Research Article

Efficiency and Generalized Convex Duality for Nondifferentiable Multiobjective Programs

Kwan Deok Bae, Young Min Kang, and Do Sang Kim

Division of Mathematical Sciences, Pukyong National University, Busan 608-737, South Korea

Correspondence should be addressed to Do Sang Kim, dskim@pknu.ac.kr

We introduce nondifferentiable multiobjective programming problems involving the support function of a compact convex set and linear functions. The concept of (properly) efficient solutions are presented. We formulate Mond-Weir-type and Wolfe-type dual problems and establish weak and strong duality theorems for efficient solutions by using suitable generalized convexity conditions. Some special cases of our duality results are given.

1. Introduction and Preliminaries

The concept of efficiency has long played an important role in economics, game theory, statistical decision theory, and in all optimal decision problems with noncomparable criteria. In 1968, Geoffrion [1] proposed a slightly restricted definition of efficiency that eliminates efficient points of a certain anomalous type and lended itself to more satisfactory characterization. He called this new definition proper efficiency. Weir [2] has used proper efficiency to establish some duality results between primal problem and Wolfe type dual problem. He extended the duality results of Wolfe [3] for scalar convex programming problems and some of the more duality results for scalar nonconvex programming problems to vector valued programs.

In 1982, five characterizations of strongly convex sets were introduced by Vial [4]. Based on this, Vial [5] studied a class of functions depending on the sign of the constant ρ. Characteristic properties of this class of sets and related it to strong and weakly convex sets are provided.

Also, Egudo [6] and Weir [2] have used proper efficiency to obtain duality relations between primal problem and Mond-Weir type dual problem. Further, Egudo [7] used the concept of efficiency to formulate duality for multiobjective non-linear programs under generalized ρ-convexity assumptions.

Duality theorems for nondifferentiable programming problem with a square root term were obtained by Lal et al. [8]. In 1996, Mond and Schechter [9] studied duality and optimality for nondifferentiable multiobjective programming problems in which each component of the objective function contains the support functions of a compact convex sets. And Kuk et al. [10] defined the concept of (V, ρ)-invexity for vector-valued functions, which is a generalization of the concept of V-invexity concept.

Recently, Yang et al. [11] introduced a class of nondifferentiable multiobjective programming problems involving the support functions of compact convex sets. They established only weak duality theorems for efficient solutions. Subsequently, Kim and Bae [12] formulated nondifferentiable multiobjective programs involving the support functions of a compact convex sets and linear functions.

In this paper, we introduce generalized convex duality for nondifferentiable multiobjective program for efficient solutions. In Section 2 and Section 3, we formulate Mond-Weir type dual and Wolfe type dual problems and establish weak and strong duality under ρ-convexity assumptions. In addition, we obtain some special cases of our duality results in Section 4. Our duality results extend and improve well known duality results.

We consider the following multiobjective programming problem:

$$
\begin{aligned}
\text{minimize} \quad & \left(f_1(x) + s(x \mid C_1), \ldots, f_p(x) + s(x \mid C_p) \right) \\
\text{subject to} \quad & g_j(x) \leqq 0, \quad j = 1, \ldots, m, \\
& h_l(x) = 0, \quad l = 1, \ldots, q.
\end{aligned} \tag{VOPE}
$$

The functions $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, p, g_j : \mathbb{R}^n \to \mathbb{R}, \ j = 1, \ldots, m$ and $h_l : \mathbb{R}^n \to \mathbb{R}, l = 1, \ldots, q$ are assumed to be differentiable. And C_i, for each $i \in P = \{1, 2, \ldots, p\}$, is a compact convex set of $\mathbb{R}^n$.

Definition 1.1. A feasible solution x^0 for (VOPE) is efficient for (VOPE) if and only if there is no other feasible x for (VOPE) such that

$$
\begin{aligned}
f_i(x) + s(x \mid C_i) < f_i\left(x^0\right) + s\left(x^0 \mid C_i\right) \quad \text{for some} \quad i \in P, \\
f_{i_0}(x) + s(x \mid C_{i_0}) \leqq f_{i_0}\left(x^0\right) + s\left(x^0 \mid C_{i_0}\right) \quad \forall i_0 \in P.
\end{aligned} \tag{1.1}
$$

Definition 1.2. Let C be a compact convex set in IR^n. The support function $s(x \mid C)$ is defined by

$$
s(x \mid C) := \max \left\{ x^T y : y \in C \right\}. \tag{1.2}
$$

The support function $s(x \mid C)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$
s(y \mid C) \geq s(x \mid C) + z^T (y - x) \quad \forall y \in C. \tag{1.3}
$$

Equivalently,

$$z^T x = s(x \mid C). \tag{1.4}$$

The subdifferential of $s(x \mid C)$ is given by

$$\partial s(x \mid C) := \left\{ z \in C : z^T x = s(x \mid C) \right\}. \tag{1.5}$$

The following definition of ρ-convex function will be used to prove weak duality theorems in Section 2 and Section 3.

