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Research Article

A Nonlinear Inequality Arising in Geometry and Calabi-Bernstein Type Problems

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A characterization for the entire solutions of a nonlinear inequality, which has a natural interpretation in terms of certain nonflat Robertson-Walker spacetimes, is given. As an application, new Calabi-Bernstein type problems are solved.

1. Introduction

Let $f : I \to \mathbb{R}$ be a positive smooth function on an open interval $I = (a,b)$, $-\infty \le a < b \le \infty$, of the real line $\mathbb{R}$, and let Ω be an open domain of $\mathbb{R}^2$. For each $u \in C^\infty(\Omega)$ such that $|Du| < f(u)$, where $|Du|$ stands for the length of the gradient Du of u, we consider the smooth function

$$H(u) = -\operatorname{div}\left(\frac{Du}{2f(u)\sqrt{f(u)^2 - |Du|^2}}\right) - \frac{f'(u)}{2\sqrt{f(u)^2 - |Du|^2}}\left(2 + \frac{|Du|^2}{f(u)^2}\right), \tag{1.1}$$

where div represents the divergence operator. The function $H(u)$ has a natural geometric interpretation as showed below. In fact, consider the graph $\{(u(x,y),x,y) : (x,y) \in \Omega\}$ of u in the 3-dimensional manifold $M = I \times \mathbb{R}^2$, endowed with the Lorentzian metric

$$\langle \cdot , \cdot \rangle = -\pi_I^*\left(dt^2\right) + f(\pi_I)^2 \pi_{\mathbb{R}^2}^*(g_0), \tag{1.2}$$

where π_I and $\pi_{\mathbb{R}^2}$ denote the projections onto I and $\mathbb{R}^2$, respectively, and g_0 is the usual Riemannian metric of $\mathbb{R}^2$. The Lorentzian manifold $(M, \langle \cdot, \cdot \rangle)$ is the warped product, in the sense of [1, page 204], with base $(I, -dt^2)$, fiber $(\mathbb{R}^2, g_0)$, and warping function f. We will call M a 3-dimensional Robertson-Walker (RW) spacetime with fiber $\mathbb{R}^2$. The induced metric from (1.2) on the graph of u is written as follows:

$$g_u = -du^2 + f(u)^2 g_0, \tag{1.3}$$

on Ω, and it is positive definite, that is, Riemannian, if and only if u satisfies $|Du| < f(u)$ on all Ω (the graph is then said to be spacelike). The unitary timelike vector field $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$ determines a time orientation on M and allows us to take for each spacelike graph (or spacelike surface) in M, a unitary normal vector field N in the same time orientation of $-\partial_t$, that is, such that $\langle N, \partial_t \rangle > 0$. On the spacelike graph of u, we have

$$N = -\frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}} \left(f(u)^2 \, \partial_t + Du \right), \tag{1.4}$$

and the function $H(u)$, given by (1.1), is the mean curvature with respect to N for the spacelike graph of u (see Section 3 for details). Note that if $u = u_0$ (constant) then (1.1) reduces to $H(u_0) = -f'(u_0)/f(u_0)$, which is the mean curvature of the spacelike surface of M defined by $t = u_0$ (it is called a spacelike slice). Thus, formula (1.1), with $H =$ constant, and the constraint $|Du| < f(u)$, constitute the constant mean curvature (CMC) spacelike graph equation in M. Note that the constraint involving the length of the gradient of u implies that the partial differential equation is elliptic. In a special case where $I = \mathbb{R}$, $\Omega = \mathbb{R}^2$ and $f = 1$, that is, when M is the Lorentz-Minkowski spacetime, there are many entire (i.e., defined on all $\mathbb{R}^2$) solutions of the CMC spacelike graph equation [2]. This suggests that, when dealing with uniqueness results of entire solutions of the CMC spacelike graph equation in RW spacetimes, a stronger assumption than $|Du| < f(u)$ is needed (see below).

More generally, in this paper we will study the following nonlinear differential inequality

$$H(u)^2 \le \frac{f'(u)^2}{f(u)^2}, \tag{\textbf{I}.1}$$

$$|Du| < \lambda f(u), \quad 0 < \lambda < 1. \tag{\textbf{I}.2}$$

The geometric meaning of (**I**.2) is that the graph of u is spacelike and $\operatorname{Sup}(|Du|/f(u)) < 1$. Moreover, (**I**.1) means that at the point of the graph of u corresponding to (x_0, y_0), the absolute value of the mean curvature, is at most the absolute value of the mean curvature of the graph of the constant function $u = u_0$, where $u_0 = u(x_0, y_0)$. Note that we only suppose here a natural comparison inequality between two mean curvature quantities, but we don't require H constant. Along the paper, inequality (**I**) will mean inequality (**I**.1) with additional assumption (**I**.2).

