Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 131240, 14 pages
doi:10.1155/2011/131240

Research Article

Size of Convergence Domains for Generalized Hausdorff Prime Matrices

T. Selmanogullari,[1] E. Savaş,[2] and B. E. Rhoades[3]

[1] *Department of Mathematics, Mimar Sinan Fine Arts University, Besiktas, 34349 Istanbul, Turkey*
[2] *Department of Mathematics, Istanbul Commerce University, Uskudar, 34672 Istanbul, Turkey*
[3] *Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA*

Correspondence should be addressed to T. Selmanogullari, tugcenmat@yahoo.com

Received 8 December 2010; Accepted 2 March 2011

Academic Editor: Q. Lan

We show that there exit E-J generalized Hausdorff matrices and unbounded sequences x such that each matrix has convergence domain $c \oplus x$.

1. Introduction

The convergence domain of an infinite matrix $A = (a_{nk})$ $(n, k = 0, 1, \ldots)$ will be denoted by (A) and is defined by $(A) := \{x = \{x_n\} \mid A_n(x) \in c\}$, where c denotes the space of convergence sequences, $A_n(x) := \sum_k a_{nk} x_k$. The necessary and sufficient conditions of Silverman and Toeplitz for a matrix to be conservative are $\lim_n a_{nk} = a_k$ exists for each k, $\lim_n \sum_{k=0}^{\infty} a_{nk} = t$ exists, and $\|A\| := \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$. A conservative matrix A is called multiplicative if each $a_k = 0$ and regular if, in addition, $t = 1$.

The E-J generalized Hausdorff matrices under consideration were defined independently by Endl ([1, 2]) and Jakimovski [3]. Each matrix $H_\mu^{(\alpha)}$ is a lower triangular matrix with nonzero entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \tag{1.1}$$

where α is real number, $\{\mu_n\}$ is a real or complex sequence and Δ is forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k)$. We will consider here only nonnegative α. For $\alpha = 0$, one obtains an ordinary Hausdorff matrix.

From [1] or [3] a E-J generalized Hausdorff matrix (for $\alpha > 0$) is regular if and only if there exists a function $\chi \in BV[0,1]$ with $\chi(1) - \chi(0+) = 1$ such that

$$\mu_n^{(\alpha)} = \int_0^1 t^{n+\alpha} d\chi(t), \tag{1.2}$$

in which case χ is called the moment generating function, or mass function, for $H_\mu^{(\alpha)}$ and $\mu_n^{(\alpha)}$ is called moment sequence.

For ordinary Hausdorff summability [4], the necessary and sufficient conditions, for regularity are that function $\chi \in BV[0,1]$, $\chi(1) - \chi(0) = 1$, $\chi(0+) = \chi(0)$, and (1.2) is satisfied with $\alpha = 0$.

As noted in [5], the set of all multiplicative Hausdorff matrices forms a commutative Banach algebra that is also an integral domain, making it possible to define the concepts of unit, prime, divisibility, associate, multiple, and factor. Hille and Tamarkin ([6, 7]), using some techniques from [8], showed that every Hausdorff matrix with moment function

$$\mu(z) = \frac{z-a}{z+b}, \qquad R(a) > 0, \qquad R(b) > 0 \tag{1.3}$$

is prime. In 1967, Rhoades [9] showed that the convergence domain of every known prime Hausdorff matrix is of the form $c \oplus x$ for a particular unbounded sequence x.

Given any unbounded sequence x, Zeller [10] constructed a regular matrix A with convergence domain $(A) = c \oplus x$. It has been shown by Parameswaran [11] that if x is any unbounded sequence such that $\{x_n - x_{n-1}\}$ is bounded, divergent, and Borel summable, then no Hausdorff matrix H exists with $(H) = c \oplus x$.

The main result of this paper is to show that there exist E-J generalized Hausdorff matrices $H_\mu^{(\alpha)}$ whose moment sequences are

$$\mu_n^{(\alpha)} = \frac{n-a}{n+b+\alpha}, \qquad R(a) > 0, \qquad R(b) > 0, \tag{1.4}$$

and unbounded sequences $x^{(\alpha)}$ such that each matrix has convergent domain $c \oplus x^{(\alpha)}$.

