete.

1. Introduction and Preliminaries

Menger proposed transferring the probabilistic notions of quantum mechanic from physics to the underlying geometry. The theory of probabilistic normed spaces (briefly, PN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The notion of a probabilistic normed space was introduced by Šerstnev [1]. Alsina, Schweizer, and Skalar gave a general definition of probabilistic normed space based on the definition of Meneger for probabilistic metric spaces in [2, 3].

Ulam propounded the first stability problem in 1940 [4]. Hyers gave a partial affirmative answer to the question of Ulam in the next year [5].

Theorem 1.1 (see [6]). *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \epsilon \tag{1.1}$$

for all $x, y \in X$. *Then the limit*

$$a(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in X$ *and* $a : X \to Y$ *is the unique additive mapping satisfying*

$$\|f(x) - a(x)\| \leq \epsilon \tag{1.3}$$

for all $x \in X$.

Hyers' theorem was generalized by Aoki [7] for additive mappings and by Th. M. Rassias [8] for linear mappings by considering an unbounded Cauchy difference. For some historical remarks see [9].

Theorem 1.2 (see [10]). *Let* X *and* Y *be two Banach spaces. Let* $\theta \in [0, \infty)$ *and let* $p \in [0, 1)$. *If a function* $f : X \to Y$ *satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta\left(\|x\|^p + \|y\|^p\right) \tag{1.4}$$

for all $x, y \in X$, *then there exists a unique linear mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \tag{1.5}$$

for all $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then the function* T *is linear.*

Theorem 1.2 was later extended for all $p \neq 1$. The stability phenomenon that was presented by Rassias is called the generalized Ulam-Hyers stability. In 1982, Rassias [11] gave a further generalization of the result of Hyers and proved the following theorem using weaker conditions controlled by a product of powers of norms.

Theorem 1.3. *Let* $f : E \to E'$ *be a mapping from a normed vector space* E *into a Banach space* E' *subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon\|x\|^p\|y\|^p \tag{1.6}$$

for all $x, y \in E$, *where* ϵ *and* p *are constants with* $\epsilon > 0$ *and* $0 \leq p < 1/2$. *Then the limit*

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{1.7}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^{2p}}\|x\|^{2p} \tag{1.8}$$

for all $x \in E$.

The above mentioned stability involving a product of powers of norms is called Ulam-Gavruta-Rassias stability by various authors (see [12–21]). In the last two decades, several forms of mixed type functional equations and their Ulam-Hyers stability are dealt with in various spaces like fuzzy normed spaces, random normed spaces, quasi-Banach spaces, quasi-normed linear spaces, and Banach algebras by various authors like in [6, 9, 14, 22–38].

Let $f : X \to Y$ be a mapping between linear spaces. The Jensen functional equation is

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y). \tag{1.9}$$

It is easy to see that f with $f(0) = 0$ satisfies the Jensen equation if and only if it is additive; compare for [39, Theorem 6]. Stability of Jensen equation has been studied at first by Kominek [36] and then by several other mathematicians example, (see [10, 33, 40–42] and references therein).

PN spaces were first defined by Šerstnev in1963 (see [1]). Their definition was generalized in [2]. We recall and apply the definition of probabilistic space briefly as given in [43], together with the notation that will be needed (see [43]). A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F from $\overline{\mathbb{R}}^+$ into $[0,1]$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0,+\infty)$; here as usual, $\overline{\mathbb{R}}^+ := [0,+\infty]$. The space of d.d.f.'s will be denoted by Δ^+, and the set of all F in Δ^+ for which $\lim_{t\to+\infty^-} F(t) = 1$ by D^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\overline{\mathbb{R}}^+$. For any $a \geq 0$, ε_a is the d.d.f. given by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases} \tag{1.10}$$

The space Δ^+ can be metrized in several ways [43], but we shall here adopt the Sibley metric d_S. If F, G are d.f.'s and h is in $]0,1[$, let $(F,G;h)$ denote the condition

$$G(x) \leq F(x+h) + h \quad \forall x \in \left]0, \frac{1}{h}\right[. \tag{1.11}$$

Then the Sibley metric d_S is defined by

$$d_S(F,G) := \inf\{h \in]0,1[: \text{both } (F,G;h) \text{ and } (G,F;h) \text{ hold}\}. \tag{1.12}$$

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+. A triangle function is a binary operation on Δ^+, namely, a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing in each place, and has ε_0 as identity, that is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$,

(TF2) $\tau(F,G) = \tau(G,F)$,

(TF3) $F \le G \Rightarrow \tau(F,H) \le \tau(G,H)$,

(TF4) $\tau(F,\varepsilon_0) = \tau(\varepsilon_0,F) = F$.