It is clear that the constant functions are entire solutions of inequality (**I**). Our main aim in this paper is to state a converse under a suitable assumption on the warping function f. In order to do that, we will work directly on spacelike surfaces instead of spacelike graphs. Recall that a spacelike surface is locally a spacelike graph and this holds globally under some extra topological hypotheses [2, Section 3]. Our main tool is a local integral estimation of the squared length of the gradient of the restriction of the warping function on a spacelike surface. If f is not locally constant (then, M is said to be proper) and $f'' \leq 0$ (which has an interesting curvature interpretation called the timelike convergence condition (TCC)), we first prove (Theorem 4.2).

Let S be a spacelike surface of a proper RW spacetime with fiber $\mathbb{R}^2$, which obeys the TCC. Suppose that the mean curvature H of S satisfies

$$H^2 \leq \frac{f'(t)^2}{f(t)^2}. \tag{1.5}$$

If B_R denotes a geodesic disc of radius R around a fixed point p in S, then, for any r such that $0 < r < R$, there exists a positive constant $C = C(p, r)$ such that

$$\int_{B_r} |\nabla f(t)|^2 \, dV \leq \frac{C}{\mu_{r,R}}, \tag{1.6}$$

where B_r is the geodesic disc of radius r around p in S, and $1/\mu_{r,R}$ is the capacity of the annulus $B_R \setminus \overline{B}_r$.

For the case in which S is analytic, we can express the local integral estimation in a more geometric way (Remark 4.5).

Recall that a (general) noncompact 2-dimensional Riemannian manifold S is parabolic if and only if $1/\mu_{r,R} \to 0$ as $R \to \infty$ [3, Section 2]. On the other hand, the Gauss curvature of the spacelike surface S is nonnegative whereas the TCC and inequality $H^2 \leq f'(t)^2/f(t)^2$ hold true (see Section 3.2). Thus, using a well-known result by Ahlfors and Blanc-Fiala-Huber [4], we obtain that if S is complete then it is parabolic. Therefore, R approaches infinity for a fixed arbitrary point p and a fixed r, obtaining that $f(t)$ is constant on S. Since the RW spacetime is proper, this implies that S must be a spacelike slice with $t = t_0$. Thus, the first application of Theorem 4.2 is to reprove uniqueness result [5, Theorem 4.5] with a local and different approach (Corollary 4.3).

It should be noted that inequality for H assumed in Theorem 4.2 holds in a natural way under some suitable hypotheses on each complete CMC spacelike surface that lies between two spacelike slices [5, Section 5]. However, note that we are not assuming here that H is constant. In fact, Theorem 4.2 provides with several uniqueness results for complete spacelike surfaces whose constant mean curvature is only bounded (Corollaries 4.6 and 4.7).

Returning to our main aim, recall that an entire spacelike graph in an RW spacetime with fiber $\mathbb{R}^2$ cannot be complete, in general (see, e.g., [6]). However, a graph of an entire function which satisfies (**I**.2) must be complete (Section 4). Therefore, as an application of the previous result we obtain the following uniqueness results in the nonparametric case (Theorems 4.8 and 4.9)

If f is not locally constant, satisfies $\mathrm{Inf}(f) > 0$ *and* $f'' \leq 0$*, then the only entire solutions of inequality* (**I**) *are the constant functions.*

If f is not locally constant and satisfies $f'' \leq 0$, then the only bounded entire solutions of inequality (**I**) *are the constant functions.*

Finally, observe that inequality (**I**) is trivially true in the maximal case; that is, for $H = 0$. Hence, our results contain new proofs of well-known Calabi-Bernstein type results (see [7, Theorem A]).

2. Preliminaries

On each RW spacetime M with fiber $\mathbb{R}^2$, the vector field $\xi := f(\pi_I)\ \partial_t$ is timelike and satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I) X, \tag{2.1}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ denotes the Levi-Civita connection of the metric (1.2), [1, Proposition 7.35]. Thus, ξ is conformal with $\mathcal{L}_\xi \langle\cdot,\cdot\rangle = 2\ f'(\pi_I)\ \langle\cdot,\cdot\rangle$, and its metrically equivalent 1-form is closed.

