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Research Article

Second Moment Convergence Rates for Uniform Empirical Processes

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Let $\{U_1, U_2, \ldots, U_n\}$ be a sequence of independent and identically distributed $U[0,1]$-distributed random variables. Define the uniform empirical process as $\alpha_n(t) = n^{-1/2}\sum_{i=1}^{n}(I\{U_i \le t\} - t), 0 \le t \le 1$, $\|\alpha_n\| = \sup_{0\le t\le 1}|\alpha_n(t)|$. In this paper, we get the exact convergence rates of weighted infinite series of $\mathrm{E}\|\alpha_n\|^2 I\{\|\alpha_n\| \ge \varepsilon(\log n)^{1/\beta}\}$.

1. Introduction and Main Results

Let $\{X, X_n; n \ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with zero mean. Set $S_n = \sum_{i=1}^{n} X_i$ for $n \ge 1$, and $\log x = \ln(x \vee e)$. Hsu and Robbins [1] introduced the concept of complete convergence. They showed that

$$\sum_{n=1}^{\infty} P\{|S_n| \ge \varepsilon n\} < \infty, \quad \varepsilon > 0 \tag{1.1}$$

if $\mathrm{E}X = 0$ and $\mathrm{E}X^2 < \infty$. The converse part was proved by the study of Erdös in [2]. Obviously, the sum in (1.1) tends to infinity as $\varepsilon \searrow 0$. Many authors studied the exact rates in terms of ε (cf. [3–5]). Chow [6] studied the complete convergence of $\mathrm{E}\{|S_n| - \varepsilon n^{\alpha}\}_{+}$, $\varepsilon > 0$. Recently, Liu and Lin [7] introduced a new kind of complete moment convergence which is interesting, and got the precise rate of it as follows.

Theorem A. *Suppose that* $\{X, X_n; n \geq 1\}$ *is a sequence of i.i.d. random variables, then*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{E}S_n^2 I\{|S_n| \geq \varepsilon n\} = 2\sigma^2 \tag{1.2}$$

holds, if and only if $\mathrm{EX} = 0$, $\mathrm{EX}^2 = \sigma^2$, *and* $\mathrm{EX}^2 \log^+ |X| < \infty$.

Other than partial sums, many authors investigated precise rates in some different cases, such as U-statistics (cf. [8, 9]) and self-normalized sums (cf. [10, 11]). Zhang and Yang [12] extended the precise asymptotic results to the uniform empirical process. We suppose $U_1, U_2, \cdots, U_n$ is the sample of $U[0,1]$ random variables and $E_n(t)$ is the empirical distribution function of it. Denote the uniform empirical process by $\alpha_n(t) = \sqrt{n}(E_n(t) - t)$, $0 \leq t \leq 1$, and the norm of a function $f(t)$ on $[0,1]$ by $\|f\| = \sup_{0 \leq t \leq 1} |f(t)|$. Let $B(t)$, $t \in [0,1]$ be the Brownian bridge. We present one result of Zhang and Yang [12] as follows.

Theorem B. *For any* $\delta > -1$, *one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} P\left\{ \|\alpha_n\| \geq \varepsilon \sqrt{\log n} \right\} = \frac{\mathrm{E}\|B\|^{2\delta+2}}{\delta + 1}. \tag{1.3}$$

Inspired by the above conclusions, we consider second moment convergence rates for the uniform empirical process in the law of iterated logarithm and the law of the logarithm. Throughout this paper, let C denote a positive constant whose values can be different from one place to another. $[x]$ will denote the largest integer $\leq x$. The following two theorems are our main results.

Theorem 1.1. *For* $0 < \beta \leq 2$, $\delta > 2/\beta - 1$, *one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta - 2/\beta}}{n} E\|\alpha_n\|^2 I\left\{ \|\alpha_n\| \geq \varepsilon (\log n)^{1/\beta} \right\} = \frac{\beta E\|B\|^{\beta(\delta+1)}}{\beta(\delta+1) - 2}. \tag{1.4}$$

Theorem 1.2. *For* $0 < \beta \leq 2$, $\delta > 2/\beta - 1$, *one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\delta - 2/\beta}}{n \log n} E\|\alpha_n\|^2 I\left\{ \|\alpha_n\| \geq \varepsilon (\log \log n)^{1/\beta} \right\} = \frac{\beta E\|B\|^{\beta(\delta+1)}}{\beta(\delta+1) - 2}. \tag{1.5}$$