Define the sequences $x^{(\alpha)}$ by

$$x_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} \quad \text{for } R(a) > 0, \tag{1.5}$$

where it is understood that if a is positive integer, then $x_n^{(\alpha)} = 0$ for $n = 0, 1, \ldots, a-1$.

If $\lambda_n^{(\alpha)}$ is the moment sequence defined by $(n-a)/(n+1+\alpha)$, $R(a) > 0$, then it is clear that $(H_\mu^{(\alpha)}) = (H_\lambda^{(\alpha)})$. Hence, it will be sufficient to prove the theorem by using $b = 1$, in (1.4). To have the convenience of regularity, we will use the sequence

$$\mu_n^{(\alpha)} = \frac{n-a}{-(a+\alpha)(n+1+\alpha)}, \tag{1.6}$$

since the constant $-1/(a+\alpha)$ does not affect the size of the convergence domain of $H_\mu^{(\alpha)}$.

2. Auxiliary Results

In order to prove the main theorem of this paper, we will need the following results.

Lemma 2.1. *Let* $A, B \in \mathbb{C}$, $d, n \in \mathbb{N} \cup \{0\}$, $d \le n$. *Then, formally, for any* n,

$$\sum_{k=d}^{n} \frac{\Gamma(A+k)}{\Gamma(B+k)} = \frac{1}{A-B+1}\left[\frac{\Gamma(A+n+1)}{\Gamma(B+n)} - \frac{\Gamma(A+d)}{\Gamma(B-1+d)}\right]. \tag{2.1}$$

Proof. Lemma 2.1 appears as formula 12 on page 138 of [12]. □

Lemma 2.2. *For* m, n *integers* $n > m+1 > a$, $x_n^{(\alpha)}$ *as in* (1.5),

$$\sum_{k=m+1}^{n} \frac{1}{x_k^{(\alpha)}(k-a)} = \frac{1}{a+\alpha}\left(\frac{1}{x_m^{(\alpha)}} - \frac{1}{x_n^{(\alpha)}}\right). \tag{2.2}$$

Proof. Using Lemma 2.1,

$$\begin{aligned}\sum_{k=m+1}^{n} \frac{1}{x_k^{(\alpha)}(k-a)} &= \sum_{k=m+1}^{n} \frac{\Gamma(k-a)}{\Gamma(k+\alpha+1)} \\ &= \frac{1}{-a-(\alpha+1)+1}\left[\frac{\Gamma(n+1-a)}{\Gamma(n+1+\alpha)} - \frac{\Gamma(m+1-a)}{\Gamma(m+1+\alpha)}\right] \\ &= \frac{1}{a+\alpha}\left(\frac{1}{x_m^{(\alpha)}} - \frac{1}{x_n^{(\alpha)}}\right).\end{aligned} \tag{2.3}$$

□

Lemma 2.3. *For* $0 \le r \le a$

$$-\sum_{j=a+1}^{n-1} \left(h_{nj}^{(\alpha)}\right)^{-1} h_{jr}^{(\alpha)} = \frac{x_n^{(\alpha)}}{\Gamma(a+\alpha+1)} - \frac{(a+\alpha+1)}{n-a}. \tag{2.4}$$

Proof. $\mu_n^{(\alpha)}$ can be written as

$$\mu_n^{(\alpha)} = \frac{-1}{a+\alpha} + \frac{a+\alpha+1}{(a+\alpha)(n+\alpha+1)}, \tag{2.5}$$

so that, for $0 \le k < n$, $h_{nk}^{(\alpha)} = (a+\alpha+1)/(a+\alpha)(n+1+\alpha)$. From Lemma 2.1 and (3.11),