Since M is 3-dimensional, its curvature is completely determined by its Ricci tensor, and this obviously depends on f; actually, M is flat if and only if f is constant [1, Corollary 7.43]. Here, we are interested in the case in which no open subset of M is flat (i.e., f is not locally constant) and, then, we will refer M as a proper RW spacetime. Moreover, we will suppose that the curvature of M satisfies a natural geometric assumption which arises from Relativity theory. This assumption is the so-called timelike convergence condition (TCC). Recall that a Lorentzian manifold (of any dimension ≥ 3) obeys the TCC if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies

$$\overline{\mathrm{Ric}}(Z,Z) \geq 0, \quad \text{for any timelike tangent vector } Z, \tag{2.2}$$

that is, such that $\langle Z,Z\rangle < 0$. This curvature condition is the mathematical translation that gravity, on average, attracts and, on 4-dimensional spacetimes, holds whenever the metric tensor satisfies the Einstein equation (with zero cosmological constant) [8]. We will also consider on M the stronger condition: $\overline{\mathrm{Ric}}(Z,Z) > 0$, for any timelike tangent vector Z. When this holds, we will say that the TCC is strict on M. Let us remark that, on 4-dimensional spacetimes, this curvature assumption indicates the presence of nonvanishing matter fields [9].

A weaker curvature condition than the TCC is the null convergence condition (NCC) which reads

$$\overline{\mathrm{Ric}}(Z,Z) \geq 0, \ \text{ for any null tangent vector } Z, \tag{2.3}$$

that is, $Z \neq 0$ which satisfies $\langle Z,Z\rangle = 0$ [8]. A clear continuity argument shows that the TCC implies the NCC (on any $n(\geq 3)$-dimensional Lorentzian manifold). Note that any Einstein Lorentzian manifold (in particular, a Lorentzian space form) always satisfies the NCC.

In the case that M is an RW spacetime with fiber $\mathbb{R}^2$, and making use again of [1, Corollary 7.43], we can express the previous curvature conditions in terms of the warping function. Thus, M obeys the TCC if and only if $f'' \leq 0$, the TCC strict if and only if $f'' < 0$ and the NCC is equivalent to $(\log f)'' \leq 0$. It is easy to see that if there exists $t_0 \in I$ with $f'(t_0) = 0$, then the NCC implies the TCC. Moreover, if $(\log f)'' \leq 0$ and there exists $t_0 \in I$ such that $f'(t_0) = 0$, then this zero of f' is unique and $\operatorname{Sup} f(t) = f(t_0)$.

3. Setup

3.1. The Restriction of the Warping Function on a Spacelike Surface

Let $x : S \to M$ be a (connected) spacelike surface in M; that is, x is an immersion and it induces a Riemannian metric on the (2-dimensional) manifold S from the Lorentzian metric (1.2). It should be noted that any spacelike surface in M is orientable and noncompact [10]. We represent the induced metric with the same symbol as the metric (1.2) does. The unitary timelike vector field $\partial_t \in \mathfrak{X}(M)$ allows us to consider $N \in \mathfrak{X}^{\perp}(S)$ as the only, globally defined, unitary timelike normal vector field on S in the same time orientation of $-\partial_t$. Thus, from the wrong way Cauchy-Schwarz inequality, (see [1, Proposition 5.30], for instance) we have $\langle N, \partial_t \rangle \geq 1$ and $\langle N, \partial_t \rangle = 1$ at a point p if and only if $N(p) = -\partial_t(p)$. By spacelike slice we mean a spacelike surface x such that $\pi_I \circ x$ is a constant. A spacelike surface is a spacelike slice if and only if it is orthogonal to ∂_t or, equivalently, orthogonal to ξ.

Denote by $\partial_t^T := \partial_t + \langle N, \partial_t \rangle N$ the tangential component of ∂_t on S. It is not difficult to see

$$\nabla t = -\partial_t^T, \tag{3.1}$$

where ∇t is the gradient of $t := \pi_I \circ x$ on S. Now, from the Gauss formula, taking into account $\xi^T = f(t)\partial_t^T$ and (3.1), the Laplacian of t satisfies

$$\Delta t = -\frac{f'(t)}{f(t)} \left\{ 2 + |\nabla t|^2 \right\} - 2H \ \langle N, \partial_t \rangle, \tag{3.2}$$

where $f(t) := f \circ t$, $f'(t) := f' \circ t$ and the function $H := -(1/2)\ \mathrm{trace}(A)$, where A is the shape operator associated to N, is called the mean curvature of S relative to N. A spacelike surface S with constant mean curvature is a critical point of the area functional under a certain volume constraint (see [11], for instance). A spacelike surface with $H = 0$ is called maximal. Note that, with our choice of N, the shape operator of the spacelike slice with $t = t_0$ is $A = (f'(t_0)/f(t_0))\ I$ and its mean curvature is $H = -f'(t_0)/f(t_0)$.

