Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 869684, 14 pages
doi:10.1155/2010/869684

Research Article

Modified Block Iterative Algorithm for Solving Convex Feasibility Problems in Banach Spaces

Shih-sen Chang,[1] Jong Kyu Kim,[2] and Xiong Rui Wang[1]

[1] *Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China*
[2] *Department of Mathematics Education, Kyungnam University Masan, Kyungnam 631-701, South Korea*

Correspondence should be addressed to Jong Kyu Kim, jongkyuk@kyungnam.ac.kr

Received 20 October 2009; Accepted 28 December 2009

Academic Editor: Yeol J. E. Cho

The purpose of this paper is to use the modified block iterative method to propose an algorithm for solving the convex feasibility problems for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings. Under suitable conditions some strong convergence theorems are established in uniformly smooth and strictly convex Banach spaces with *Kadec-Klee property*. The results presented in the paper improve and extend some recent results.

1. Introduction

The problem of finding a point in the intersection of closed and convex subsets $\{C_i\}_{i=1}^{m}$ of a Banach space is a frequently appearing problem in diverse areas of mathematics and physical sciences. This problem is commonly referred to as the *convex feasibility problem* (CFP). There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [1]. The advantage of a Hilbert space H is that the projection P_C onto a closed convex subset C of H is nonexpansive. So projection methods have dominated in the iterative approaches to (CFP) in Hilbert space. In 1993, Kitahara and Takahashi [2] deal with the convex feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach space (see also, O'Hara et al. [3] and Chang et al. [4]). It is known that if C is a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E, then the *generalized projection* Π_C from E onto C is relatively nonexpansive. In 2005, Matsushita and Takahashi [5] reformulated the definition of the notion and obtained weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping. Recently, Qin et al. [6], Zhou and

Tan [7], Wattanawitoon and Kumam [8], Li and Su [9], and Takahashi and Zembayashi [10] extend the notion from relatively nonexpansive mappings or quasi-ϕ-nonexpansive mappings to quasi-ϕ-asymptotically nonexpansive mappings and also prove some weak and strong convergence theorems to approximate a common fixed point of finite or infinite family of quasi-ϕ-nonexpansive mappings or quasi-ϕ-asymptotically nonexpansive mappings.

It should be noted that the *block iterative algorithm* is a method which often used by many authors to solve the convex feasibility problem (see, e.g., Kikkawa and Takahashi [11], Aleyner and Reich [12]). Recently, some authors by using the block iterative scheme to establish strong convergence theorems for a finite family of relativity nonexpansive mappings in Hilbert space or finite-dimensional Banach space (see, e.g., Aleyner and Reich [12], Plubtieng and Ungchittrakool [13, 14]) or uniformly smooth and uniformly convex Banach spaces (see, e.g., Sahu et al. [15] and Ceng et al. [16–18]).

Motivated and inspired by these facts, the purpose of this paper is to use the modified block iterative method to propose an iterative algorithm for solving *the convex feasibility problems* for an infinite family of quasi-ϕ-asymptotically nonexpansive. Under suitable conditions some strong convergence theorems are established in a uniformly smooth and strictly convex Banach space with *Kadec-Klee property*. The results presented in the paper improve and extend the corresponding results in Aleyner and Reich [12], Plubtieng and Ungchittrakool [13, 14], and Chang et al. [19].

2. Preliminaries

Throughout this paper we assume that E is a real Banach space with the dual E^* and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \left\| f^* \right\|^2 \right\}, \quad x \in E. \tag{2.1}$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T, and use $\mathcal{R}$ and $\mathcal{R}^+$ to denote the set of all real numbers and the set of all nonnegative real numbers, respectively. We also denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of a sequence $\{x_n\}$, respectively.

A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex* if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 \le 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \ge \epsilon$. E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\left\|x + ty\right\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. E is said to be *uniformly smooth* if the above limit exists uniformly in $x, y \in U$.

Remark 2.1. The following basic properties can be found in Cioranescu [20].

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-$weak^*$-continuous.

(iii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one, and onto.