$$
\begin{aligned}
-\sum_{j=a+1}^{n-1}\left(h_{nj}^{(\alpha)}\right)^{-1}h_{jr}^{(\alpha)} &= -\sum_{j=a+1}^{n-1}\frac{-x_n^{(\alpha)}(a+\alpha)(a+\alpha+1)^2}{x_j^{(\alpha)}(j-a)(a+\alpha)(j+\alpha+1)} \\
&= (a+\alpha+1)^2 x_n^{(\alpha)}\sum_{j=a+1}^{n-1}\frac{\Gamma(j-a)}{\Gamma(j+2+\alpha)} \\
&= \frac{(a+\alpha+1)^2 x_n^{(\alpha)}}{(a+\alpha+1)}\left(\frac{1}{\Gamma(a+\alpha+2)}-\frac{\Gamma(n-a)}{\Gamma(n+1+\alpha)}\right) \\
&= \frac{x_n^{(\alpha)}}{\Gamma(a+\alpha+1)}-\frac{(a+\alpha+1)}{n-a}.
\end{aligned}
\tag{2.6}
$$

□

3. Main Result

Theorem 3.1. *If for fixed a and b the matrix $H_\mu^{(\alpha)}$ is defined by* (1.4) *and a sequence $x^{(\alpha)}$ by* (1.5), *then $(H_\mu^{(\alpha)}) = c \oplus x^{(\alpha)}$.*

Proof. We will first show that $c \oplus x^{(\alpha)} \subseteq (H_\mu^{(\alpha)})$.

We can write the matrix $H_\mu^{(\alpha)} = (-1/(a+\alpha))(I - H_\lambda^{(\alpha)})$, where the diagonal entries of $H_\lambda^{(\alpha)}$ are

$$
\lambda_n^{(\alpha)} = \frac{a+\alpha+1}{n+\alpha+1}. \tag{3.1}
$$

For each n and k,

$$
\begin{aligned}
\Delta^{n-k}\lambda_k^{(\alpha)} &= (a+\alpha+1)\int_0^1 t^{k+\alpha}(1-t)^{n-k}dt \\
&= \frac{(a+\alpha+1)\Gamma(k+\alpha+1)\Gamma(n-k+1)}{\Gamma(n+\alpha+2)}.
\end{aligned}
\tag{3.2}
$$

Therefore,

$$
\begin{aligned}
\left(H_\lambda^{(\alpha)}\right)_{n,k} &= \binom{n+\alpha}{n-k}\Delta^{n-k}\lambda_k^{(\alpha)} \\
&= \binom{n+\alpha}{n-k}\frac{(a+\alpha+1)\Gamma(k+\alpha+1)\Gamma(n-k+1)}{\Gamma(n+\alpha+2)} \\
&= \frac{a+\alpha+1}{n+\alpha+1}.
\end{aligned}
\tag{3.3}
$$

Define $y_n = -u_n/(a+\alpha)$, where

$$u_n = x_n^{(\alpha)} - \sum_{k=0}^{n} h_{n,k}^{(\alpha)} x_k^{(\alpha)}. \tag{3.4}$$

From Lemma 2.1,

$$\begin{aligned} u_n &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} - \frac{a+\alpha+1}{(n+\alpha+1)(\alpha+a+1)}\left[\frac{\Gamma(n+\alpha+2)}{\Gamma(n-a+1)} - \frac{\Gamma(\alpha+1)}{\Gamma(-a)}\right] \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)}[1-1] + \frac{\Gamma(\alpha+1)}{\Gamma(-a)(n+\alpha+1)} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(-a)(n+\alpha+1)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.5}$$

This argument is valid provided a is not a positive integer. If a is a positive integer, then $x_k^{(\alpha)} = 0$ for $0 \le k \le a-1$.

Then, $u_n = 0$ for $0 \le n \le a-1$, and for $n \ge a$, from Lemma 2.1, we get

$$\begin{aligned} u_n &= x_n^{(\alpha)} - \frac{(a+\alpha+1)}{(n+\alpha+1)} \sum_{k=a}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k-a+1)} \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} - \frac{(a+\alpha+1)}{(n+\alpha+1)}\left[\frac{\Gamma(a+\alpha+1)}{\Gamma(1)}\right] - \frac{1}{n+\alpha+1}\left[\frac{\Gamma(n+\alpha+2)}{\Gamma(n-a+1)} - \frac{\Gamma(a+\alpha+2)}{\Gamma(1)}\right] \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)} - \frac{\Gamma(a+\alpha+2)}{(n+1+\alpha)} - \frac{\Gamma(n+\alpha+2)}{(n+1+\alpha)\Gamma(n-a+1)} + \frac{\Gamma(a+\alpha+2)}{(n+1+\alpha)} \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)}\left[1 - \frac{(n+1+\alpha)}{(n+1+\alpha)}\right] \\ &= 0 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.6}$$