Definition 1.3 (see [4, 5]). A function $f_i : \mathbb{R}^n \to \mathbb{R}$ is said to be ρ_i-convex if there exists a real number ρ_i such that for each $x, u \in \mathbb{R}^n$ and $0 \leqq \lambda_i \leqq 1$,

$$f_i(\lambda_i x + (1 - \lambda_i) u) \leqq \lambda_i f_i(x) + (1 - \lambda_i) f_i(u) - \rho_i \lambda_i (1 - \lambda_i) \|x - u\|^2. \tag{1.6}$$

For a differentiable function $f_i : \mathbb{R}^n \to \mathbb{R}$, f_i is ρ_i-convex if and only if for all $x, u \in \mathbb{R}^n$,

$$f_i(x) - f_i(u) \geqq (x - u)^T \nabla f_i(u) + \rho_i \|x - u\|^2. \tag{1.7}$$

If ρ_i is positive then f_i is said to be strongly convex [4] and if ρ_i is negative then f_i is said to be weakly convex [5].

In this paper, the proofs of strong duality theorems will invoke the following.

Lemma 1.4. *(Chankong and Haimes [13, Theorem 4.1]) x^0 is an efficient solution for (VOPE) if and only if x^0 solves the following:*

$$
\begin{aligned}
\left(P_k\left(\epsilon^0 \right) \right) \quad &\textit{minimize} \quad && f_k(x) + s(x \mid C_k) \\
&\textit{subject to} \quad && f_i(x) + s(x \mid C_i) \\
& && \leqq f_i\left(x^0 \right) + s\left(x^0 \mid C_i \right), \quad \forall i \neq k, \\
& && g_j(x) \leqq 0, \quad j = 1, \ldots, m, \\
& && h_l(x) = 0, \quad l = 1, \ldots, q
\end{aligned} \tag{1.8}
$$

for each $k = 1, \ldots, p$.

2. Mond-Weir-Type Duality

We introduce a Mond-Weir type dual programming problem and establish weak and strong duality theorems.

$$\text{Maximize} \quad f(u) + u^T w = \left(f_1(u) + u^T w_1, \ldots, f_p(u) + u^T w_p \right) \qquad \text{(MVODE)}$$

subject to

$$\sum_{i=1}^{p} \tau_i \left(\nabla f_i(u) + w_i \right) + \nabla y^T g(u) + \nabla z^T h(u) = 0, \qquad (2.1)$$

$$y^T g(u) + z^T h(u) \geqq 0,$$

$$y_j \geqq 0, \quad j = 1, \ldots, m, \ w_i \in C_i, \ i = 1, \ldots, p,$$

$$\tau_i \geqq 0, \quad i = 1, \ldots, p, \quad \sum_{i=1}^{p} \tau_i = 1. \qquad (2.2)$$

Theorem 2.1 (Weak Duality). *Assume that for all feasible x for* (VOPE) *and all feasible* (u, τ, w, y, z) *for* (MVODE), $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \ldots, p$, *are ρ_i–convex, $g_j(\cdot)$, $j = 1, \ldots, m$, are σ_j-convex and $h_l(\cdot)$, $l = 1, \ldots, q$, are affine. If also any of the following conditions holds*

(a) $\tau_i > 0$, *for all* $i \in P$ *and* $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geqq 0$;
(b) $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j > 0$,

then the following cannot hold:

$$f_i(x) + s(x \mid C_i) < f_i(u) + u^T w_i, \quad \text{for some } i \in P$$

$$f_{i_0}(x) + s(x \mid C_{i_0}) \leqq f_{i_0}(u) + u^T w_{i_0}, \quad \forall i_0 \in P. \qquad (2.3)$$