A direct computation from (3.1) and (3.2) gives

$$\Delta f(t) = -2\ \frac{f'(t)^2}{f(t)} + f(t)\left(\log f\right)''(t)|\nabla t|^2 - 2f'(t)H\langle N, \partial_t \rangle, \tag{3.3}$$

for any spacelike surface in M.

3.2. The Gauss Curvature of a Spacelike Surface

From the Gauss equation of a spacelike surface S in M and taking in mind the expression for the Ricci tensor of M [1, Corollary 7.43], the Gauss curvature K of S satisfies

$$K = \frac{f'(t)^2}{f(t)^2} - (\log f)''(t)|\nabla t|^2 - 2H^2 + \frac{1}{2}\,\mathrm{trace}\left(A^2\right), \tag{3.4}$$

where

$$\frac{f'(t)^2}{f(t)^2} - (\log f)''(t)|\nabla t|^2 \tag{3.5}$$

is, at any $p \in S$, the sectional curvature in M of the tangent plane $dx_p(T_pS)$.

Now, the Cauchy-Schwarz inequality for symmetric operators implies $(\mathrm{trace}\ A)^2 \leq 2\,\mathrm{trace}(A^2)$, and therefore, we have $H^2 \leq (1/2)\,\mathrm{trace}(A^2)$. If M obeys the NCC and we assume the spacelike surface satisfies $H^2 \leq f'(t)^2/f(t)^2$, then formula (3.4) gives $K \geq 0$.

4. Main Results

If B_r and B_R ($r < R$) denote geodesic balls centered at the point p of a Riemannian manifold, we recall that $1/\mu_{r,R} := \int_{A_{r,R}} |\nabla \omega_{r,R}|^2 dV$ is the capacity of the annulus $A_{r,R} := B_R \setminus \overline{B}_r$, being $\omega_{r,R}$ the harmonic measure of ∂B_R (see [3, Section 2] for instance). First of all, we recall the following technical result.

Lemma 4.1 (see [12, Lemma 2.2]). *Let S be an $n(\geq 2)$-dimensional Riemannian manifold and let $v \in C^2(S)$ which satisfies $v\Delta v \geq 0$. Let B_R be a geodesic ball of radius R in S. For any r such that $0 < r < R$, one has*

$$\int_{B_r} |\nabla v|^2\, dV \leq \frac{4\mathrm{Sup}_{B_R} v^2}{\mu_{r,R}}, \tag{4.1}$$

where B_r denotes the geodesic ball of radius r around p in S and $1/\mu_{r,R}$ is the capacity of the annulus $B_R \setminus \overline{B}_r$.

Now, we are in a position to prove the announced local integral estimation.

Theorem 4.2. *Let S be a spacelike surface of a proper RW spacetime with fiber $\mathbb{R}^2$, which obeys the TCC. Suppose that the mean curvature H of S satisfies*

$$H^2 \leq \frac{f'(t)^2}{f(t)^2}. \tag{4.2}$$

If B_R denotes a geodesic disc of radius R around a fixed point p in S, then, for any r such that $0 < r < R$, there exists a positive constant $C = C(p, r)$ such that

$$\int_{B_r} |\nabla f(t)|^2 \, dV \leq \frac{C}{\mu_{r,R}}, \tag{4.3}$$

where B_r is the geodesic disc of radius r around p in S, and $1/\mu_{r,R}$ is the capacity of the annulus $B_R \setminus \overline{B}_r$.

Proof. Let Θ be the hyperbolic angle between N and $-\partial_t$, therefore $\langle N, \partial_t \rangle^2 = \cosh^2\Theta$ and $|\nabla t|^2 = \sinh^2\Theta$. Now, from (3.3) we obtain

$$\frac{1}{f(t)} \, \Delta f(t) \leq \left(H^2 - \frac{f'(t)^2}{f(t)^2} \right) \cosh^2\Theta + \frac{f''(t)}{f(t)} \, \sinh^2\Theta. \tag{4.4}$$

The first term of the right-hand side of (4.4) is nonpositive, because of (4.2), and the second one is also nonpositive using the TCC. Therefore, we obtain $\Delta f(t) \leq 0$.

Now, let us consider the function $v := \operatorname{arccot}(f(t)) : S \to (\pi, 2\pi)$. A direct computation from (4.4) gives $v \Delta v \geq 0$. Finally, the result follows making use of Lemma 4.1. □

As a first application of Theorem 4.2 we reprove the following well-known uniqueness result, using a different approach.