Since $H_\mu^{(\alpha)}$ is regular, $c \subseteq (H_\mu^{(\alpha)})$. Thus, $c \oplus x^{(\alpha)} \subseteq (H_\mu^{(\alpha)})$.

To prove the converse, we will use Zeller's technique to construct a regular matrix A with $(A) = c \oplus x^{(\alpha)}$ and then show that $(H_\mu^{(\alpha)}) \subseteq (A)$.

Set $P_0 = 0$ and define a sequence $\{P_n\}$ inductively by selecting P_{n+1} to be smallest integer $P > P_n$ such that $|x_P^{(\alpha)}| \ge 2\,|x_{P_n}^{(\alpha)}|$. (Such a construction is clearly possible, since $x^{(\alpha)}$ is not bounded.) Let $q_n^{(\alpha)} = 1 - x_{P_{n-1}}^{(\alpha)}/x_{P_n}^{(\alpha)}$, $n = 1, 2, \ldots$. Define a matrix B by

$$\begin{aligned} b_{00} &= 1, \\ b_{n,n-1} &= \frac{1}{q_n^{(\alpha)}}, \quad n \ge 1, \\ b_{n,n} &= -\frac{x_{P_{n-1}}^{(\alpha)}}{q_n^{(\alpha)} x_{P_n}^{(\alpha)}}, \quad n \ge 1, \\ b_{n,k} &= 0 \quad \text{otherwise}. \end{aligned} \tag{3.7}$$

Now, define the matrix A as follows:

$$a_{P_n,P_k} = b_{nk},$$
$$a_{P_n,k} = 0, \qquad k \neq P_i \text{ for any integer } i, \tag{3.8}$$
$$a_{nn} = 1, \qquad n \neq P_i \text{ for any integer } i.$$

If $n \neq P_i$ for any integer i, then there exists an integer r such that $P_r < n < P_{r+1}$. For this r, define

$$a_{n,P_{r-1}} = \frac{x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}},$$
$$a_{n,P_r} = \frac{-x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}}. \tag{3.9}$$

Set $a_{nk} = 0$ otherwise. From [10], A is regular and $(A) = c \oplus x^{(\alpha)}$. There are three cases to consider, based on whether a is real number and not a positive integer, a is positive integer, or a is complex.

Proof of Case I. If a is real and not a positive integer, the E-J generalized Hausdorff matrix $H_\mu^{(\alpha)}$ generated by (1.6) has a unique two sided inverse $(H_\mu^{(\alpha)})^{-1} = ((h_{nk}^{(\alpha)})^{-1})$ with generating sequence

$$\frac{1}{\mu_n^{(\alpha)}} = \frac{-(a+\alpha)(n+1+\alpha)}{(n-a)} = -(a+\alpha) - \frac{(a+\alpha)(a+\alpha+1)}{n-a}. \tag{3.10}$$

For $k < n$,

$$\left(h_{nk}^{(\alpha)}\right)^{-1} = \binom{n+\alpha}{n-k} \Delta^{n-k} \frac{1}{\mu_k^{(\alpha)}}$$
$$= \frac{-(a+\alpha)(a+\alpha+1)\Gamma(n+\alpha+1)\Gamma(k-a)}{\Gamma(k+\alpha+1)\Gamma(n-a+1)} \tag{3.11}$$
$$= \frac{-x_n^{(\alpha)}(a+\alpha)(a+\alpha+1)}{x_k^{(\alpha)}(k-a)},$$

$$\left(h_{nn}^{(\alpha)}\right)^{-1} = \frac{-(a+\alpha)(n+1+\alpha)}{n-a}. \tag{3.12}$$

To show that $(H_\mu^{(\alpha)}) \subseteq (A)$, it will be sufficent to show that $D = A(H_\mu^{(\alpha)})^{-1}$ is a regular matrix. Each column of $(H_\mu^{(\alpha)})^{-1}$ is essentially a scalar multiple of (1.5), so it is obvious that each column of $(H_{\mu}^{(\alpha)})^{-1}$ belongs to the convergence domain of A. However, it will be necessary to calculate the terms of D explicitly, since we must show that $t = 1$ and that D has finite norm.