Proof. Suppose contrary to the result that (2.3) hold; then for $\tau_i > 0$ for each $i = 1, \ldots, p$, (2.3) imply

$$\sum_{i=1}^{p} \tau_i \left(f_i(x) + s(x \mid C_i) \right) < \sum_{i=1}^{p} \tau_i \left(f_i(u) + u^T w_i \right) \quad \text{for some } i \in P \qquad (2.4)$$

and for $\tau_i \geqq 0, i = 1, \ldots, p$, (2.3) imply

$$\sum_{i=1}^{p} \tau_i \left(f_{i_0}(x) + s(x \mid C_{i_0}) \right) \leqq \sum_{i=1}^{p} \tau_i \left(f_{i_0}(u) + u^T w_{i_0} \right) \quad \forall i_0 \in P. \qquad (2.5)$$

Since $x^T w_i \leqq s(x \mid C_i)$, $i = 1, \ldots, p$, then (2.4) and (2.5) imply

$$\sum_{i=1}^{p} \tau_i \left(f_i(x) + x^T w_i \right) < \sum_{i=1}^{p} \tau_i \left(f_i(u) + u^T w_i \right),$$

$$\sum_{i=1}^{p} \tau_i \left(f_{i_0}(x) + x^T w_{i_0} \right) \leqq \sum_{i=1}^{p} \tau_i \left(f_{i_0}(u) + u^T w_{i_0} \right). \qquad (2.6)$$

From ρ_i-convexity of $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \ldots, p$, we have

$$(x - u)^T \left(\sum_{i=1}^p \tau_i (\nabla f_i(u) + w_i) \right) + \left(\sum_{i=1}^p \tau_i \rho_i \right) \|x - u\|^2 < 0 \quad \text{for some } i \in P, \qquad (2.7)$$

$$(x - u)^T \left(\sum_{i=1}^p \tau_i (\nabla f_{i_0}(u) + w_{i_0}) \right) + \left(\sum_{i=1}^p \tau_i \rho_i \right) \|x - u\|^2 \leqq 0 \quad \forall i_0 \in P \qquad (2.8)$$

respectively. Also, since (u, τ, w, y, z) is feasible for (MVODE) and x is feasible for (VOPE), we have

$$\left(\sum_{j=1}^m y_j g_j(x) + \sum_{l=1}^q z_l h_l(x) \right) - \left(\sum_{j=1}^m y_j g_j(u) + \sum_{l=1}^q z_l h_l(u) \right) \leqq 0. \qquad (2.9)$$

Since $g_j(\cdot)$, $j = 1, \ldots, m$ are σ_j-convex and $h_l(\cdot)$, $l = 1, \ldots, q$ are affine, then (2.9) imply

$$(x - u)^T \left(\sum_{j=1}^m y_j \nabla g_j(u) + \sum_{l=1}^q z_l \nabla h_l(u) \right) + \left(\sum_{j=1}^m y_j \sigma_j \right) \|x - u\|^2 \leqq 0. \qquad (2.10)$$

Adding (2.7), (2.10) and then applying hypothesis (a), we get

$$(x - u)^T \left(\sum_{i=1}^p \tau_i (\nabla f_i(u) + w_i) + \sum_{j=1}^m y_j \nabla g_j(u) + \sum_{l=1}^q z_l \nabla h_l(u) \right) < 0, \qquad (2.11)$$

which contradicts (2.1). Also, adding (2.8) and (2.10) and then applying hypothesis (b), we get (2.11). This contradicts to (2.1). Hence (2.3) cannot hold. □

It is easy to derive the following result from the corresponding one by Egudo [7].

Corollary 2.2. *Assume that the conclusion of Theorem 2.1 holds between (VOPE) and (MVODE). If $(u^0, \tau^0, w^0, y^0, z^0)$ is feasible for (MVODE) such that u^0 is feasible for (VOPE) and $u^{0T} w_i^0 = s(u^0 \mid C_i)$, $i = 1, \ldots, p$, then u^0 is efficient for (VOPE) and $(u^0, \tau^0, w^0, y^0, z^0)$ is efficient for (MVODE).*

