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Research Article

Some Estimates of Integrals with a Composition Operator

Bing Liu

Department of Mathematical Science, Saginaw Valley State University, University Center, MI 48710, USA

Correspondence should be addressed to Bing Liu, sgbing987@hotmail.com

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We give some estimates of integrals with a composition operator, namely, composition of homotopy, differential, and Green's operators $T \circ d \circ G$, with the Lipschitz and BMO norms. We also have estimates of those integrals with a singular factor.

1. Introduction

The purpose of this paper is to establish the Poincaré-type inequalities for the composition of the homotopy operator T, differential operator d, and Green's operator G under Lipschitz and BMO norms. One of the reasons that we consider this composition operator is due to the Hodge theorem. It is well known that Hodge decomposition theorem plays important role in studying harmonic analysis and differential forms; see [1–3]. It gives a relationship of the three key operators in harmonic analysis, namely, Green's operator G, the Laplacian operator Δ, and the harmonic projection operator H. This relationship offers us a tool to apply the composition of the three operators under the consideration to certain harmonic forms and to obtain some estimates for certain integrals which are useful in studying the properties of the solutions of PDEs. We also consider the integrals of this composition operator with a singular factor because of their broad applications in solving differential and integral equations; see [4].

We first give some notations and definitions which are commonly used in many books and papers; for example, see [1, 4–12]. We use M to denote a Riemannian, compact, oriented, and C^∞ smooth manifold without boundary on $\mathbb{R}^n$. Let $\wedge^l M$ be the lth exterior power of the cotangent bundle, and let $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M and $\mathcal{W}(\wedge^l M) = \{u \in L^1_{\text{loc}}(\wedge^l M) : u \text{ has generalized gradient}\}$. The harmonic l-fields are defined by $\mathcal{H}(\wedge^l M) = \{u \in \mathcal{W}(\wedge^l M) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of

$\mathscr{H}$ in L^1 is defined by $\mathscr{H}^{\perp} = \{u \in L^1 : \langle u, h\rangle = 0 \text{ for all } h \in \mathscr{H}\}$. Then, Green's operator G is defined as $G : C^{\infty}(\wedge^l M) \to \mathscr{H}^{\perp} \cap C^{\infty}(\wedge^l M)$ by assigning $G(u)$ as the unique element of $\mathscr{H}^{\perp} \cap C^{\infty}(\wedge^l M)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^{\infty}(\wedge^l M)$ onto $\mathscr{H}$ so that $H(u)$ is the harmonic part of u. In this paper, we also assume that Ω is a bounded and convex domain in $\mathbb{R}^n$. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. The operator K_y with the case $y = 0$ was first introduced by Cartan in [3]. Then, it was extended to the following version in [13]. To each $y \in \Omega$ there corresponds a linear operator $K_y : C^{\infty}(\Omega, \wedge^l) \to C^{\infty}(\Omega, \wedge^{l-1})$ defined by $(K_y u)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})\, dt$ and the decomposition $u = d(K_y u) + K_y(du)$. A homotopy operator $T : C^{\infty}(\Omega, \wedge^l) \to C^{\infty}(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$:

$$Tu = \int_{\Omega} \phi(y) K_y u dy, \tag{1.1}$$

where $\phi \in C_0^{\infty}(\Omega)$ is normalized so that $\int \phi(y)\, dy = 1$. We are particularly interested in a class of differential forms which are solutions of the well-known nonhomogeneous A-harmonic equation:

$$d^{*} A(x, du) = B(x, du), \tag{1.2}$$

where $A, B : \quad \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfy the conditions: $|A(x,\xi)| \le a|\xi|^{s-1}$, $\langle A(x,\xi), \xi\rangle \ge |\xi|^s$ and $|B(x,\xi)| \le b|\xi|^{s-1}$ for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ and $b > 0$ are constants, and $1 < s < \infty$ is a fixed exponent associated with the equation. A significant progress has been made recently in the study of different versions of the harmonic equations; see [1, 4–12].