Moreover, a triangle function is *continuous* if it is continuous in the metric space (Δ^+, d_S).

Typical continuous triangle functions are $\Pi_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t))$, and $\Pi_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t))$. Here T is a continuous t-norm, that is, a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable, and has 1 as identity; T^* is a continuous t-conorm, namely, a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x,1-y)$. For example, $T(x,y) = \min(x,y) = M(x,y)$ and $T^*(x,y) = \max(x,y)$ or $T(x,y) = \pi(x,y) = xy$ and $T^*(x,y) = \pi^*(x,y) = x+y-xy$.

Note that $\prod_M(F,G)(x) = \min\{F(x),G(x)\}$ for $F,G \in \Delta^+$ and $x \in \mathbb{R}^+$.

Definition 1.4. A *Probabilistic Normed space* (briefly, PN space) is a quadruple (X,ν,τ,τ^*), where X is a real vector space, τ and τ^* are continuous triangle functions with $\tau \le \tau^*$ and ν is a mapping (the *probabilistic norm*) from X into Δ^+ such that for every choice of p and q in X the following hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$ (θ is the null vector in X),

(N2) $\nu_{-p} = \nu_p$,

(N3) $\nu_{p+q} \ge \tau(\nu_p, \nu_q)$,

(N4) $\nu_p \le \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A PN space is called a Šerstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right) \tag{1.13}$$

holds for every $\alpha \ne 0 \in \mathbb{R}$ and $x > 0$. When here is a continuous t-norm T such that $\tau = \Pi_T$ and $\tau^* = \Pi_{T^*}$, the PN space (X,ν,τ,τ^*) is called Meneger PN space (briefly, MPN space), and is denoted by (X,ν,τ).

Let (X,ν,τ) be an MPN space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1 \tag{1.14}$$

for all $t > 0$. In this case x is called the limit of $\{x_n\}$.

The sequence x_n in MPN Space (X,ν,τ) is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$ there exist some n_0 such that $\nu(x_n - x_m)(\delta) > 1 - \varepsilon$ for all $m,n \ge n_0$.

Clearly, every convergent sequence in an MPN space is Cauchy. If each Cauchy sequence is convergent in an MPN space (X,ν,τ), then (X,ν,τ) is called Meneger Probabilistic Banach space (briefly, MPB space).

2. Stability of Jensen Mapping in Šerstnev MPN Spaces

In this section, we provide a generalized Ulam-Hyers stability theorem in a Šerstnev MPN space.

Theorem 2.1. *Let* X *be a real linear space and let* f *be a mapping from* X *to a Šerstnev MPB space* (Y, ν, Π_M) *such that* $f(0) = 0$*. Suppose that* φ *is a mapping from* X *into a Šerstnev MPN space* (Z, ω, Π_M) *such that*

$$\nu\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\ \right)(t) \geq \Pi_M\{\omega(\varphi(x)), \omega(\varphi(y))\}(t), \tag{2.1}$$

for all $x, y \in X - \{0\}$ *and positive real number* t*. If* $\varphi(3x) = \alpha\varphi(x)$ *for some real number* α *with* $0 < |\alpha| < 3$*, then there is a unique additive mapping* $T : X \to Y$ *such that* $T(x) = \lim_{n\to\infty} 3^{-n} f(3^n)$ *and*

$$\nu(T(x) - f(x)) \geq \psi_x(t), \tag{2.2}$$

where

$$\psi_x(t) := \Pi_M\{\Pi_M\{\omega(\varphi(x)), \omega(\varphi(-x))\}, \Pi_M\{\omega(\varphi(3x)), \omega(\varphi(-x))\}\}(3t). \tag{2.3}$$