Remark 1.3. It is well known that $P\{\|B\| \geq x\} = 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2k^2x^2}$, $x > 0$ (see Csörgő and Révész [13, page 43]). Therefore, by Fubini's theorem we have

$$\begin{aligned}
\mathrm{E}\|B\|^{\beta(\delta+1)} &= \beta(\delta+1)\int_0^{\infty} x^{\beta(\delta+1)-1}P\{\|B\| \geq x\}dx \\
&= 2\beta(\delta+1)\int_0^{\infty} x^{\beta(\delta+1)-1}\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2k^2x^2}dx \\
&= \frac{\beta(\delta+1)\Gamma(\beta(\delta+1)/2)}{2^{\beta(\delta+1)/2}}\sum_{k=1}^{\infty}(-1)^{k+1}k^{-\beta(\delta+1)}.
\end{aligned} \tag{1.6}$$

Consequently, explicit results of (1.4) and (1.5) can be calculated further.

2. The Proofs

In order to prove Theorem 1.1, we present several propositions first.

Proposition 2.1. *For* $\beta > 0$, $\delta > -1$, *one has*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{\beta(\delta+1)}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\left\{\|B\| \geq \varepsilon(\log n)^{1/\beta}\right\} = \frac{E\|B\|^{\beta(\delta+1)}}{\delta+1}. \tag{2.1}$$

Proof. We calculate that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{\beta(\delta+1)}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\left\{\|B\| \geq \varepsilon(\log n)^{1/\beta}\right\} \\
&\quad= \lim_{\varepsilon\searrow 0}\varepsilon^{\beta(\delta+1)}\int_2^{\infty}\frac{(\log y)^{\delta}}{y}P\left\{\|B\| \geq \varepsilon(\log y)^{1/\beta}\right\}dy \\
&\quad= \beta\int_0^{\infty} t^{\beta(\delta+1)-1}P\{\|B\| \geq t\}dt \\
&\quad= \frac{\mathrm{E}\|B\|^{\beta(\delta+1)}}{\delta+1}.
\end{aligned} \tag{2.2}$$

□

Proposition 2.2. *For* $\beta > 0$, $\delta > -1$, *one has*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{\beta(\delta+1)}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}\left|P\left\{\|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta}\right\} - P\left\{\|B\| \geq \varepsilon(\log n)^{1/\beta}\right\}\right| = 0. \tag{2.3}$$

Proof. Following [4], set $A(\varepsilon) = [\exp(M/\varepsilon^{\beta})]$, where $M > 1$. Write

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \left| P\left\{ \|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta} \right\} - P\left\{ \|B\| \geq \varepsilon(\log n)^{1/\beta} \right\} \right| \\
&\quad = \sum_{n \leq A(\varepsilon)} \frac{(\log n)^{\delta}}{n} \left| P\left\{ \|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta} \right\} - P\left\{ \|B\| \geq \varepsilon(\log n)^{1/\beta} \right\} \right| \\
&\qquad + \sum_{n > A(\varepsilon)} \frac{(\log n)^{\delta}}{n} \left| P\left\{ \|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta} \right\} - P\left\{ \|B\| \geq \varepsilon(\log n)^{1/\beta} \right\} \right| \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{2.4}$$

It is wellknown that $\alpha_n(\cdot) \xrightarrow{d} B(\cdot)$ (see Csörgő and Révész [13, page 17]). By continuous mapping theorem, we have $\|\alpha_n\| \xrightarrow{d} \|B\|$. As a result, it follows that

$$\Delta_n := \sup_x \left| P\{\|\alpha_n\| \geq x\} - P\{\|B\| \geq x\} \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

Using the Toeplitz's lemma (see Stout [14, pages 120-121]), we can get $\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)} I_1 = 0$. For I_2, it is obvious that

$$\begin{aligned}
I_2 &\leq \sum_{n > A(\varepsilon)} \frac{(\log n)^{\delta}}{n} P\left\{ \|B\| \geq \varepsilon(\log n)^{1/\beta} \right\} + \sum_{n > A(\varepsilon)} \frac{(\log n)^{\delta}}{n} P\left\{ \|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta} \right\} \\
&=: I_3 + I_4.
\end{aligned} \tag{2.6}$$