A function $f \in L^1_{\text{loc}}(\Omega, \mu)$ is said to be in $BMO(\Omega, \mu)$ if there is a constant C such that $(1/\mu(B)) \int_B |f - f_B|\, d\mu \le C$ for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant. BMO norm of l-forms is defined as the following. Let $\omega \in L^1_{\text{loc}}(M, \wedge^l)$, $l = 0, 1, \ldots, n$. We say $\omega \in BMO(M, \wedge^l)$ if

$$\|\omega\|_{*,M} = \sup_{\sigma Q \subset M} |Q|^{-1} \left\|\omega - \omega_Q\right\|_{1,Q} < \infty \tag{1.3}$$

for some $\sigma \ge 1$. Similar way to define the Lipschitz norm for $\omega \in L^1_{\text{loc}}(M, \wedge^l)$, $l = 0, 1, \ldots, n$, we say $\omega \in \operatorname{loc}\operatorname{Lip}_k(M, \wedge^l)$, $0 \le k \le 1$, if

$$\|\omega\|_{\operatorname{loc}\operatorname{Lip}_k, M} = \sup_{\sigma Q \subset M} |Q|^{-(n+k)/n} \left\|\omega - \omega_Q\right\|_{1,Q} < \infty \tag{1.4}$$

for some $\sigma \ge 1$.

We will use the following results.

Lemma 1.1 (see [7]). *If* $u \in C^\infty(\wedge^l(\mathbb{R}^n))$, $l = 0, 1, \ldots, n$, $1 < s < \infty$, *then for any bounded ball* $B \subset \mathbb{R}^n$,

$$\|T \circ d \circ G(u)\|_{s,B} \leq C|B| \operatorname{diam}(B)\|u\|_{s,B}, \tag{1.5}$$

$$\|T \circ d \circ G(u)\|_{W^{1,s}(B)} \leq C|B|\|u\|_{s,B}. \tag{1.6}$$

One also has the Poincaré type inequality:

$$\|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{s,B} \leq C|B| \operatorname{diam}(B)\|u\|_{s,B}. \tag{1.7}$$

Lemma 1.2 (see [5]). *Let* $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a solution of the A-harmonic equation in a bounded, convex domain* M, *and let* T *be* $C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ *the homotopy operator defined in* (1.1). *Then, there exists a constant* C, *independent of* u, *such that*

$$\|T(u)\|_{\operatorname{loc}\operatorname{Lip}_k, M} \leq C\|u\|_{s,M}, \tag{1.8}$$

where k *is a constant with* $0 \leq k \leq 1$.

Lemma 1.3 (see [4]). *Let* $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a bounded domain* Ω, *let* H *be the projection operator and let* T *be the homotopy operator. Then, there exists a constant* C, *independent of* u, *such that*

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^s \frac{1}{|x - x_B|^\alpha} dx\right)^{1/s} \leq C|B|^\gamma \left(\int_{\sigma B} |u|^s \frac{1}{|x - x_B|^\lambda} dx\right)^{1/s} \tag{1.9}$$

for all balls B *with* $\sigma B \subset \Omega$ *and any real numbers* α *and* λ *with* $\alpha > \lambda \geq 0$, *where* $\gamma = 1 + 1/n - (\alpha - \lambda)/ns$ *and* x_B *is the center of ball* B *and* $\sigma > 1$ *is a constant.*

2. The Estimates for Lipschitz and *BMO* Norms

We first give an estimate of the composition operator with the Lipschitz norm $\|\cdot\|_{\operatorname{loc}\operatorname{Lip}_k, M}$.

Theorem 2.1. *Let* $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a solution of the A-harmonic equation* (1.2) *in a bounded, convex domain* M, *and let* T *be* $C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ *the homotopy operator defined in* (1.1) *and* G *Green's operator. Then, there exists a constant* C, *independent of* u, *such that*

$$\|T \circ d \circ G(u)\|_{\operatorname{loc}\operatorname{Lip}_k, M} \leq C\|u\|_{s,M}, \tag{2.1}$$

where k *is a constant with* $0 \leq k \leq 1$.