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(v) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Next we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. In the sequel we always use $\phi : E \times E \to \mathcal{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.3}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{2.4}$$

Following Alber [21], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.5}$$

Lemma 2.2 (see [21]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$ *for all $x \in C$ and $y \in E$;*

(b) *if $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \Longleftrightarrow \langle z - y, Jx - Jz\rangle \ge 0, \quad \forall y \in C, \tag{2.6}$$

(c) *for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Remark 2.3. If E is a real Hilbert space H, then $\phi(x, y) = \|x - y\|^2$ and Π_C is the metric projection P_C of H onto C.

Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, $T : C \to C$ a mapping, and $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all asymptotic fixed points of T by $\widetilde{F}(T)$.

Definition 2.4. (1) A mapping $T : C \to C$ is said to be *relatively nonexpansive* [5] if $F(T) \neq \emptyset$, $F(T) = \widetilde{F}(T)$, and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C,\ p \in F(T). \tag{2.7}$$

(2) A mapping $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 2.5. (1) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T). \tag{2.8}$$

(2) A mapping $T : C \to C$ is is said to be *quasi-ϕ-asymptotically nonexpansive* [7], if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, \ x \in C, \ p \in F(T). \tag{2.9}$$

Remark 2.6. (1) From the definition, it is easy to know that each relatively nonexpansive mapping is closed.

(2) The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass and the class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true.

Next, we give some examples which are closed and quasi-ϕ-asymptotically nonexpansive mappings.

Example 2.7 (see [7]). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ a maximal monotone mapping such that $A^{-1}0$ (the set of zero points of A) is nonempty. Then the mapping $J_r = (J + rA)^{-1} J$ is closed and quasi-ϕ-asymptotically nonexpansive from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.8. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset $C \subset E$. Then Π_C is relative nonexpansive, which in turn is a closed and quasi-ϕ-nonexpansive mapping, and so it is a closed and quasi-ϕ-asymptotically nonexpansive mapping.

Lemma 2.9 (see [13, 22]). *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{n=1}^{N} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$,*

$$\left\| \sum_{n=1}^{N} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{N} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.10}$$

Lemma 2.10. *Let E be a uniformly convex Banach space, $r > 0$ a positive number and $B_r(0)$ a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.11}$$

Proof. Since $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and $\lambda_i > 0$ for all $i \geq 1$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, we have

$$\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\| \leq \sum_{i=1}^{\infty} \lambda_i \|x_i\| \leq r. \tag{2.12}$$

Hence, for any given $\epsilon > 0$ and any given positive integers i, j with $i < j$, it follows from (2.12) that there exists a positive integer $N > j$ such that $\| \sum_{i=N+1}^{\infty} \lambda_i x_i \| \leq \epsilon$. Letting $\sigma_N = \sum_{i=1}^{N} \lambda_i$, by Lemma 2.9, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\|^2 &= \left\| \sigma_N \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} + \sum_{i=N+1}^{\infty} \lambda_i x_i \right\|^2 \leq \left(\sigma_N \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\| + \left\| \sum_{i=N+1}^{\infty} \lambda_i x_i \right\| \right)^2 \\
&\leq \sigma_N^2 \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\|^2 + \epsilon^2 + 2\epsilon\sigma_N \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\| \\
&\leq \sigma_N^2 \sum_{i=1}^{N} \frac{\lambda_i}{\sigma_N} \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) + \epsilon \left(\epsilon + 2\sigma_N \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\| \right) \\
&\leq \sum_{i=1}^{N} \lambda_i \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) + \epsilon \left(\epsilon + 2 \left\| \sum_{i=1}^{N} \lambda_i x_i \right\| \right) \\
&\leq \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) + \epsilon \left(\epsilon + 2 \left\| \sum_{i=1}^{N} \lambda_i x_i \right\| \right).
\end{aligned} \tag{2.13}$$

Since $\epsilon > 0$ is arbitrary, the conclusion of Lemma 2.10 is proved. □

Lemma 2.11. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Then $F(T)$ is a closed convex subset of C.*

Proof. Letting $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ (as $n \to \infty$), we prove that $p \in F(T)$. In fact, from the definition of T, we have

$$\phi(p_n, Tp) \leq k_1 \phi(p_n, p) \longrightarrow 0 \quad (\text{as} \quad n \longrightarrow \infty). \tag{2.14}$$

Therefore we have

$$\begin{aligned}
\lim_{n \to \infty} \phi(p_n, Tp) &= \lim_{n \to \infty} \left(\|p_n\|^2 - 2\langle p_n, JTp \rangle + \|Tp\| \right) \\
&= \|p\|^2 - 2\langle p, JTp \rangle + \|Tp\| = \phi(p, Tp) = 0,
\end{aligned} \tag{2.15}$$

that is, $p \in F(T)$.