Theorem 2.3 (Strong Duality). *If x^0 be efficient for (VOPE) and assume that x^0 satisfies a constraint qualification [14, pages 102-103] for (1.8) for at least one $k = 1, \ldots, p$. Then there exist $\tau^0 \in \mathbb{R}^p, y^0 \in \mathbb{R}^m, z^0 \in \mathbb{R}^q$ and $w_i^0 \in C_i$, $i = 1, \ldots, p$ such that $(x^0, \tau^0, w^0, y^0, z^0)$ is feasible for (MVODE) and $(x^0)^T w_i^0 = s(x^0 \mid C_i), i = 1, \ldots, p$. If also weak duality (Theorem 2.1) holds between (VOPE) and (MVODE), then $(x^0, \tau^0, w^0, y^0, z^0)$ is efficient for (MVODE).*

Proof. Since x^0 is an efficient solution of (VOPE), by Lemma 1.4, x^0 solves (1.8) for each $k = 1, \ldots, p$. By hypothesis there exists at least one $k = 1, \ldots, p$ such that x^0 satisfies a constraint qualification [14, pages 102,103] for (1.8). From the Kuhn-Tucker necessary conditions [14], we obtain $\tau_i \geqq 0$ for all $i \neq k$, $0 \leqq y \in \mathbb{R}^m, z \in \mathbb{R}^q$ and $w_i^0 \in C_i, i = 1, \ldots, p$ such that

$$\nabla f_k\left(x^0\right) + w_k + \sum_{i \neq k} \tau_i\left(\nabla f_i\left(x^0\right) + w_i\right) + \sum_{j=1}^m y_j \nabla g_j\left(x^0\right) + \sum_{l=1}^q z_l \nabla h_l\left(x^0\right) = 0, \qquad (2.12)$$

$$w_i^T x^0 = s\left(x^0 \mid C_i\right), \quad i = 1, \ldots, p, \qquad (2.13)$$

$$\sum_{j=1}^m y_j g_j\left(x^0\right) = 0. \qquad (2.14)$$

Now dividing (2.12) and (2.14) by $1 + \sum_{i \neq k} \tau_i$ and defining

$$
\begin{aligned}
\tau_k^0 &= \frac{1}{1 + \sum_{i \neq k} \tau_i} > 0, \qquad \tau_i^0 = \frac{\tau_i}{1 + \sum_{i \neq k} \tau_i} \geqq 0, \\
y^0 &= \frac{y}{1 + \sum_{i \neq k} \tau_i} \geqq 0, \qquad z^0 = \frac{z}{1 + \sum_{i \neq k} \tau_i},
\end{aligned}
\qquad (2.15)
$$

we conclude that $(x^0, \tau^0, w^0, y^0, z^0)$ is feasible for (MVODE). The efficiency of $(x^0, \tau^0, w^0, y^0, z^0)$ for (MVODE) now follows from Corollary 2.2. □

3. Wolfe Type Duality

We introduce a Wolfe type dual programming problem and establish weak and strong duality theorems.

Maximize $\quad f(u) + u^T w + \left(y^T g(u)\right)e + \left(z^T h(u)\right)e$

$$= \left(f_1(u) + u^T w_1 + y^T g(u) + z^T h(u), \ldots, f_p(u) + u^T w_p + y^T g(u) + z^T h(u)\right)$$
(WVODE)

subject to

$$\sum_{i=1}^p \tau_i\left(\nabla f_i(u) + w_i\right) + \nabla y^T g(u) + \nabla z^T h(u) = 0, \qquad (3.1)$$

$$y_j \geqq 0, \quad j = 1, \ldots, m, \ w_i \in C_i, \ i = 1, \ldots, p,$$

$$\tau_i \geqq 0, \quad i = 1, \ldots, p, \quad \sum_{i=1}^p \tau_i = 1. \qquad (3.2)$$

Theorem 3.1 (Weak Duality). *Assume that for all feasible x for* (VOPE) *and all feasible* (u, τ, w, y, z) *for* (WVODE), $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \ldots, p$, *are ρ_i-convex,* $g_j(\cdot)$, $j = 1, \ldots, m$, *are σ_j-convex and* $h_l(\cdot)$, $l = 1, \ldots, q$, *are affine. If also any of the following conditions holds:*

(a) $\tau_i > 0$, *for all* $i \in P$ *and* $\sum_{i=1}^p \tau_i \rho_i + \sum_{j=1}^m y_j \sigma_j \geqq 0$;

(a) $\sum_{i=1}^p \tau_i \rho_i + \sum_{j=1}^m y_j \sigma_j > 0$,

then the following cannot hold:

$$f_i(x) + s(x \mid C_i) < f_i(u) + u^T w_i + y^T g(u) + z^T h(u), \quad \text{for some } i \in P,$$
$$f_{i_0}(x) + s(x \mid C_{i_0}) \leqq f_{i_0}(u) + u^T w_{i_0} + y^T g(u) + z^T h(u), \quad \forall i_0 \in P. \tag{3.3}$$