Notice that $A(\varepsilon) - 1 \geq \sqrt{A(\varepsilon)}$, for a small ε. Via the similar argument in [4] we have

$$\begin{aligned}
\varepsilon^{\beta(\delta+1)} I_3 &\leq \varepsilon^{\beta(\delta+1)} \sum_{n > A(\varepsilon)} \frac{(\log n)^{\delta}}{n} P\left\{ \|B\| \geq \varepsilon(\log n)^{1/\beta} \right\} \\
&\leq C \int_{(M/2)^{1\beta}}^{\infty} y^{\beta(\delta+1)-1} P\{\|B\| \geq y\} dy \longrightarrow 0, \quad \text{as } M \longrightarrow \infty.
\end{aligned} \tag{2.7}$$

From Kiefer and Wolfowitz [15], we have

$$P\{\|\alpha_n\| \geq x\} \leq C e^{-Cx^2}. \tag{2.8}$$

Therefore,

$$\begin{aligned}
\varepsilon^{\beta(\delta+1)} I_4 &\leq C\varepsilon^{\beta(\delta+1)} \sum_{n>A(\varepsilon)} \frac{(\log n)^{\delta}}{n} \exp\left\{-C\varepsilon^2 (\log n)^{2/\beta}\right\} \\
&\leq C\varepsilon^{\beta(\delta+1)} \int_{\sqrt{A(\varepsilon)}}^{\infty} \frac{(\log x)^{\delta}}{x} \exp\left\{-C\varepsilon^2 (\log x)^{2/\beta}\right\} dx \\
&\leq C \int_{C(M/2)^{2/\beta}}^{\infty} y^{\beta(\delta+1)/2-1} e^{-y} dy \longrightarrow 0, \quad \text{as } M \longrightarrow \infty.
\end{aligned} \tag{2.9}$$

From (2.6), (2.7), and (2.9), we get $\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)} I_2 = 0$. Proposition 2.2 has been proved. □

Proposition 2.3. *For $\beta > 0$, $\delta > 2/\beta - 1$, one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-2/\beta}}{n} \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|B\| \geq y\} dy = \frac{2E\|B\|^{\beta(\delta+1)}}{(\delta+1)(\beta(\delta+1)-2))}. \tag{2.10}$$

Proof. The calculation here is analogous to (2.1), so it is omitted here. □

Proposition 2.4. *For $0 < \beta \leq 2$, $\delta > 2/\beta - 1$, one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-2/\beta}}{n} \left| \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\} dy - \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|B\| \geq y\} dy \right| = 0. \tag{2.11}$$

Proof. Like [4] and Proposition 2.2, we divide the summation into two parts,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(\log n)^{\delta-2\beta}}{n} \left| \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\} dy - \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|B\| \geq y\} dy \right| \\
&\quad = \sum_{n\leq A(\varepsilon)} \frac{(\log n)^{\delta-2\beta}}{n} \left| \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\} dy - \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|B\| \geq y\} dy \right| \\
&\qquad + \sum_{n>A(\varepsilon)} \frac{(\log n)^{\delta-2/\beta}}{n} \left| \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\} dy - \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|B\| \geq y\} dy \right| \\
&\quad =: J_1 + J_2.
\end{aligned} \tag{2.12}$$