Next we prove that $F(T)$ is convex. For any $p, q \in F(T)$, $t \in (0,1)$, putting $w = tp + (1-t)q$, we prove that $w \in F(T)$. Indeed, in view of the definition of $\phi(x,y)$ we have

$$\begin{aligned}
\phi(w, T^n w) &= \|w\|^2 - 2\langle w, JT^n w\rangle + \|T^n w\|^2 \\
&= \|w\|^2 - 2t\langle p, JT^n w\rangle - 2(1-t)\langle q, JT^n w\rangle + \|T^n w\|^2 \\
&= \|w\|^2 + t\phi(p, T^n w) + (1-t)\phi(q, T^n w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&\le \|w\|^2 + tk_n\phi(p, w) + (1-t)k_n\phi(q, w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&= (k_n - 1)\left(t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2\right).
\end{aligned} \tag{2.16}$$

Since $k_n \to 1$, we have $\phi(w, T^n w) \to 0$ (as $n \to \infty$). From (2.4) we have $\|T^n w\| \to \|w\|$. Consequently $\|JT^n w\| \to \|Jw\|$. This implies that $\{JT^n w\}$ is a bounded sequence. Since E is reflexive, E^* is also reflexive. So we can assume that

$$JT^n w \rightharpoonup f_0 \in E^*. \tag{2.17}$$

Again since E is reflexive, we have $J(E) = E^*$. Therefore there exists $x \in E$ such that $Jx = f_0$. By virtue of the weakly lower semicontinuity of norm $\|\cdot\|$, we have

$$\begin{aligned}
0 = \liminf_{n\to\infty} \phi(w, T^n w) &= \liminf_{n\to\infty} \left(\|w\|^2 - 2\langle w, J(T^n w)\rangle + \|T^n w\|^2\right) \\
&= \liminf_{n\to\infty} \left(\|w\|^2 - 2\langle w, J(T^n w)\rangle + \|J(T^n w)\|^2\right) \\
&\ge \|w\|^2 - 2\langle w, f_0\rangle + \|f_0\|^2 \\
&= \|w\|^2 - 2\langle w, Jx\rangle + \|Jx\|^2 \\
&= \|w\|^2 - 2\langle w, Jx\rangle + \|x\|^2 = \phi(w, x),
\end{aligned} \tag{2.18}$$

that is, $w = x$ which implies that $f_0 = Jw$. Thus from (2.17) we have $JT^n w \rightharpoonup Jw \in E^*$. Since $\|JT^n w\| \to \|Jw\|$ and E^* has the Kadec-Klee property, we have $JT^n w \to Jw$. Since E is uniformly smooth and strictly convex, by Remark 2.1(ii) it yields that $J^{-1} : E^* \to E$ is hemi-continuous. Therefore $T^n w \rightharpoonup w$. Again since $\|T^n w\| \to \|w\|$, by using the Kadec-Klee property of E, we have $T^n w \to w$. This implies that $TT^n w = T^{n+1} w \to w$. Since T is closed, we have $w = Tw$. This completes the proof of Lemma 2.11. □

3. Main Results

In this section, we will use the modified block iterative method to propose an iterative algorithm for solving the convex feasibility problem for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings in uniformly smooth and strictly convex Banach spaces with the *Kadec-Klee property*.

Definition 3.1. (1) Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings. $\{S_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly quasi -ϕ-asymptotically nonexpansive mappings*, if $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$

$$\phi(p, S_i^n x) \leq k_n \phi(p, x), \quad \forall p \in \bigcap_{n=1}^{\infty} F(S_n),\ x \in C,\ \forall n \geq 1. \tag{3.1}$$

(2) A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\left\|S^n x - S^n y\right\| \leq L\left\|x - y\right\|, \quad \forall x, y \in C. \tag{3.2}$$