Proof. Without loss of generality we may assume that $0 < \alpha < 3$. Replacing y by $-x$ in (2.1) we get

$$\nu(-f(x) - f(-x))(t) \geq \Pi_M\{\omega(\varphi(x)), \omega(\varphi(-x))\}(t) \tag{2.4}$$

and replacing x by $-x$ and y by $3x$ in (2.1), we obtain

$$\nu(2f(x) - f(-x) - f(3x))(t) \geq \Pi_M\{\omega(\varphi(-x)), \omega(\varphi(3x))\}(t). \tag{2.5}$$

Thus

$$\nu(3f(x) - f(3x))(t) \geq \Pi_M\{\Pi_M\{\omega(\varphi(x)), \omega(\varphi(-x))\}, \Pi_M\{\omega(\varphi(3x)), \omega(\varphi(-x))\}\}(t) \tag{2.6}$$

and so

$$\nu\left(f(x) - 3^{-1}f(3x)\right)(t) \geq \psi_x(t). \tag{2.7}$$

By our assumption, we have

$$\psi_{3x}(t) = \psi_x\left(\frac{1}{\alpha}t\right). \tag{2.8}$$

Replacing x by $3^n x$ in (2.7) and applying (2.8), we get

$$\begin{aligned}\nu\Big(f(3^n x)3^{-n} - f\Big(3^{n+1}x\Big)3^{-n-1}\Big)\left(\frac{\alpha^n}{3^n}t\right) &= \nu\Big(f(3^n) - f\Big(3^{n+1}x\Big)3^{-1}\Big)(\alpha^n t)\\ &\geq \psi_{3^n x}(\alpha^n t) = \psi_x(t).\end{aligned} \tag{2.9}$$

Thus for each $n > m$, we have

$$\begin{aligned}&\nu\big(f(3^m x)3^{-m} - f(3^n x)3^{-n}\big)\left(\frac{\alpha^m}{3^m}t\right)\\ &\quad = \nu\left(\sum_{k=m}^{n-1}\Big(f\Big(3^k x\Big)3^{-k} - f\Big(3^{k+1}x\Big)3^{-k-1}\Big)\right)\left(\frac{\alpha^m}{3^m}t\right)\\ &\quad \geq \Pi_M\Bigg\{\nu\Big(f(3^m x)3^{-m} - f\Big(3^{m+1}x\Big)3^{-m-1}\Big)\left(\frac{\alpha^m}{3^m}t\right),\\ &\qquad\quad \nu\left(\sum_{k=m+1}^{n-1} f\Big(3^k x\Big)3^{-k} - f\Big(3^{k+1}x\Big)3^{-k-1}\right)\left(\frac{\alpha^{m+1}}{3^{m+1}}t\right)\Bigg\}\\ &\quad \geq \psi_x(t).\end{aligned} \tag{2.10}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since

$$\lim_{t\to\infty}\psi_x(t) = 1, \tag{2.11}$$

there is some $t_0 > 0$ such that $\psi_x(t_0) > 1 - \varepsilon$. Since

$$\lim_{m\to\infty}\left(\frac{\alpha^m}{3^m}t_0\right) = 0, \tag{2.12}$$

there is some $n_0 \in \mathbb{N}$ such that $(\alpha^m/3^m)t_0 < \delta$ for all $m \geq n_0$. Thus for all $n > m \geq n_0$ we have

$$\begin{aligned}\nu\big(f(3^m x)3^{-m} - f(3^n x)3^{-n}\big)(\delta) &\geq \nu\big(f(3^m x)3^{-m} - f(3^n x)3^{-n}\big)\left(\frac{\alpha^m}{3^m}t_0\right)\\ &\geq \psi_x(t_0) > 1 - \varepsilon.\end{aligned} \tag{2.13}$$

This shows that $\{3^{-n}f(3^n x)\}$ is a Cauchy sequence in (Y, ν, Π_M). Since (Y, ν, Π_M) is complete, $\{f(3^n x)3^{-n}\}$ converges to some $T(x) \in Y$. Thus we can well define a mapping $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty} 3^{-n}f(3^n x). \tag{2.14}$$