Proof. Suppose contrary to the result that (3.3) hold. Then since x is feasible for (VOPE) and $0 \leqq y \in \mathbb{R}^m$, $z \in \mathbb{R}^q$, (3.3) imply

$$f_i(x) + s(x \mid C_i) + y^T g(x) + z^T h(x) < f_i(u) + u^T w_i + y^T g(u) + z^T h(u), \quad \text{for some } i \in P,$$
$$f_{i_0}(x) + s(x \mid C_{i_0}) + y^T g(x) + z^T h(x) \leqq f_{i_0}(u) + u^T w_{i_0} + y^T g(u) + z^T h(u), \quad \forall i_0 \in P. \tag{3.4}$$

Since $x^T w_i \leqq s(x \mid C_i)$, $i = 1, \ldots, p$, (3.4) yield

$$f_i(x) + x^T w_i + y^T g(x) + z^T h(x) < f_i(u) + u^T w_i + y^T g(u) + z^T h(u), \quad \text{for some } i \in P$$
$$f_{i_0}(x) + x^T w_{i_0} + y^T g(x) + z^T h(x) \leqq f_{i_0}(u) + u^T w_{i_0} + y^T g(u) + z^T h(u), \quad \forall i_0 \in P. \tag{3.5}$$

Now if hypothesis (a) holds, then from $\tau_i > 0$ for all $i \in P$, (3.5) we obtain

$$\sum_{i=1}^p \tau_i \left(f_i(x) + x^T w_i \right) + y^T g(x) \sum_{i=1}^p \tau_i + z^T h(x) \sum_{i=1}^p \tau_i$$
$$< \sum_{i=1}^p \tau_i \left(f_i(u) + u^T w_i \right) + y^T g(u) \sum_{i=1}^p \tau_i + z^T h(u) \sum_{i=1}^p \tau_i, \tag{3.6}$$

and since $\sum_{i=1}^p \tau_i = 1$, this inequality reduces to

$$\sum_{i=1}^p \tau_i \left(f_i(x) + x^T w_i - f_i(u) - u^T w_i \right) + y^T g(x) + z^T h(x) - y^T g(u) - z^T h(u) < 0. \tag{3.7}$$

Now from (3.7), ρ_i-convexity of $f_i(\cdot) + (\cdot)^T w_i, i = 1, \ldots, p$, σ_j-convexity of $g_j(\cdot)$, $j = 1, \ldots, m$ and $h_l(\cdot)$, $l = 1, \ldots, q$, is affine, we obtain

$$(x - u)^T \left(\sum_{i=1}^{p} \tau_i (\nabla f_i(u) + w_i) + \nabla y^T g(u) + \nabla z^T h(u) \right) + \left(\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \right) \|x - u\|^2 < 0$$

$$(3.8)$$

and since by hypothesis (a), $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geqq 0$, this implies

$$(x - u)^T \left(\sum_{i=1}^{p} \tau_i (\nabla f_i(u) + w_i) + \nabla y^T g(u) + \nabla z^T h(u) \right) < 0 \qquad (3.9)$$

which contradicts (3.1). Also from (3.5), $\tau_i \geqq 0$, $i = 1, \ldots, p$, and $\sum_{i=1}^{p} \tau_i = 1$, we obtain

$$\sum_{i=1}^{p} \tau_i \left(f_i(x) + x^T w_i - f_i(u) - u^T w_i \right) + y^T g(x) + z^T h(x) - y^T g(u) - z^T h(u) \leqq 0, \qquad (3.10)$$

and since $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \cdots, p$, are ρ_i-convex, $g_j(\cdot)$, $j = 1, \ldots, m$, are σ_j-convex and $h_l(\cdot)$, $l = 1, \ldots, q$, are affine, (3.10) implies

$$(x - u)^T \left(\sum_{i=1}^{p} \tau_i (\nabla f_i(u) + w_i) + \nabla y^T g(u) + \nabla z^T h(u) \right) + \left(\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \right) \|x - u\|^2 \leqq 0.$$

$$(3.11)$$

Now by hypothesis (b), $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j > 0$, hence (3.11) implies (3.9), again contradicting (3.1). □

The following result can be easily driven from the corresponding one by Egudo [7].