Corollary 4.3 (see [5, Theorem 4.5]). *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$ and which obeys the TCC. The only complete spacelike surfaces S in M whose mean curvature H satisfies*

$$H^2 \leq \frac{f'(t)^2}{f(t)^2} \tag{4.5}$$

on all S, are the spacelike slices.

As mentioned in Section 2, if M obeys the NCC and there exists $t_0 \in I$ such that $f'(t_0) = 0$, then M also obeys the TCC. On the other hand, any maximal surface in M clearly satisfies (4.2), hence we reprove and extend (with a different approach) the parametric version of the Calabi-Bernstein type result [7, Corollary 5.1].

Corollary 4.4. *Let M be a proper RW spacetime, with fiber $\mathbb{R}^2$, which obeys the NCC and assume there exists $t_0 \in I$ such that $f'(t_0) = 0$. Then, the only complete maximal surface in M is the spacelike slice $t = t_0$.*

Remark 4.5. Let us assume f is analytic (i.e., the metric (1.2) is analytic) and nonconstant. Take the spacelike surface S to be the graph of an analytic function. Then, under the assumptions of Theorem 4.2 and using [13, Lemma 2.1 and inequality (2.4)], we can rewrite the integral estimation as

$$\int_{B_r} |\nabla f(t)|^2 \, dV \leq \frac{\widetilde{C}}{\log(R/r)}, \tag{4.6}$$

where $\widetilde{C} = \widetilde{C}(p,r)$ is a positive constant.

Corollary 4.6. *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$, which obeys the TCC. Suppose that its warping function satisfies $f' > 0$. If $\lim_{s\to b}(f'(s)^2/f(s)^2)R$, then the only complete spacelike surfaces such that*

$$H^2 \leq \lim_{s\to b} \frac{f'(s)^2}{f(s)^2} \tag{4.7}$$

are the spacelike slices. Moreover, if the TCC is strict on M, then there is no such a spacelike surface.

Proof. From our assumptions, the TCC and $f' > 0$, we have $\operatorname{Inf} f'(s)^2/f(s)^2 = \lim_{s\to b} f'(s)^2/f(s)^2$. Therefore, the mean curvature of the spacelike surface satisfies (4.2). Thus, Theorem 4.2 can be then claimed to conclude the integral estimation (4.3). The proof ends making $R \to \infty$ in this formula. □

Analogously we can state the following corollary.

Corollary 4.7. *Let M be a proper RW spacetime with fiber $\mathbb{R}^2$, which obeys the TCC. Suppose that its warping function satisfies $f' < 0$. If $\lim_{s\to a}(f'(s)^2/f(s)^2)R$, then the only complete spacelike surfaces such that*

$$H^2 \leq \lim_{s\to a} \frac{f'(s)^2}{f(s)^2} \tag{4.8}$$

are the spacelike slices. Moreover, if the TCC is strict on M, then there is no such a spacelike surface.

Finally, we show the announced uniqueness results of inequality (**I**).

Theorem 4.8. *If f is not locally constant, has $\operatorname{Inf}(f) > 0$ and satisfies $f'' \leq 0$, then the only entire solutions to inequality* (**I**) *are the constant functions.*

Proof. The graph $\Sigma = \{(u(x,y),x,y) : (x,y) \in \mathbb{R}\}$ of any entire solution u to inequality (**I**) is a spacelike surface and the constraint (**I**.2) may be expressed as follows:

$$\langle N, \partial_t \rangle < \frac{1}{\sqrt{1-\lambda^2}}. \tag{4.9}$$

Hence, the induced metric g_u, given in (1.3), satisfies

$$g_u((a,b),(a,b)) \geq \left(1-\lambda^2\right) f(u)^2 \left(a^2+b^2\right), \tag{4.10}$$

from (4.9), for all $(a,b)R^2$. On the other hand, we have $\text{Inf}\, f(u) > 0$, and therefore, previous inequality indicates that g_u is complete. Now, the result follows from the parametric case. □

With an analogous reasoning we obtain the following theorem.

Theorem 4.9. *If f is not locally constant and satisfies $f'' \leq 0$, then the only bounded entire solutions of inequality* (**I**) *are the constant functions.*

Remark 4.10. Observe that Theorem 4.9 trivially holds true if H is assumed to be identically zero. Therefore, Theorem 4.9 reproves the well-known uniqueness result for the maximal surface equation [7, Theorem A]. On the other hand, Theorem 4.8, partially extends [14, Theorem 7.1].

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