Moreover, if we put $m = 0$ in (2.10), then we obtain

$$\nu\big(f(x) - f(3^n x)3^{-n}\big)(t) \geq \psi_x(t). \tag{2.15}$$

Next we will show that T is additive. Let $x, y \in X$. Then we have

$$\begin{aligned}
&\nu\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y)\right)(t) \\
&\quad \geq \Pi_M\left\{\Pi_M\left\{\nu\left(2T\left(\frac{x+y}{2}\right) - 2f\left(\frac{x+y}{2}3^n\right)3^{-n}\right), \nu\left(f(3^n x)3^{-n} - T(x)\right)\right\}(t),\right. \\
&\qquad \left.\Pi_M\left\{\nu\left(f(3^n y)3^{-n} - T(y)\right), \nu\left(2f\left(\frac{x+y}{2}3^n\right)3^{-n} - f(3^n x)3^{-n} - f(3^n y)3^{-n}\right)\right\}(t)\right\}.
\end{aligned} \tag{2.16}$$

But we have

$$\begin{gathered}
\lim_{n\to\infty} \nu\left(2T\left(\frac{x+y}{2}\right) - 2f\left(\frac{x+y}{2}3^n\right)3^{-n}\right)(t) = 1, \\
\lim_{n\to\infty} \nu\left(f(3^n x)3^{-n} - T(x)\right)(t) = 1, \\
\lim_{n\to\infty} \nu\left(f(3^n y)3^{-n} - T(y)\right)(t) = 1,
\end{gathered} \tag{2.17}$$

and by (2.1) we have

$$\begin{aligned}
&\nu\left(2f\left(\frac{x+y}{2}3^n\right)3^{-n} - f(3^n x)3^{-n} - f(3^n y)3^{-n}\right)(t) \\
&\quad = \nu\left(2f\left(\frac{x+y}{2}3^n\right) - f(3^n x) - f(3^n y)\right)(3^n t) \\
&\quad \geq \Pi_M\{\omega(\varphi(3^n x)), \omega(\varphi(3^n y))\}(3^n t) \\
&\quad = \Pi_M\{\omega(\varphi(x)), \omega(\varphi(y))\}\left(\frac{3^n}{\alpha^n}t\right),
\end{aligned} \tag{2.18}$$

which tends to 1 as $n \to \infty$. Therefore

$$\nu\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y)\right)(t) = 1, \tag{2.19}$$

for each $x, y \in X$ and $t > 0$. Thus T satisfies the Jensen equation and so it is additive.

Next, we approximate the difference between f and T in the Šerstnev MPN space (Y, ν, Π_M). For every $x \in X$ and $t > 0$, by (2.15), for large enough n, we have

$$\nu(T(x) - f(x))(t) \geq \Pi_M\{\nu(T(x) - f(3^n x)3^{-n}), \nu(f(3^n x)3^{-n} - f(x))\}(t) \geq \psi_x(t). \tag{2.20}$$

The uniqueness assertion can be proved by standard fashion. Let $T' : X \to Y$ be another additive mapping, which satisfies the required inequality. Then for each $x \in X$ and $t > 0$,

$$\nu(T(x) - T'(x))(t) \geq \Pi_M\{\nu(T(x) - f(x)), \nu(T'(x) - f(x))\}(t) \geq \psi_x(t). \tag{2.21}$$

Therefore by the additivity of T and T',

$$\nu\big(T(x)-T'(x)\big)(t)=\nu\big(T(3^n x)-T'(3^n x)\big)(3^n t)\geq \varphi_x\left(\frac{3^n}{\alpha^n}t\right), \tag{2.22}$$

for all $x\in X$, $t>0$, and $n\in\mathbb{N}$. Since $0<\alpha<3$,

$$\lim_{n\to\infty}\left(\frac{3^n}{\alpha^n}\right)=\infty. \tag{2.23}$$

Hence the right-hand side of the above inequality tends to 1 as $n\to\infty$. It follows that $T(x)=T'(x)$ for all $x\in X$. □

Remark 2.2. One can prove a similar result for the case that $|\alpha|>3$. In this case, the additive mapping T is defined by $T(x):=\lim_{n\to\infty}3^{-n}f(3^{-n}x)$.