First, consider J_1,

$$
\begin{aligned}
J_1 &\le \sum_{n\le A(\varepsilon)} \frac{(\log n)^{\delta-2/\beta}}{n} \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2y \left| P\{\|\alpha_n\| \ge y\} - P\{\|B\| \ge y\} \right| dy \\
&\le \sum_{n\le A(\varepsilon)} \frac{(\log n)^{\delta}}{n} \int_0^{\infty} 2(x+\varepsilon) \left| P\left\{\|\alpha_n\| \ge (x+\varepsilon)(\log n)^{1/\beta}\right\} - P\left\{\|B\| \ge (x+\varepsilon)(\log n)^{1/\beta}\right\} \right| dx \\
&\le \sum_{n\le A(\varepsilon)} \frac{(\log n)^{\delta}}{n} \Bigg(\int_0^{(\log n)^{-1/\beta}\Delta_n^{-1/4}} 2(x+\varepsilon) \Big| P\left\{\|\alpha_n\| \ge (x+\varepsilon)(\log n)^{1/\beta}\right\} \\
&\qquad\qquad - P\left\{\|B\| \ge (x+\varepsilon)(\log n)^{1/\beta}\right\} \Big| dx \\
&\qquad + \int_{(\log n)^{-1/\beta}\Delta_n^{-1/4}}^{\infty} 2(x+\varepsilon) P\left\{\|B\| \ge (x+\varepsilon)(\log n)^{1/\beta}\right\} dx \\
&\qquad + \int_{(\log n)^{(-1/\beta)}\Delta_n^{-1/4}}^{\infty} 2(x+\varepsilon) P\left\{\|\alpha_n\| \ge (x+\varepsilon)(\log n)^{1/\beta}\right\} dx \Bigg) \\
&=: \sum_{n\le A(\varepsilon)} \frac{(\log n)^{\delta}}{n} (J_{11} + J_{12} + J_{13}).
\end{aligned}
\tag{2.13}
$$

Since $n \le A(\varepsilon)$ means $\varepsilon < (M/\log n)^{1/\beta}$, it follows

$$
\begin{aligned}
(\log n)^{2/\beta} J_{11} &\le (\log n)^{2/\beta} \int_0^{(\log n)^{-1/\beta}\Delta_n^{-1/4}} 2(x+\varepsilon)\Delta_n dx \\
&\le (\log n)^{2/\beta} \Delta_n \left((\log n)^{-1/\beta}\Delta_n^{-1/4} + (\log n)^{-1/\beta} M^{1/\beta} \right)^2 \\
&\le \left(\Delta_n^{1/4} + M^{1/\beta}\Delta_n^{1/2} \right)^2 \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}
\tag{2.14}
$$

By Lemma 2.1 in Zhang and Yang [12], we have $P\{\|B\| \ge x\} \le 2e^{-2x^2}$. For J_{12}, it is easy to get

$$
\begin{aligned}
(\log n)^{2/\beta} J_{12} &\le (\log n)^{2/\beta} \int_{\varepsilon(\log n)^{1/\beta}+\Delta_n^{-1/4}}^{\infty} (\log n)^{-2/\beta} \cdot 2y P\{\|B\| \ge y\} dy \\
&\le C \int_{\Delta_n^{-1/4}}^{\infty} 2y \exp\left\{-2y^2\right\} dy \\
&\le C \exp\left\{-2\Delta_n^{-1/2}\right\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{2.15}
$$

In the same way, by the inequality $P\{\|\alpha_n\| \geq x\} \leq Ce^{-Cx^2}$, we can get

$$(\log n)^{2/\beta} J_{13} \leq C \exp\left\{-C\Delta_n^{-1/2}\right\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.16}$$

Put the three parts together, we get that $(\log n)^{2/\beta}(J_{11} + J_{12} + J_{13}) \to 0$ uniformly in ε as $n \to \infty$. Using Toeplitz's lemma again, we have $\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} J_1 = 0$.

In the sequel, we verify $\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} J_2 = 0$. It is easy to see that

$$\begin{aligned} J_2 &\leq \sum_{n>A(\varepsilon)} \frac{(\log n)^{\delta-2/\beta}}{n} \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2xP\{\|B\| \geq x\} dx \\ &\quad + \sum_{n>A(\varepsilon)} \frac{(\log n)^{\delta-2/\beta}}{n} \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2xP\{\|\alpha_n\| \geq x\} dx \\ &=: J_{21} + J_{22}. \end{aligned} \tag{2.17}$$

