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Research Article

Sharpening the Becker-Stark Inequalities

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In this paper, we establish a general refinement of the Becker-Stark inequalities by using the power series expansion of the tangent function via Bernoulli numbers and the property of a function involving Riemann's zeta one.

1. Introduction

Steckin [1] (or see Mitrinovic [2, 3.4.19, page 246]) gives us a result as follows.

Theorem 1.1 (see [1, Lemma 2.1]). *If* $0 < x < \pi/2$, *then*

$$\frac{4}{\pi}\frac{x}{\pi - 2x} < \tan x. \tag{1.1}$$

Later, Becker and Stark [3] (or see Kuang [4, 5.1.102, page 248]) obtain the following two-sided rational approximation for $(\tan x)/x$.

Theorem 1.2. *Let* $0 < x < \pi/2$, *then*

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}. \tag{1.2}$$

Furthermore, 8 and π^2 *are the best constants in* (1.2).

In fact, we can obtain the following further results.

Theorem 1.3. *Let $0 < x < \pi/2$, then*

$$\frac{\pi^2 + ((4(8-\pi^2))/\pi^2)x^2}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + (\pi^2/3 - 4)x^2}{\pi^2 - 4x^2}. \tag{1.3}$$

Furthermore, $\alpha = (4(8-\pi^2))/\pi^2$ and $\beta = \pi^2/3 - 4$ are the best constants in (1.3).

In this paper, in the form of (1.2) and (1.3) we shall show a general refinement of the Becker-Stark inequalities as follows.

Theorem 1.4. *Let $0 < x < \pi/2$, and let $N \geq 0$ be a natural number. Then*

$$\frac{P_{2N}(x) + \alpha x^{2N+2}}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{P_{2N}(x) + \beta x^{2N+2}}{\pi^2 - 4x^2} \tag{1.4}$$

holds, where $P_{2N}(x) = a_0 + a_1 x^2 + \cdots + a_N x^{2N}$, and

$$a_n = \frac{2^{2n+2}(2^{2n+2}-1)\pi^2}{(2n+2)!}|B_{2n+2}| - \frac{4 \cdot 2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where B_{2n} are the even-indexed Bernoulli numbers.

Furthermore, $\alpha = (8 - a_0 - a_1(\pi/2)^2 - \cdots - a_N(\pi/2)^{2N})/(\pi/2)^{2N+2}$ and $\beta = a_{N+1}$ are the best constants in (1.4).

2. Four Lemmas

Lemma 2.1. *The function $(1 - 1/2^n)\zeta(n)(n = 1, 2, \ldots)$ is decreasing, where $\zeta(n)$ is Riemann's zeta function.*

Proof. $(1 - 1/2^n)\zeta(n) = \zeta(n) - \zeta(n)/2^n$ is equivalent to the function $\lambda(n) = \sum_{k=0}^{\infty} 1/(2k+1)^n$, which is decreasing. □

Lemma 2.2 (see [5, Theorem 3.4]). *Let $\zeta(n)$ be Riemann's zeta function and B_{2n} the even-indexed Bernoulli numbers. Then*

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|, \quad n = 1, 2, \ldots. \tag{2.1}$$

Lemma 2.3 (see [6, 1.3.1.4 (1.3)]). *Let $|x| < \pi/2$. Then*

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(-1)^{n-1} B_{2n} x^{2n-1} = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}| x^{2n-1}. \tag{2.2}$$

Lemma 2.4. *Let $F(x) = (\pi^2 - 4x^2)(\tan x/x)$ and $|x| < \pi/2$. Then $F(x) = \pi^2 + \sum_{n=1}^{+\infty} a_n x^{2n}$, where*

$$a_n = \frac{2^{2n+2}(2^{2n+2}-1)\pi^2}{(2n+2)!}|B_{2n+2}| - \frac{4 \cdot 2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}| < 0, \quad n = 1, 2, \ldots. \tag{2.3}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned} F(x) &= \left(\pi^2 - 4x^2\right)\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-2} \\ &= \pi^2 + \sum_{n=1}^{+\infty}\left[\frac{2^{2n+2}(2^{2n+2}-1)\pi^2}{(2n+2)!}|B_{2n+2}| - \frac{4 \cdot 2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|\right]x^{2n} \\ &:= \pi^2 + \sum_{n=1}^{+\infty} a_n x^{2n}. \end{aligned} \tag{2.4}$$

Since $(1 - (1/2^{2n}))\zeta(2n)$ is decreasing by Lemma 2.1, it follows that

$$\frac{2^{2n+2}-1}{4}\zeta(2n+2) < \left(2^{2n}-1\right)\zeta(2n). \tag{2.5}$$

From Lemma 2.2, we get

$$\frac{\pi^2(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}| < \frac{(2^{2n}-1)}{(2n)!}|B_{2n}|, \tag{2.6}$$

which implies that $a_n < 0$ for $n = 1, 2, \ldots$. □

3. Proofs of Theorems

Proof of Theorem 1.4. Let

$$G(x) = \frac{((\tan x)/x)(\pi^2 - 4x^2) - (a_0 + a_1x^2 + \cdots + a_N x^{2N})}{x^{2N+2}}. \tag{3.1}$$

Then

$$G(x) = \frac{F(x) - (a_0 + a_1x^2 + \cdots + a_N x^{2N})}{x^{2N+2}} = \frac{\sum_{n=N+1}^{+\infty} a_n x^{2n}}{x^{2N+2}} = \sum_{k=0}^{+\infty} a_{N+1+k}x^{2k}. \tag{3.2}$$

By Lemma 2.4, we have $a_n < 0$ for $n \in \mathbb{N}^+$, and $G(x)$ is decreasing on $(0, \pi/2)$.

At the same time, $\alpha = \lim_{x\to(\pi/2)^-}G(x) = (8 - a_0 - a_1(\pi/2)^2 - \cdots - a_N(\pi/2)^{2N})/(\pi/2)^{2N+2}$ by (3.1), and $\beta = \lim_{x\to 0^+}G(x) = a_{N+1}$ by (3.2), so α and β are the best constants in (1.4). □

Proof of Theorem 1.3. Let $N = 0$ in Theorem 1.4; we obtain that $\alpha = (4(8 - \pi^2))/\pi^2$ and $\beta = \pi^2/3 - 4$. Then the proof of Theorem 1.3 is complete. □

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