Theorem 3.2. *Let E be a uniformly smooth and strictly convex Banach space with Kleac-Klee property and C a nonempty closed convex subsets of E. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Suppose that for each $i \geq 1$, S_i is uniformly L_i-Lipschitz continuous and that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(S_i)$ is a nonempty and bounded subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrary}, \quad C_0 = C, \\ &y_n = J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right), \\ &C_{n+1} = \left\{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \xi_n\right\}, \\ &x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{aligned} \tag{3.3}$$

where $\xi_n = \sup_{u \in \mathcal{F}} (k_n - 1)\phi(u, x_n)$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto the set C_{n+1} and for each $i \geq 0$, $\{\alpha_{n,i}\}$ is a sequence in $[0, 1]$ satisfying the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all $n \geq 0$;*

(b) $\liminf_{n \to \infty} \alpha_{n,0} \cdot \alpha_{n,i} > 0$ *for all $i \geq 1$.*

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$.

Proof. We divide the proof of Theorem 3.2 into five steps.

Step 1. We first prove that $\mathcal{F}$ and C_n both are closed and convex subset of C for all $n \geq 0$.

In fact, It follows from Lemma 2.11 that $F(S_i)$, $i \geq 1$, is closed and convex. Therefore $\mathcal{F}$ is a closed and convex subset in C. Furthermore, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$. Since the inequality $\phi(v, y_n) \leq \phi(v, x_n) + \xi_n$ is equivalent to

$$2\langle v, J x_n - J y_n\rangle \leq \|x_n\|^2 - \left\|y_n\right\|^2 + \xi_n, \tag{3.4}$$

therefore, we have

$$C_{n+1} = \left\{v \in C_n : 2\langle v, J x_n - J y_n\rangle \leq \|x_n\|^2 - \left\|y_n\right\|^2 + \xi_n\right\}. \tag{3.5}$$

This implies that C_{n+1} is closed and convex. The desired conclusions are proved. These in turn show that $\Pi_{\mathcal{F}} x_0$ and $\Pi_{C_n} x_0$ are well defined.

Step 2. We prove that $\{x_n\}$ is a bounded sequence in C.

By the definition of C_n, we have $x_n = \Pi_{C_n} x_0$ for all $n \geq 0$. It follows from Lemma 2.2(a) that

$$\begin{aligned}\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) &\leq \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0) \\ &\leq \phi(u, x_0), \quad \forall n \geq 0, \ u \in \mathcal{F}.\end{aligned} \tag{3.6}$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (2.4), $\{x_n\}$ is bounded. Denote

$$M = \sup_{n \geq 0} \{\|x_n\|\} < \infty. \tag{3.7}$$

Step 3. Next, we prove that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i) \subset C_n$ for all $n \geq 0$.

It is obvious that $\mathcal{F} \subset C_0 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 0$. Since E is uniformly smooth, E^* is uniformly convex. For any given $u \in \mathcal{F} \subset C_n$ and for any positive integer $j > 0$, from Lemma 2.10 we have

$$\begin{aligned}\phi(u, y_n) &= \phi\left(u, J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right)\right) \\ &= \|u\|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n\rangle + \left\|\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right\|^2 \\ &\leq \|u\|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n\rangle + \alpha_{n,0}\|Jx_n\|^2 \\ &\quad + \sum_{i=1}^{\infty} \alpha_{n,i}\left\|JS_i^n x_n\right\|^2 - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \qquad \text{(by Lemma 2.10)} \\ &= \|u\|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n\rangle + \alpha_{n,0}\|x_n\|^2 \\ &\quad + \sum_{i=1}^{\infty} \alpha_{n,i}\left\|S_i^n x_n\right\|^2 - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &= \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(u, S_i^n x_n) - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} k_n \phi(u, x_n) - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &\leq k_n \phi(u, x_n) - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &\leq \phi(u, x_n) + \sup_{u \in \mathcal{F}}(k_n - 1)\phi(u, x_n) - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &= \phi(u, x_n) + \xi_n - \alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \qquad \forall u \in \mathcal{F}.\end{aligned} \tag{3.8}$$

Hence $u \in C_{n+1}$ and so $\mathcal{F} \subset C_n$ for all $n \geq 0$. By the way, from the definition of $\{\xi_n\}$, (2.4), and (3.7), it is easy to see that

$$\xi_n = \sup_{u \in \mathcal{F}} (k_n - 1)\phi(u, x_n) \leq \sup_{u \in \mathcal{F}} (k_n - 1)(\|u\| + M)^2 \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.9}$$

Step 4. Now, we prove that $\{x_n\}$ converges strongly to some point $p \in \mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i)$.