Proof. From Lemma 1.1, we have

$$\|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{s,B} \leq C|B| \operatorname{diam}(B)\|u\|_{s,B} \tag{2.2}$$

for all balls $B \subset M$. By Hölder inequality with $1 = 1/s + (s-1)/s$, we have

$$\begin{aligned}
&\|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{1,B} \\
&\quad = \int |T \circ d \circ G(u) - (T \circ d \circ G(u))_B| dx \\
&\quad \le \left(\int_B |T \circ d \circ G(u) - (T \circ d \circ G(u))_B|^s dx\right)^{1/s} \left(\int_B 1^{s/(s-1)} dx\right)^{(s-1)/s} \\
&\quad = |B|^{(s-1)/s} \|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{s,B} \\
&\quad \le |B|^{1-1/s} \left(C_1 |B| \operatorname{diam}(B) \|u\|_{s,B}\right) \\
&\quad \le C_2 |B|^{2-1/s+1/n} \|u\|_{s,B}.
\end{aligned} \tag{2.3}$$

By the definition of Lipschitz norm and noticing that $1 - k/n - 1/s + 1/n > 0$, we have

$$\begin{aligned}
&\|T \circ d \circ G(u)\|_{\operatorname{loc}\operatorname{Lip}_k, M} \\
&\quad = \sup_{\sigma B \subset M} |B|^{-(n+k)/n} \|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{1,B} \\
&\quad = \sup_{\sigma B \subset M} |B|^{-1-k/n} \|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{1,B} \\
&\quad \le \sup_{\sigma B \subset M} |B|^{-1-k/n} C_2 |B|^{2-1/s+1/n} \|u\|_{s,B} \\
&\quad = C_2 \sup_{\sigma B \subset M} |B|^{-1-k/n+2-1/s+1/n} \|u\|_{s,B} \\
&\quad \le C_2 \sup_{\sigma B \subset M} |M|^{1-1/s-k/n+1/n} \|u\|_{s,B} \\
&\quad \le C_3 \sup_{\sigma B \subset M} \|u\|_{s,\sigma B} \\
&\quad \le C_3 \|u\|_{s,M}.
\end{aligned} \tag{2.4}$$

Theorem 2.1 is proved. □

We learned from [5] that the BMO norm and the Lipschitz norm are related in the following inequality.

Lemma 2.2 (see [5]). *If a differential form is $u \in \operatorname{loc}\operatorname{Lip}_k(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, $0 \le k \le 1$, in a bounded domain Ω, then $u \in BMO(\Omega, \wedge^l)$ and*

$$\|u\|_{*,\Omega} \le C \|u\|_{\operatorname{loc}\operatorname{Lip}_k, \Omega}, \tag{2.5}$$

where C is a constant.

Applying $T(d(G(u)))$ to (2.5), then using Theorem 2.1, we have the following.

Theorem 2.3. *Let $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the A-harmonic equation* (1.2) *in a bounded, convex domain M, and let T be $C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ the homotopy operator defined in* (1.1)*, and let G be the Green's operator. Then, there exists a constant C, independent of u, such that*

$$\|T(d(G(u)))\|_{*,M} \leq C\|u\|_{s,M}. \tag{2.6}$$

3. The Lipschitz and *BMO* Norms with a Singular Factor

We considered the integrals with singular factors in [4]. Here, we will give estimates to Poincaré type inequalities with singular factors in the Lipschitz and BMO norms. If we use the formula (1.7) in Lemma 1.1 and follow the same proof of Lemma 3 in [4], we obtain the following theorem.