We estimate J_{22} first, by noticing $0 < \beta \leq 2$ and (2.8), it follows

$$\begin{aligned} J_{22} &\leq \sum_{n>A(\varepsilon)} \frac{(\log n)^{\delta-2/\beta}}{n} \int_n^{\infty} 2\varepsilon(\log y)^{1/\beta} P\left\{\|\alpha_n\| \geq \varepsilon(\log y)^{1/\beta}\right\} \frac{\varepsilon}{\beta y} (\log y)^{1/\beta-1} dy \\ &\leq C \int_{A(\varepsilon)}^{\infty} \frac{(\log x)^{\delta-2/\beta}}{x} \int_x^{\infty} \frac{\varepsilon^2 (\log y)^{2/\beta-1}}{y} \exp\left\{-C\varepsilon^2 (\log y)^{2/\beta}\right\} dy\, dx \\ &\leq C \int_{A(\varepsilon)}^{\infty} \frac{\varepsilon^2 (\log y)^{2/\beta-1}}{y} \exp\left\{-C\varepsilon^2 (\log y)^{2/\beta}\right\} (\log y)^{\delta-2/\beta+1} dy \\ &\leq C\varepsilon^2 \int_{A(\varepsilon)}^{\infty} \frac{(\log y)^{\delta}}{y} \exp\left\{-C\varepsilon^2 \log y\right\} dy \\ &\leq C\varepsilon^2 \frac{\log^{\delta}(A(\varepsilon))}{(A(\varepsilon))^{C\varepsilon^2}} \leq C\varepsilon^{2-\beta\delta}. \end{aligned} \tag{2.18}$$

Therefore, we get $\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} J_{22} = 0$. So far, we only need to prove $\lim_{\varepsilon \searrow 0} \varepsilon^{\beta(\delta+1)-2} J_{21} = 0$. Use the inequality $P\{\|B\| \geq x\} \leq 2e^{-2x^2}$ again and follow the proof of J_{22}, we can get this result. The proof of the proposition is completed now. □

Proof of Theorem 1.1. According to Fubini's theorem, it is easy to get

$$EXI\{X \geq a\} = aP\{X \geq a\} + \int_a^{\infty} P\{X \geq x\} dx, \tag{2.19}$$

for $a > 0$. Therefore, we have

$$\mathrm{E}\|\alpha_n\|^2 I\left\{\|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta}\right\} = \varepsilon^2(\log n)^{2/\beta} P\left\{\|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta}\right\} + \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\}dy. \tag{2.20}$$

From Proposition 2.1– 2.4, we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-2/\beta}}{n} \mathrm{E}\|\alpha_n\|^2 I\left\{\|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta}\right\} \\
&= \lim_{\varepsilon\searrow 0} \varepsilon^{\beta(\delta+1)} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left\{\|\alpha_n\| \geq \varepsilon(\log n)^{1/\beta}\right\} \\
&\quad + \lim_{\varepsilon\searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-2/\beta}}{n} \int_{\varepsilon(\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\}dy \\
&= \frac{\beta \mathrm{E}\|B\|^{\beta(\delta+1)}}{\beta(\delta+1)-2}.
\end{aligned} \tag{2.21}$$

a □

Proof of Theorem 1.2. From (2.19), we have

$$\begin{aligned}
&\varepsilon^{\beta(\delta+1)-2} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta-2/\beta}}{n\log n} \mathrm{E}\|\alpha_n\|^2 I\left\{\|\alpha_n\| \geq \varepsilon(\log\log n)^{1/\beta}\right\} \\
&= \varepsilon^{\beta(\delta+1)-2} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta-2/\beta}}{n\log n} \int_{\varepsilon(\log\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\}dy \\
&\quad + \varepsilon^{\beta(\delta+1)} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta}}{n\log n} P\left\{\|\alpha_n\| \geq \varepsilon(\log\log n)^{1/\beta}\right\}.
\end{aligned} \tag{2.22}$$

Via the similar argument in Proposition 2.1 and 2.2,

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\beta(\delta+1)} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta}}{n\log n} P\left\{\|\alpha_n\| \geq \varepsilon\|\log\log n\|^{1/\beta}\right\} = \frac{\mathrm{E}\|B\|^{\beta(\delta+1)}}{\delta+1}. \tag{2.23}$$

Also, by the analogous proof of Proposition 2.3 and 2.4,

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\beta(\delta+1)-2} \sum_{n=3}^{\infty} \frac{(\log\log n)^{\delta-2/\beta}}{n\log n} \int_{\varepsilon(\log\log n)^{1/\beta}}^{\infty} 2yP\{\|\alpha_n\| \geq y\}dy = \frac{2\mathrm{E}\|B\|^{\beta(\delta+1)}}{(\delta+1)(\beta(\delta+1)-2)}. \tag{2.24}$$

Combine (2.22), (2.23), and (2.24)together, we get the result of Theorem 1.2. □

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