If $k \neq P_i$ for any integer i, and r denotes the integer such that $P_{r-1} < k < P_r$, then from the definition of A,

$$
\begin{aligned}
d_{P_n,k} &= \sum_{j=r}^{n} a_{P_n,P_j} \left(h_{P_j,k}^{(\alpha)}\right)^{-1} \\
&= b_{n,n-1}\left(h_{P_{n-1},k}^{(\alpha)}\right)^{-1} + b_{nn}\left(h_{P_n,k}^{(\alpha)}\right)^{-1} \\
&= 0.
\end{aligned}
\tag{3.13}
$$

If $k = P_r$ for $r < n-1$, then

$$
d_{P_n,P_r} = \sum_{j=r}^{n} a_{P_n,P_j}\left(h_{P_j,P_r}^{(\alpha)}\right)^{-1} = 0. \tag{3.14}
$$

For $k = P_{n-1}$,

$$
\begin{aligned}
d_{P_n,P_{n-1}} &= a_{P_n,P_{n-1}}\left(h_{P_{n-1},P_{n-1}}^{(\alpha)}\right)^{-1} + a_{P_n,P_n}\left(h_{P_n,P_{n-1}}^{(\alpha)}\right)^{-1} \\
&= \frac{1}{q_n^{(\alpha)}}\left(\frac{-(a+\alpha)(P_{n-1}+\alpha+1)}{P_{n-1}-a}\right) + \left(\frac{-x_{P_{n-1}}^{(\alpha)}}{q_n^{(\alpha)} x_{P_n}^{(\alpha)}}\right)\left(\frac{-(a+\alpha)(a+\alpha+1)x_{P_n}^{(\alpha)}}{x_{P_{n-1}}^{(\alpha)}(P_{n-1}-a)}\right) \\
&= \frac{-(a+\alpha)}{q_n^{(\alpha)}}.
\end{aligned}
\tag{3.15}
$$

For $P_{n-1} < k < P_n$,

$$
d_{P_n,k} = a_{P_n,P_n}\left(h_{P_n,k}^{(\alpha)}\right)^{-1} = \frac{(a+\alpha)(1+a+\alpha)x_{P_{n-1}}^{(\alpha)}}{(k-a)q_n^{(\alpha)}x_k^{(\alpha)}}, \tag{3.16}
$$

$$
d_{P_n,P_n} = a_{P_n,P_n}\left(h_{P_n,P_n}^{(\alpha)}\right)^{-1} = \frac{(a+\alpha)(1+\alpha+P_n)x_{P_{n-1}}^{(\alpha)}}{(P_n-a)q_n^{(\alpha)}x_{P_n}^{(\alpha)}}. \tag{3.17}
$$

For $n \neq P_i$ for any i, if we now let r denote the integer such that $P_r < n < P_{r+1}$, then for $0 < k < P_{r-1,}$

$$
\begin{aligned}
d_{nk} &= \sum_{j=k}^{n} a_{n,j}\left(h_{j,k}^{(\alpha)}\right)^{-1} \\
&= a_{n,P_{r-1}}\left(h_{P_{r-1},k}^{(\alpha)}\right)^{-1} + a_{n,P_r}\left(h_{P_r,k}^{(\alpha)}\right)^{-1} + a_{n,n}\left(h_{n,k}^{(\alpha)}\right)^{-1} = 0.
\end{aligned}
\tag{3.18}
$$