Theorem 3.1. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a bounded domain Ω, let G be Green's operator, and let T be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |T(d(G(u))) - (T(d(G(u))))_B|^s \frac{1}{|x - x_B|^\alpha} dx\right)^{1/s} \leq C|B|^\gamma \left(\int_{\sigma B} |u|^s \frac{1}{|x - x_B|^\lambda} dx\right)^{1/s} \tag{3.1}$$

for all balls B with $\sigma B \subset \Omega$ and any real numbers α and λ with $\alpha > \lambda \geq 0$, where $\gamma = 1 + 1/n - (\alpha - \lambda)/ns$ and x_B is the center of ball B and $\sigma > 1$ is a constant.

We extend Theorem 3.1 to the Lipschitz norm with a singular factor and have the following result.

Theorem 3.2. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the non-homogeneous A-harmonic equation in a bounded and convex domain Ω, let G be Green's operator, and let T be the homotopy operator. Then, there exists a constant $C(n, s, \alpha, \lambda, \Omega)$, independent of u, such that*

$$\|T(d(G(u)))\|_{\mathrm{loc\,Lip}_k, \Omega, w_1} \leq C(n, s, \alpha, \lambda, \Omega)\|u\|_{s, \Omega, w_2} \tag{3.2}$$

for all balls B with $\sigma B \subset \Omega$, $\sigma > 1$, where $w_1 = 1/|x - x_B|^\alpha$ and $w_2 = \sup_{\sigma B \subset \Omega} 1/|x - x_B|^\lambda$, and α, λ are real numbers with $(s-1)n + \lambda \geq \alpha s > \lambda \geq 0$. Here x_B is the center of the ball B.

Proof. Equation (3.2) is equivalent to

$$\sup_{\sigma B \subset \Omega} |B|^{-(n+k)/n} \int_B |T(d(G(u))) - (T(d(G(u))))_B| w_1 dx \leq C(n, s, \alpha, \lambda, \Omega)\left(\int_\Omega |u|^s w_2 dx\right)^{1/s}. \tag{3.3}$$

By using Theorem 3.1, we have

$$\begin{aligned}
&\left(\int_B |T(d(G(u))) - (T(d(G(u))))_B| \frac{1}{|x-x_B|^{\alpha}}dx\right)\\
&\quad\le \left(\int_B \left(|T(d(G(u))) - (T(d(G(u))))_B| \frac{1}{|x-x_B|^{\alpha}}\right)^s dx\right)^{1/s} \left(\int_B 1^{s/(s-1)}dx\right)^{(s-1)/s}\\
&\quad= |B|^{(s-1)/s}\left(\int_B |T(d(G(u))) - (T(d(G(u))))_B|^s |x-x_B|^{-\alpha s}dx\right)^{1/s}\\
&\quad\le C_1|B|^{(s-1)/s}|B|^{\gamma_1}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}dx\right)^{1/s},
\end{aligned} \tag{3.4}$$

where $\gamma_1 = 1 + 1/n - (\alpha s - \lambda)/ns$. Notice that $-(n+k)/n + (s-1)/s + 1 + 1/n - (\alpha s - \lambda)/ns = (1-k)/n + (s-1)/s - (\alpha s - \lambda)/ns > 0$ as $(s-1)n \ge \alpha s - \lambda > 0$. Thus,

$$\begin{aligned}
&\sup_{\sigma B\subset\Omega}|B|^{-(n+k)/n}\int_B |T(d(G(u))) - (T(d(G(u))))_B| \frac{1}{|x-x_B|^{\alpha}}dx\\
&\quad\le \sup_{\sigma B\subset\Omega}|B|^{-(n+k)/n}C_1|B|^{(s-1)/s}|B|^{\gamma_1}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}dx\right)^{1/s}\\
&\quad\le C_2\sup_{\sigma B\subset\Omega}|\Omega|^{-(n+k)/n+(s-1)/s+\gamma_1}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}dx\right)^{1/s}\\
&\quad\le C_3\sup_{\sigma B\subset\Omega}\left(\int_{\sigma B}|u|^s|x-x_B|^{-\lambda}dx\right)^{1/s}\\
&\quad\le C_4\left(\int_{\Omega}|u|^s\sup_{\sigma B\subset\Omega}|x-x_B|^{-\lambda}dx\right)^{1/s}\\
&\quad= C_4\left(\int_{\Omega}|u|^s w_2 dx\right)^{1/s}.
\end{aligned} \tag{3.5}$$

We have completed the proof of Theorem 3.2. □

We also obtain a similar version of the Poincaré type inequality with a singular factor for the BMO norm.