Now we examine some conditions under which the additive mapping found in Theorem 2.1 is to be continuous. We use a known strategy of Hyers [5] (see also [44]).

Theorem 2.3. *Let* X *be a linear space. Let* (Y,ν,Π_M) *be a Šerstnev MPN space and let* $f:X\to Y$ *be a mapping with* $f(0)=0$*. Suppose that* $\delta>0$ *is a positive real number and* z_0 *is a fixed vector in a Šerstnev MPN space* (Z,ω,Π_M) *such that*

$$\nu\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)(t)\geq\omega(\delta z_0)(t), \tag{2.24}$$

for all $x,y\in X-\{0\}$ *and positive real number* t*. Then there is a unique additive mapping* $T:X\to Y$ *such that*

$$\nu\big(T(x)-f(x)\big)(t)\geq\omega(\delta z_0)(3t). \tag{2.25}$$

Moreover, if (X,ν',Π_M) *is a Šerstnev MPN space and* f *is continuous at a point, then* T *is continuous on* X*.*

Proof. Using Theorem 2.1 with $\varphi(x)=\delta z_0$, we deduce the existence of the required additive mapping T. Let us put $\beta=3/\delta$. Suppose that f is continuous at a point x_0. If T were not continuous at a point, then there would be a sequence x_n in X such that

$$\lim_{n\to\infty}\nu'(x_n)(t)=1,\qquad \lim_{n\to\infty}\nu(T(x_n))(t)\neq 1. \tag{2.26}$$

By passing to a subsequence if necessary, we may assume that

$$\lim_{n\to\infty}\nu'(x_n)(t)=1, \tag{2.27}$$

and there are $t_0 > 0$ and $\varepsilon > 0$ such that

$$\nu(T(x_n))(t_0) < 1 - \varepsilon \quad \forall n. \tag{2.28}$$

Since $\lim_{t\to\infty}\omega(z_0)(\beta t) = 1$, there is t_1 such that $\omega(z_0)(\beta t_1) \geq 1 - \varepsilon$. There is a positive integer k such that $t_1/k < t_0$. We have

$$\nu(T(kx_n + x_0) - T(x_0))(t_1) = \nu(T(x_n))\left(\frac{t_1}{k}\right) \leq \nu(T(x_n))(t_0) < 1 - \varepsilon. \tag{2.29}$$

On the other hand

$$\begin{aligned}\nu(T(kx_n + x_0) - T(x_0))(t_1) \geq \Pi_M\{\Pi_M\{\nu(T(kx_n + x_0) - f(kx_n + x_0))\ ,\\ \nu(f(kx_n + x_0) - f(x_0))\}, \nu(f(x_0) - T(x_0))\}(t_1).\end{aligned} \tag{2.30}$$

By (2.25) we have

$$\begin{gathered}\nu(T(kx_n + x_0) - f(kx_n + x_0))(t_1) \geq \omega(z_0)(\beta t_1),\\ \nu(f(x_0) - T(x_0))(t_1) \geq \omega(z_0)(\beta t_1),\end{gathered} \tag{2.31}$$

and we have

$$\lim_{n\to\infty} \nu(f(kx_n + x_0) - f(x_0))(t_1) = 1. \tag{2.32}$$

Therefore for sufficiently large n,

$$\nu(T(kx_n + x_0) - T(x_0))(t_1) \geq \omega(z_0)(\beta t_1) \geq 1 - \varepsilon, \tag{2.33}$$

which contradicts (2.29). □

3. Completeness of Šerstnev MPN Spaces

This section contains two results concerning the completeness of a Šerstnev MPN space. Those are versions of a theorem of Schwaiger [45] stating that a normed space E is complete if, for each $f : \mathbb{N} \to E$ whose Cauchy difference $f(x + y) - f(x) - f(y)$ is bounded for all $x, y \in \mathbb{N}$, there exists an additive mapping $T : \mathbb{N} \to E$ such that $f(x) - T(x)$ is bounded for all $x \in \mathbb{N}$.