For $k = P_{r-1}$,

$$\begin{aligned}
d_{n,P_{r-1}} &= a_{n,P_{r-1}}\left(h^{(\alpha)}_{P_{r-1},P_{r-1}}\right)^{-1} + a_{n,P_r}\left(h^{(\alpha)}_{P_r,P_{r-1}}\right)^{-1} + a_{n,n}\left(h^{(\alpha)}_{n,P_{r-1}}\right)^{-1} \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{P_{r-1}-a}\left(\frac{P_{r-1}+\alpha+1}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}} - \frac{(a+\alpha+1)}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}}\frac{x^{(\alpha)}_{P_r}}{x^{(\alpha)}_{P_{r-1}}} + \frac{a+\alpha+1}{x^{(\alpha)}_{P_{r-1}}}\right) \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{P_{r-1}-a}\left(\frac{P_{r-1}+\alpha+1}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}} - \frac{(a+\alpha+1)}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}}\right) \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}}.
\end{aligned} \tag{3.19}$$

For $P_{r-1} < k < P_r$,

$$\begin{aligned}
d_{nk} &= a_{n,P_r}\left(h^{(\alpha)}_{P_r,k}\right)^{-1} + a_{n,n}\left(h^{(\alpha)}_{n,k}\right)^{-1} \\
&= \frac{(a+\alpha)(a+\alpha+1)x_n^{(\alpha)}x^{(\alpha)}_{P_{r-1}}}{x_k^{(\alpha)}\left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)(k-a)}.
\end{aligned} \tag{3.20}$$

For $k = P_r$,

$$\begin{aligned}
d_{n,P_r} &= a_{n,P_r}\left(h^{(\alpha)}_{P_r,P_r}\right)^{-1} + a_{n,n}\left(h^{(\alpha)}_{n,P_r}\right)^{-1} \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{(P_r-a)}\left[\frac{-(P_r+\alpha+1)}{x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}} + \frac{a+\alpha+1}{x^{(\alpha)}_{P_r}}\right] \\
&= \frac{-(a+\alpha)x_n^{(\alpha)}}{(P_r-a)x^{(\alpha)}_{P_r}\left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)}\left[-(P_r+\alpha+1)x^{(\alpha)}_{P_r} + \left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)(a+\alpha+1)\right].
\end{aligned} \tag{3.21}$$

The quantity in brackets is equal to $-(P_r-a)x^{(\alpha)}_{P_r} - x^{(\alpha)}_{P_{r-1}}(a+\alpha+1)$, giving

$$d_{n,P_r} = \frac{(a+\alpha)x_n^{(\alpha)}}{x^{(\alpha)}_{P_r-}x^{(\alpha)}_{P_{r-1}}} + \frac{x_n^{(\alpha)}}{x^{(\alpha)}_{P_r}}\frac{(a+\alpha)(a+\alpha+1)x^{(\alpha)}_{P_{r-1}}}{(P_r-a)\left(x^{(\alpha)}_{P_r}-x^{(\alpha)}_{P_{r-1}}\right)}. \tag{3.22}$$

For $P_r < k < n$,

$$d_{n,k} = a_{n,n}\left(h^{(\alpha)}_{n,k}\right)^{-1} = \frac{-x_n^{(\alpha)}}{x_k^{(\alpha)}}\frac{(a+\alpha)(a+\alpha+1)}{(k-a)}, \tag{3.23}$$

and finally,

$$d_{n,n} = \frac{-(a+\alpha)(n+1+\alpha)}{n-a}. \tag{3.24}$$

By using (3.13)–(3.17),

$$\begin{aligned}\sum_{k=0}^{P_n} d_{P_n,k} &= d_{P_n,P_{n-1}} + \sum_{k=P_{n-1}+1}^{P_n-1} d_{P_n,k} + d_{P_n,P_n} \\ &= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1 + x_{P_{n-1}}^{(\alpha)}(a+\alpha+1)\sum_{k=P_{n-1}+1}^{P_n-1}\frac{1}{x_k^{(\alpha)}(k-a)} + \frac{(P_n+\alpha+1)x_{P_{n-1}}^{(\alpha)}}{(P_n-a)x_{P_n}^{(\alpha)}}\right].\end{aligned} \tag{3.25}$$