Theorem 3.3. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the non-homogeneous A-harmonic equation in a bounded and convex domain Ω, let G be Green's operator, and let T be the homotopy operator. Then, there exists a constant $C(n, s, \alpha, \lambda, \Omega)$, independent of u, such that*

$$\|T(d(G(u)))\|_{*,\Omega,w_1} \le C(n, s, \alpha, \lambda, \Omega)\|u\|_{s,\Omega,w_2} \tag{3.6}$$

for all balls B with $\sigma B \subset \Omega$, $\sigma > 1$, where $w_1 = 1/|x-x_B|^{\alpha}$ and $w_2 = \sup_{\sigma B\subset\Omega}1/|x-x_B|^{\lambda}$, and α, λ are real numbers with $(s-1)n + \lambda \ge \alpha s > \lambda \ge 0$. Here x_B is the center of the ball B.

We omit the proof since it is the same as the proof of Theorem 3.2.

4. The Weighted Inequalities

In this section, we introduce weighted versions of the Poincaré type inequality with the Lipschitz and BMO norms.

Definition 4.1. We say that a weight w belongs to the $A_r(M)$ class, $1 < r < \infty$ and write $w \in A_r(M)$, if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|} \int_B w dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} < \infty \tag{4.1}$$

for any ball $B \subset M$.

Definition 4.2. We say $\omega \in \operatorname{loc}\operatorname{Lip}_k(\Omega, \wedge^l, w^\alpha)$, $0 \le k \le 1$ for $\omega \in L^1_{\mathrm{loc}}(\Omega, \wedge^l, \omega^\alpha)$, $l = 0, 1, \ldots, n$, if

$$\|\omega\|_{\operatorname{loc}\operatorname{Lip}_k, \Omega, w^\alpha} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-(n+k)/n} \|\omega - \omega_Q\|_{1, Q, w^\alpha} < \infty \tag{4.2}$$

for some $\sigma > 1$, where the measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight, and α is a real number. Similarly, for $\omega \in L^1_{\mathrm{loc}}(\Omega, \wedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $\omega \in BMO(\Omega, \wedge^l, w^\alpha)$ if

$$\|\omega\|_{*, \Omega, w^\alpha} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-1} \|\omega - \omega_Q\|_{1, Q, w^\alpha} < \infty. \tag{4.3}$$

Lemma 4.3 (see [7]). *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 0, \ldots, n$, $1 < s < \infty$, be a smooth differential form satisfying equation* (1.2) *in a bounded domain Ω, and let $T : L^s_{\mathrm{loc}}(\Omega, \wedge^l) \to L^s_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ be the homotopy operator defined in* (1.1). *Assume that $\rho > 1$ and $w \in A_r(\Omega)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|T \circ d \circ G(u) - (T \circ d \circ G(u))_B\|_{s, B, w^\alpha} \le C|B| \operatorname{diam}(B) \|u\|_{s, \rho B, w^\alpha} \tag{4.4}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

We extend the Lemma 4.3 to the version with the Lipschitz norm as the following.

Theorem 4.4. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 0, \ldots, n$, $1 < s < \infty$, be a solution of* (1.2) *in a bounded domain, convex Ω, and let T be the homotopy operator defined in* (1.1), *where the measure μ is defined by $d\mu = w^\alpha dx$ and $w \in A_r(\Omega)$ for some $r > 1$ with $w(x) \ge \epsilon > 0$ for any $x \in \Omega$. Then, there exists a constant C, independent of u, such that*

$$\|T \circ d \circ G(u)\|_{\operatorname{loc}\operatorname{Lip}_k, \Omega, w^\alpha} \le C\|u\|_{s, \Omega, w^\alpha}, \tag{4.5}$$

where k and α are constants with $0 \le k \le 1$ and $0 < \alpha < 1$.