Corollary 3.2. *Assume that the conclusion of Theorem 3.1 holds between* (VOPE) *and* (WVODE). *If* $(u^0, \tau^0, w^0, y^0, z^0)$ *is feasible for* (WVODE) *such that* u^0 *is feasible for* (VOPE), $y^{0T} g(u^0) = 0$ *and* $(u^0)^T w_i^0 = s(u^0 \mid C_i)$, $i = 1, \ldots, p$, *then* u^0 *is efficient for* (VOPE) *and* $(u^0, \tau^0, w^0, y^0, z^0)$ *is efficient for* (WVODE).

Theorem 3.3 (Strong Duality). *If* x^0 *be efficient for* (VOPE) *and assume that* x^0 *satisfies a constraint qualification [14, pages 102,103] for* (1.8) *for at least one* $k = 1, \ldots, p$. *Then there exist* $\tau^0 \in \mathbb{R}^p, y^0 \in \mathbb{R}^m, z^0 \in \mathbb{R}^q$ *and* $w_i^0 \in C_i$, $i = 1, \ldots, p$ *such that* $(x^0, \tau^0, w^0, y^0, z^0)$ *is feasible for* (WVODE) *and* $(y^0)^T g(x^0) = 0$, $(x^0)^T w_i^0 = s(x^0 \mid C_i)$, $i = 1, \ldots, p$. *If also weak duality (Theorem 3.1) holds between* (VOPE) *and* (WVODE), *then* $(x^0, \tau^0, w^0, y^0, z^0)$ *is efficient for*

(WVODE).

Proof. Since x^0 is an efficient solution of (VOPE), from Lemma 1.4, x^0 solves (1.8) for all $k = 1, \ldots, p$. By hypothesis there exists a $k \in P = \{1, \ldots, p\}$ for which x^0 satisfies a constraint qualification [14, pages 102-103] for (1.8). Now from the Kuhn-Tucker necessary conditions [14], there exist $\tau_i \geqq 0$ for all $i \neq k$, $0 \leqq y \in \mathbb{R}^m$, $z \in \mathbb{R}^q$ and $w_i^0 \in C_i, i = 1, \ldots, p$ such that

$$\nabla f_k\left(x^0\right) + w_k + \sum_{i \neq k} \tau_i\left(\nabla f_i\left(x^0\right) + w_i\right) + \sum_{j=1}^{m} y_j \nabla g_j\left(x^0\right) + \sum_{l=1}^{q} z_l \nabla h_l\left(x^0\right) = 0, \qquad (3.12)$$

$$w_i^T x^0 = s\left(x^0 \mid C_i\right), \quad i = 1, \ldots, p, \qquad (3.13)$$

$$\sum_{j=1}^{m} y_j g_j\left(x^0\right) = 0 \qquad (3.14)$$

Now dividing (3.12) and (3.14) by $1 + \sum_{i \neq k} \tau_i$ and defining

$$\tau_k^0 = \frac{1}{1 + \sum_{i \neq k} \tau_i} > 0, \qquad \tau_i^0 = \frac{\tau_i}{1 + \sum_{i \neq k} \tau_i} \geqq 0,$$

$$y^0 = \frac{y}{1 + \sum_{i \neq k} \tau_i} \geqq 0, \qquad z^0 = \frac{z}{1 + \sum_{i \neq k} \tau_i}, \qquad (3.15)$$

we conclude that $(x^0, \tau^0, w^0, y^0, z^0)$ is feasible for (WVODE).

The efficiency of $(x^0, \tau^0, w^0, y^0, z^0)$ for (WVODE) now follows from Corollary 3.2. □

4. Special Cases

We give some special cases of our duality results.

(1) If support functions are excepted and $h = 0$, then our dual programs are reduced to the duals in Egudo [7].

(2) Let $C_i = \{B_i w : w^T B_i w \leqq 1\}$. Then $s(x \mid C_i) = (x^T B_i x)^{1/2}$ and the sets $C_i, i = 1, \ldots, p$, are compact and convex. If $h = 0$, then (VOPE), (MVODE) and (WVODE) reduce to the corresponding (VP), $(VDP)_2$ and $(VDP)_1$ in Lal et al. [8], respectively.

(3) If we replace ρ-convexity by generalized (F, ρ)-convexity, then our weak duality theorems reduce to the corresponding ones in Yang et al. [11].

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