In fact, since $\{x_n\}$ is bounded in C and E is reflexive, we may assume that $x_n \rightharpoonup p$. Again since C_n is closed and convex for each $n \geq 1$, it is easy to see that $p \in C_n$ for each $n \geq 0$. Since $x_n = \Pi_{C_n} x_0$, from the definition of Π_{C_n}, we have

$$\phi(x_n, x_0) \leq \phi(p, x_0), \quad \forall n \geq 0. \tag{3.10}$$

Since

$$\begin{aligned} \liminf_{n \to \infty} \phi(x_n, x_0) &= \liminf_{n \to \infty} \left\{ \|x_n\|^2 - 2\langle x_n, J x_0 \rangle + \|x_0\|^2 \right\} \\ &\geq \|p\|^2 - 2\langle p, J x_0 \rangle + \|x_0\|^2 = \phi(p, x_0), \end{aligned} \tag{3.11}$$

we have

$$\phi(p, x_0) \leq \liminf_{n \to \infty} \phi(x_n, x_0) \leq \limsup_{n \to \infty} \phi(x_n, x_0) \leq \phi(p, x_0). \tag{3.12}$$

This implies that $\lim_{n \to \infty} \phi(x_n, x_0) = \phi(p, x_0)$, that is, $\|x_n\| \to \|p\|$. In view of the Kadec-Klee property of E, we obtain that

$$\lim_{n \to \infty} x_n = p. \tag{3.13}$$

Now we prove that $p \in \bigcap_{i=1}^{\infty} F(S_i)$.In fact, by the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. Therefore by Lemma 2.2(a) we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \longrightarrow 0 \quad (\text{as } \ n \longrightarrow \infty). \end{aligned} \tag{3.14}$$

In view of $x_{n+1} \in C_{n+1}$ and note the construction of C_{n+1} we obtain that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \xi_n \longrightarrow 0 \quad (\text{as } \ n \longrightarrow \infty). \tag{3.15}$$

From (2.4) it yields $(\|x_{n+1}\| - \|y_n\|)^2 \to 0$. Since $\|x_{n+1}\| \to \|p\|$, we have

$$\|y_n\| \longrightarrow \|p\| \quad (\text{as } \ n \longrightarrow \infty), \tag{3.16}$$

Hence we have

$$\|Jy_n\| \longrightarrow \|Jp\| \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.17}$$

This implies that $\{Jy_n\}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can assume that $Jy_n \rightharpoonup f_0 \in E^*$. In view of the reflexive of E, we see that $J(E) = E^*$. Hence there exists $x \in E$ such that $Jx = f_0$. Since

$$\begin{aligned}\phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2.\end{aligned} \tag{3.18}$$

Taking $\liminf_{n\to\infty}$ on the both sides of equality above and in view of the weak lower semicontinuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, f_0\rangle + \|f_0\|^2 = \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 = \phi(p, x),\end{aligned} \tag{3.19}$$

that is, $p = x$. This implies that $f_0 = Jp$, and so $Jy_n \rightharpoonup Jp$. It follows from (3.17) and the Kadec-Klee property of E^* that $Jy_n \to Jp$ (as $n \to \infty$). Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $y_n \rightharpoonup p$. It follows from (3.16) and the Kadec-Klee property of E that

$$\lim_{n\to\infty} y_n = p. \tag{3.20}$$

From (3.13) and (3.20) we have that

$$\|x_n - y_n\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.21}$$

Since J is uniformly continuous on any bounded subset of E, we have

$$\|Jx_n - Jy_n\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.22}$$

For any $j \geq 1$ and any $u \in \mathcal{F}$, it follows from (3.8), (3.13), and (3.20) that

$$\alpha_{n,0}\alpha_{n,j}\, g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \leq \phi(u, x_n) - \phi(u, y_n) + \xi_n \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.23}$$

In view of condition (b) $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,j} > 0$, we see that

$$g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.24}$$

It follows from the property of g that

$$\left\|Jx_n - JS_j^n x_n\right\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.25}$$

Since $x_n \to p$ and J is uniformly continuous, it yieads $Jx_n \to Jp$. Hence from (3.25) we have

$$JS_j^n x_n \longrightarrow Jp \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.26}$$