Proof. First, by using the Hölder inequality and inequality (4.4), we see that

$$\begin{aligned}
&\|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&\quad= \int_B |T(d(G(u))) - (T(d(G(u))))_B| d\mu \\
&\quad\le \left(\int |T(d(G(u))) - (T(d(G(u))))_B|^s d\mu\right)^{1/s} \left(1^{s/(s-1)} d\mu\right)^{(s-1)/s} \\
&\quad= (\mu(B))^{(s-1)/s} \|T(d(G(u))) - (Td(G(u)))_B\|_{s,B,w^\alpha} \\
&\quad\le (\mu(u))^{1-1/s} C_1 |B| \operatorname{diam}(B) \|u\|_{s,B,w^\alpha} \\
&\quad\le C_2 (\mu(u))^{1-1/s} |B|^{1+1/n} \|u\|_{s,B,w^\alpha}.
\end{aligned} \tag{4.6}$$

Since $\mu(B) = \int_B w^\alpha\, dx \ge \int_B \epsilon^\alpha\, dx \ge C_3|B|$, we have $1/\mu(B) \le C_4/|B|$. Then,

$$\begin{aligned}
\|T(d(G(u)))\|_{\mathrm{loc\,Lip}_k,\Omega,w^\alpha} &= \sup_{\rho B\subset\Omega} (\mu(B))^{-(n+k)/n} \|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&\le \sup_{\rho B\subset\Omega} (\mu(B))^{-1-k/n} C_2 (\mu(u))^{1-1/s} |B|^{1+1/n} \|u\|_{s,B,w^\alpha} \\
&= \sup_{\rho B\subset\Omega} C_2 (\mu(B))^{-k/n-1/s} |B|^{1+1/n} \|u\|_{s,B,w^\alpha} \\
&\le C_5 \sup_{\rho B\subset\Omega} (|B|)^{-k/n-1/s+1+1/n} \|u\|_{s,B,w^\alpha} \\
&\le C_5 \sup_{\rho B\subset\Omega} |\Omega|^{-k/n-1/s+1+1/n} \|u\|_{s,B,w^\alpha} \\
&\le C_5 |\Omega|^{-k/n-1/s+1+1/n} \sup_{\rho B\subset\Omega} \|u\|_{s,B,w^\alpha} \\
&\le C_6 \|u\|_{s,\Omega,w^\alpha}
\end{aligned} \tag{4.7}$$

due to $-k/n-1/s+1+1/n = (1-k)/n+(1-1/s) > 0$ and $|\Omega| < \infty$. Theorem 4.4 is proved. □

Similarly, we have the weighted version for the *BMO* norm.

Theorem 4.5. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 0, \dots, n$, $1 < s < \infty$, be a solution of* (1.2) *in a bounded domain, convex Ω, and let T be the homotopy operator defined in* (1.1)*, where the measure μ is defined by $d\mu = w^\alpha dx$ and $w \in A_r(\Omega)$ for some $r > 1$ with $w(x) \ge \epsilon > 0$ for any $x \in \Omega$. Then, there exists a constant C, independent of u, such that*

$$\|T \circ d \circ G(u)\|_{*,\Omega,w^\alpha} \le C\|u\|_{s,\Omega,w^\alpha}, \tag{4.8}$$

where α is a constant with $0 < \alpha < 1$.