Definition 3.1. Let (X, ν, τ) be an MPN space and let $\alpha \in (0, 1)$. A mapping $f_\alpha : \mathbb{N} \to X$ is said to be α-approximately Jensen-type if

$$\nu(2f_\alpha(x + y) - f_\alpha(2x) - f_\alpha(2y))(\beta) \geq \alpha, \tag{3.1}$$

for some $\beta > 0$ and all $x, y \in \mathbb{N}$.

In order to prove our next results, we need to put the following conditions on an MPN space.

Definition 3.2. An MPN space (X, ν, τ) is called *definite* if

$$\nu(x)(t) > \quad \forall t > 0 \quad \text{implies that } x = 0 \tag{3.2}$$

holds. It is called *pseudodefinite* if for each $\alpha \in (0,1)$ the following condition holds:

$$\nu(x)(t) > \alpha \quad \forall t > 0 \quad \text{implies that } x = 0. \tag{3.3}$$

Clearly a definite MPN space is pseudodefinite.

Theorem 3.3. *Let (X, ν, Π_M) be a pseudodefinite Šerstnev MPN space. Suppose that for each $\alpha \in (0,1)$ and each α-approximately Jensen-type $f_\alpha : \mathbb{N} \to X$ there exist numbers $\delta_\alpha > 0$, $n_\alpha \in \mathbb{N}$, and an additive mapping $T_\alpha : \mathbb{N} \to X$ such that*

$$\nu(T_\alpha(n) - f_\alpha(n))(\delta_\alpha) > \alpha, \tag{3.4}$$

for all $n \geq n_\alpha$. Then (X, ν, Π_M) is a Šerstnev MPB-space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, ν, Π_M). Temporarily fix $\alpha \in (0,1)$. There is an increasing sequence n_k of positive integers such that $n_k \geq k$ and

$$\nu(x_n - x_m)\left(\frac{1}{2k}\right) \geq \alpha \quad \text{for } n, m \geq n_k. \tag{3.5}$$

Put $y_k = x_{n_k}$ and define $f_\alpha : \mathbb{N} \to X$ by $f_\alpha(k) = k y_k (k \in \mathbb{N})$. Then by (3.5) we have

$$\begin{aligned}
&\nu(2f_\alpha(j+k) - f_\alpha(2j) - f_\alpha(2k))(1) \\
&\quad = \nu(2(j+k)y_{j+k} - 2jy_{2j} - 2ky_{2k})(1) \\
&\quad \geq \Pi_M\{\nu(2j(y_{j+k} - y_{2j})), \nu(2k(y_{j+k} - y_{2k}))\}(1) \geq \alpha,
\end{aligned} \tag{3.6}$$

for each $j, k \in \mathbb{N}$. Thus f_α is α-approximately Jensen-type. By our assumption, there exist numbers $\delta_\alpha > 0$, $n_\alpha \in \mathbb{N}$, and an additive mapping $T_\alpha : \mathbb{N} \to X$ such that

$$\nu(T_\alpha(n) - f_\alpha(n))(\delta_\alpha) > \alpha, \tag{3.7}$$

for all $n \geq n_\alpha$. Since T_α is additive, $T_\alpha(n) = nT_\alpha(1)$. Hence

$$\nu(T_\alpha(1) - y_n)\left(\frac{\delta_\alpha}{n}\right) > \alpha, \quad \text{for } n \in \mathbb{N}. \tag{3.8}$$

Let $\varepsilon > 0$. Then there is some $n_0 \geq n_\alpha$ such that

$$\nu(x_n - x_m)(\varepsilon) \geq \alpha, \tag{3.9}$$

for all $m, n \geq n_0$. Take some $k_0 \in \mathbb{N}$ such that $k_0 \geq n_0$ and $\delta_\alpha/k_0 < \varepsilon/2$. It follows that $n_{k_0} \geq k_0 \geq n_0 \geq n_\alpha$. Let $\alpha \neq \beta$, then, for large enough n,

$$\nu(T_\alpha(1) - x_n)(\varepsilon) \geq \Pi_M\left\{\nu\left(x_n - x_{n_{k_0}}\right), \nu\left(y_{k_0} - T_\alpha(1)\right)\right\}(\varepsilon) \geq \min\{\alpha, \beta\}, \tag{3.10}$$

for each $\varepsilon > 0$. By (3.3), $T_\alpha(1) = T_\beta(1)$. Put $x = T_\alpha(1)$. Then for each $\alpha \in (0,1)$ and $\varepsilon > 0$,

$$\nu(x - x_n)(\varepsilon) \geq \alpha, \tag{3.11}$$

for sufficiently large n. This means that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1. \tag{3.12}$$