Since $J^{-1} : E^* \to E$ is hemi-continuous, it follows that

$$S_j^n x_n \rightharpoonup p, \quad \text{for each} \quad j \geq 1. \tag{3.27}$$

On the other hand, for each $j \geq 1$ we have

$$\left|\left\|S_j^n x_n\right\| - \|p\|\right| = \left|\left\|J\left(S_j^n x_n\right)\right\| - \|Jp\|\right| \leq \left\|J\left(S_j^n x_n\right) - Jp\right\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.28}$$

This together with (3.27) shows that

$$S_j^n x_n \to p \qquad \text{for each } j \geq 1. \tag{3.29}$$

Furthermore, by the assumption that for each $j \geq 1$, S_j is uniformly L_i-Lipschitz continuous, hence we have

$$\begin{aligned}\left\|S_j^{n+1} x_n - S_j^n x_n\right\| &\leq \left\|S_j^{n+1} x_n - S_j^{n+1} x_{n+1}\right\| + \left\|S_j^{n+1} x_{n+1} - x_{n+1}\right\| + \|x_{n+1} - x_n\| + \left\|x_n - S_j^n x_n\right\| \\ &\leq (L_j + 1)\|x_{n+1} - x_n\| + \left\|S_j^{n+1} x_{n+1} - x_{n+1}\right\| + \left\|x_n - S_j^n x_n\right\|.\end{aligned} \tag{3.30}$$

This together with (3.13) and (3.29), yields $\|S_j^{n+1} x_n - S_j^n x_n\| \to 0$ (as $n \to \infty$). Hence from (3.29) we have $S_j^{n+1} x_n \to p$, that is, $S_j S^n x_n \to p$. In view of (3.29) and the closeness of S_j, it yields that $S_j p = p$, for all $j \geq 1$. This implies that $p \in \bigcap_{i=1}^{\infty} F(S_j)$.

Step 5. Finally we prove that $x_n \to p = \Pi_{\mathcal{F}} x_0$.

Let $w = \Pi_{\mathcal{F}} x_0$. Since $w \in \mathcal{F} \subset C_n$ and $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0. \tag{3.31}$$

This implies that

$$\phi(p, x_0) = \lim_{n \to \infty} \phi(x_n, x_0) \leq \phi(w, x_0). \tag{3.32}$$

In view of the definition of $\Pi_{\mathcal{F}} x_0$, from (3.32) we have $p = w$. Therefore, $x_n \to p = \Pi_{\mathcal{F}} x_0$. This completes the proof of Theorem 3.2. $\square$

The following theorem can be obtained from Theorem 3.2 immediately.

Theorem 3.3. *Let E be a uniformly smooth and strictly convex Banach space with Kadec-Klee property , C a nonempty closed convex subset of E. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and quasi-ϕ-nonexpansive mappings. Suppose that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(S_i)$ is a nonempty subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrary}, \quad C_0 = C, \\
&y_n = J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i x_n\right), \\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0,
\end{aligned} \tag{3.33}$$

where $\{\alpha_{n,i}\}$, for each $i \ge 0$, is a sequence in $[0,1]$ satisfying the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \ge 0$;

(b) $\liminf_{n\to\infty} \alpha_{n,0} \cdot \alpha_{n,i} > 0$ *for all* $i \ge 1$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$.

Proof. Since $\{S_i\}_{i=1}^{\infty} : C \to C$ is an infinite family of closed quasi-ϕ-nonexpansive mappings, it is an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_n = 1\}$. Hence $\xi_n = \sup_{u\in\mathcal{F}}(k_n - 1)\phi(u, x_n) = 0$. Therefore the conditions appearing in Theorem 3.2: "$\mathcal{F}$ is a bounded subset in C" and "for each $i \ge 1$, S_i is uniformly L_i-Lipschitz continuous" are of no use here. In fact, by the same methods as given in the proofs of (3.13), (3.20) and (3.29), we can prove that $x_n \to p$, $y_n \to p$ and $S_j x_n \to p$ (as $n \to \infty$) for each $j \ge 1$. By virtue of the closeness of mapping S_j for each $j \ge 1$, it yields that $p \in F(S_j)$ for each $j \ge 1$, that is, $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Therefore all conditions in Theorem 3.2 are satisfied. The conclusion of Theorem 3.3 is obtained from Theorem 3.2 immediately. □

Remark 3.4. Theorems 3.2 and 3.3 improve and extend the corresponding results in Aleyner and Reich [12], Plubtieng and Ungchittrakool [13, 14] and Chang et al. [19] in the following aspects.