Proof. We only need to prove that

$$\|T(d(G(u)))\|_{*,\Omega,w^\alpha} \le C\|T(d(G(u)))\|_{\mathrm{loc\,Lip}_k,\Omega,w^\alpha}. \tag{4.9}$$

As a matter of fact,

$$
\begin{aligned}
\|T(d(G(u)))\|_{*,\Omega,w^\alpha} &= \sup_{\rho B\subset\Omega}(\mu(B))^{-1}\|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&= \sup_{\rho B\subset\Omega}(\mu(B))^{k/n}(\mu(B))^{-(n+k)/n}\|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&\le \sup_{\rho B\subset\Omega}(\mu(\Omega))^{k/n}(\mu(B))^{-(n+k)/n}\|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&\le (\mu(\Omega))^{k/n}\sup_{\rho B\subset\Omega}(\mu(B))^{-(n+k)/n}\|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&\le C_1\sup_{\rho B\subset\Omega}(\mu(B))^{-(n+k)/n}\|T(d(G(u))) - (T(d(G(u))))_B\|_{1,B,w^\alpha} \\
&= C_1\|T(d(G(u)))\|_{\mathrm{loc\,Lip}_k,\Omega,w^\alpha}.
\end{aligned} \tag{4.10}
$$

□

5. Applications

Example 5.1. We consider the homogeneous case of (1.2) as $B(x, du) = 0$ and $A(x, \xi) = \xi|\xi|^{s-2}$, $s > 1$. Let u be a 0-form. Then, the operator A satisfies the required conditions of (1.2) and (1.2) is reduced to the s-harmonic equation:

$$
\operatorname{div}\left(\nabla u|\nabla u|^{s-2}\right) = 0. \tag{5.1}
$$

For example, $u = |x|^{(s-n)/(s-1)} \in \mathbb{R}^n$, as $2-1/n < s < n$ and $u = -\log|x|$ as $s = n$ is a solution of s-harmonic equation (5.1). Then, u also satisfies the results proved in the Theorems 2.1–4.5. Let us consider a special case. Set $s = 2, n = 3$, and let Ω be the unit sphere in $\mathbb{R}^3$. In particular, one could think of u as square root of an attraction force between two objects of masses m and M, respectively. Then, $u^2 = mMg/(x_1^2 + x_2^2 + x_3^2)$, where g is the gravitational constant. It would be very complicated to estimate the $\|T(d(G(u)))\|_{\mathrm{loc\,Lip}_k,\Omega}$ or $\|T(d(G(u)))\|_{*,\Omega}$ directly. To estimate their upper bounds by estimating $\|u\|_s$ is much easier. As a matter of fact, by using the spherical coordinates, we have

$$
\|u\|_{2,\Omega} = \sqrt{mMg}\left(\int_\Omega |x|^{-2}dx\right)^{1/2} = \sqrt{mMg}\left(2\pi\int_0^\pi\int_0^1 \rho^{-2+2}\sin\phi\, d\rho\, d\phi\right)^{1/2} = 2\sqrt{mMg\pi}. \tag{5.2}
$$

Example 5.2 (see [5]). Let $f(x) = (f^1, f^2, \ldots, f^n) : \Omega \to \mathbb{R}^n$ be a K-quasiregular mapping, $K \ge 1$; that is, if f^i are in the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega)$, for $i = 1, 2, \ldots, n$, and the norm of the corresponding Jacobi matrix $|Df(x)| = \max\{|Df(x)h| : h = 1\}$ satisfies $|Df(x)|^n \le KJ(x, f)$, where $J(x, f) = \det Df(x)$ is the Jacobian determinant of the f, then, each of the functions $u = f^i(x)$, $i = 1, 2, \ldots, n$ or $u = \log|f(x)|$, is a generalized solution of the quasilinear elliptic equation:

$$
\operatorname{div} A(x, \nabla u) = 0, \quad A = (A_1, A_2, \ldots, A_n) \tag{5.3}
$$

in $\Omega - f^{-1}(0)$, where $A_i(x,\xi) = \partial/\partial\xi_i(\sum_{i,j=1}^{n}\theta_{i,j}(x)\xi_i\xi_j)^{n/2}$ and $\theta_{i,j}$ are some functions that satisfy $C_1(K)|\xi|^2 \le \sum_{i,j}^{n}\theta_{i,j}\xi_i\xi_j \le C_2(K)|\xi|^2$ for some constants $C_1(K), C_2(K) > 0$. Then, all of functions u defined here also satisfy the results in Theorems 2.1–4.5.

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