□

Definition 3.4. Let (X, ν, Π_M) be a Šerstnev MPN space and let $f : \mathbb{N} \to X$ be a mapping. Assume that, for each $\alpha \in (0,1)$, there are numbers $n_\alpha \in \mathbb{N}$ and $\delta > 0$ such that

$$\nu\left(2f(n+m) - f(2n) - f(2m)\right)(\delta) \geq \alpha, \tag{3.13}$$

for each $n, m \geq n_\alpha$. Then f is said to be an approximately Jensen-type mapping.

Theorem 3.5. *Let (X, ν, Π_M) be a Šerstnev MPN space such that for every approximately Jensen-type mapping $f : \mathbb{N} \to X$ there is an additive mapping $T : \mathbb{N} \to X$ such that*

$$\lim_{n\to\infty} \nu\left(T(n) - f(n)\right)(t) = 1 \tag{3.14}$$

for each $t > 0$. Then (X, ν, Π_M) is a Šerstnev MPB-space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, ν, Π_M). Take a sequence $\{\alpha_n\}$ in interval $(0,1)$ such that $\{\alpha_n\}$ increasingly tends to 1. For each $k \in \mathbb{N}$ one can find some $n_k \in \mathbb{N}$ such that

$$\nu(x_m - x_n)(1/2k) \geq \alpha_k \tag{3.15}$$

for each $n, m \geq n_k$. Let $y_k = x_{n_k}$ for each $k \geq 1$. Define $f : \mathbb{N} \to X$ by $f(k) := k y_k$, for $k \in \mathbb{N}$. If $\alpha \in (0,1)$, take some $m_0 \in \mathbb{N}$ such that $\alpha_{m_0} > \alpha$ and let $n_\alpha = m_0$. Then for each $n \geq m \geq n_\alpha$, we have

$$\begin{aligned}
&\nu(2f(n+m) - f(2n) - f(2m))(1) \\
&\quad = \nu(2(n+m)y_{n+m} - 2ny_{2n} - 2my_{2m})(1) \\
&\quad \geq \Pi_M\{\nu(2n(y_{n+m} - y_{2n})), \nu(2m(y_{n+m} - y_{2m}))\}(1) \\
&\quad \geq \min\left\{\nu(y_{n+m} - y_{2n})\left(\frac{1}{2n}\right), \nu(y_{n+m} - y_{2m})\left(\frac{1}{2m}\right)\right\} \\
&\quad \geq \min\{\alpha_n, \alpha_m\} \geq \alpha.
\end{aligned} \tag{3.16}$$

Therefore f is an approximately Jensen-type mapping. By our assumption, there is an additive mapping $T : \mathbb{N} \to X$ such that

$$\lim_{n \to \infty} \nu(T(n) - f(n))(t) = 1. \tag{3.17}$$

This means that

$$\lim_{n \to \infty} \nu(T(1) - y_n)\left(\frac{t}{n}\right) = 1. \tag{3.18}$$

Hence the subsequence $\{y_n\}$ of the Cauchy sequence $\{x_n\}$ converges to $x = T(1)$. Hence $\{x_n\}$ also converges to x. □

4. Conclusions

In this work, we have analyzed a generalized Ulam-Hyers theorem in Šerstnev PN spaces endowed with Π_M. We have proved that if an approximate Jensen mapping in a Šerstnev PN space is continuous at a point then we can approximate it by an anywhere continuous Jensen mapping. Also, as a version of Schwaiger, we have showed that if every approximate Jensen-type mapping from natural numbers into a Šerstnev PN-space can be approximate by an additive mapping then the norm of Šerstnev PN-space is complete.

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