(a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (note that each uniformly convex Banach space must have the Kadec-Klee property).

(b) For the mappings, we extend the mappings from nonexpansive mappings, relatively nonexpansive mappings or quasi-ϕ-nonexpansive mapping to an infinite family of quasi-ϕ-asymptotically mappings;

(c) For the algorithms, we propose a new modified block iterative algorithms which are different from ones given in [12–14, 19] and others.

Acknowledgment

This work was supported by the Natural Science Foundation of Yibin University (no. 2009Z3) and the Kyungnam University Research Fund 2009.

References

[1] P. L. Combettes, "The convex feasibility problem in inage recovery," in *Advances in Imaging and Electron Physics*, P. Hawkes, Ed., vol. 95, pp. 155–270, Academic Press, New York, NY, USA, 1996.

[2] S. Kitahara and W. Takahashi, "Image recovery by convex combinations of sunny nonexpansive retractions," *Topological Methods in Nonlinear Analysis*, vol. 2, no. 2, pp. 333–342, 1993.

[3] J. G. O'Hara, P. Pillay, and H.-K. Xu, "Iterative approaches to convex feasibility problems in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 9, pp. 2022–2042, 2006.

[4] S.-S. Chang, J.-C. Yao, J. K. Kim, and L. Yang, "Iterative approximation to convex feasibility problems in Banach space," *Fixed Point Theory and Applications*, vol. 2007, Article ID 46797, 19 pages, 2007.

[5] S. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[6] X. Qin, Y. J. Cho, and S. M. Kang, "Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[7] H. Zhou and B. Tan, "Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings," *Journal of Applied Mathematics and Computing*. In press.

[8] K. Wattanawitoon and P. Kumam, "Strong convergence theorems by a new hybrid projection algorithm for fixed point problems and equilibrium problems of two relatively quasi-nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 1, pp. 11–20, 2009.

[9] H. Y. Li and Y. F. Su, "Strong convergence theorems by a new hybrid for equilibrium problems and variational inequality problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 847–855, 2009.

[10] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 45–57, 2009.

[11] M. Kikkawa and W. Takahashi, "Approximating fixed points of nonexpansive mappings by the block iterative method in Banach spaces," *International Journal of Computational and Numerical Analysis and Applications*, vol. 5, no. 1, pp. 59–66, 2004.

[12] A. Aleyner and S. Reich, "Block-iterative algorithms for solving convex feasibility problems in Hilbert and in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 427–435, 2008.

[13] S. Plubtieng and K. Ungchittrakool, "Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 583082, 19 pages, 2008.

[14] S. Plubtieng and K. Ungchittrakool, "Strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 3, pp. 431–450, 2007.

[15] D. R. Sahu, H.-K. Xu, and J.-C. Yao, "Asymptotically strict pseudocontractive mappings in the intermediate sense," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3502–3511, 2009.

[16] L. C. Ceng, A. Petrusel, and J. C. Yao, "A hybrid method for Lipschitz continuous monotone mappings and asymptotically strict pseudocontractive mappings in the intermediate sense," to appear in *Journal of Nonlinear and Convex Analysis*.

[17] L. C. Ceng, A. Petrusel, and J. C. Yao, "Iterative approximation of fixed points for asymptotically strict pseudocontractive type mappings in the intermediate sense," to appear in *Taiwanese Journal of Mathematics*.

[18] L. C. Ceng, D. R. Sahu, and J. C. Yao, "Implicit iterative algorithms for asymptotically nonexpansive mappings nonexpansive mappings in the intermediate sense and Lipschitz-continuous monotone mappings," to appear in *Journal of Computational and Applied Mathematics*.

[19] S.-S. Chang, H. W. J. Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3307–3319, 2009.

[20] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[21] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartosator, Ed., vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Dekker, New York, NY, USA, 1996.

[22] S. S. Chang, "On the generalized mixed equilibrium problem in Banach spaces," *Journal of Applied Mathematics and Mechanics*, vol. 30, no. 9, pp. 1105–1112, 2009.