By using Lemma 2.2, and noting that

$$\begin{aligned}&\frac{P_n+\alpha+1}{P_n-a} = 1 + \frac{1+a+\alpha}{P_n-a}, \\ &\sum_{k=0}^{P_n} d_{P_n,k} = \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1 + x_{P_{n-1}}^{(\alpha)}\frac{(a+\alpha+1)}{a+\alpha}\left(\frac{1}{x_{P_{n-1}}^{(\alpha)}} - \frac{1}{x_{P_n-1}^{(\alpha)}}\right) + \frac{1+a+\alpha}{P_n-a}\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}} + \frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right] \\ &\qquad = \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1 + \frac{a+\alpha+1}{a+\alpha} - \frac{a+\alpha+1}{a+\alpha}\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n-1}^{(\alpha)}} + \frac{1+a+\alpha}{P_n-a}\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}} + \frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right].\end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned}-\frac{a+\alpha+1}{a+\alpha}\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n-1}^{(\alpha)}} + \frac{1+a+\alpha}{P_n-a}\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}} &= (a+\alpha+1)\left[-\frac{x_{P_{n-1}}^{(\alpha)}}{(a+\alpha)x_{P_n-1}^{(\alpha)}} + \frac{x_{P_{n-1}}^{(\alpha)}}{(P_n-a)x_{P_n}^{(\alpha)}}\right] \\ &= (a+\alpha+1)\left[-\frac{x_{P_{n-1}}^{(\alpha)}\Gamma(P_n-a)}{(a+\alpha)\Gamma(P_n+\alpha)} + \frac{x_{P_{n-1}}^{(\alpha)}\Gamma(P_n-a+1)}{(P_n-a)\Gamma(P_n+\alpha+1)}\right] \\ &= (a+\alpha+1)\frac{\Gamma(P_n-a)}{\Gamma(P_n+\alpha)}\left[\frac{-x_{P_{n-1}}^{(\alpha)}}{(a+\alpha)} + \frac{x_{P_{n-1}}^{(\alpha)}}{(P_n+\alpha)}\right] \\ &= -\frac{(a+\alpha+1)\Gamma(P_n-a+1)\ x_{P_{n-1}}^{(\alpha)}}{\Gamma(P_n+\alpha+1)(a+\alpha)}.\end{aligned} \tag{3.27}$$

Finally,

$$\begin{aligned}
\sum_{k=0}^{P_n} d_{P_n,k} &= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1+\frac{(a+\alpha+1)}{a+\alpha}+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}-\frac{(a+\alpha+1)\Gamma(P_n-a+1)x_{P_{n-1}}^{(\alpha)}}{(a+\alpha)\Gamma(P_n+\alpha+1)}\right] \\
&= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1+\frac{(a+\alpha+1)}{a+\alpha}\left(1-\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right)+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right] \\
&= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[-1+(1+a+\alpha)\left(\frac{q_n^{(\alpha)}}{a+\alpha}\right)+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right] \\
&= -\frac{(a+\alpha)}{q_n^{(\alpha)}}\left(1-\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right)+(1+a+\alpha)=1.
\end{aligned} \tag{3.28}$$

For $n \neq P_i$ for any i, r the integer such that $P_r < n < P_{r+1}$, and using (3.18)–(3.24), we have

$$\begin{aligned}
\sum_{k=0}^{n} d_{nk} = \frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}&\left[-1+x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)\sum_{k=P_{r-1}+1}^{P_r-1}\frac{1}{x_k^{(\alpha)}(k-a)}+1+\frac{x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}(P_r-a)}\right] \\
&+\sum_{k=P_r+1}^{n-1}\frac{-x_n^{(\alpha)}(a+\alpha)(a+\alpha+1)}{x_k^{(\alpha)}(k-a)}-\frac{(a+\alpha)(n+1+\alpha)}{(n-a)}.
\end{aligned} \tag{3.29}$$

Writing $(n+1+\alpha)/(n-a) = 1+(x_n^{(\alpha)}(1+a+\alpha)/x_n^{(\alpha)}(n-a))$ and using Lemma 2.2, the quantity in brackets, which we call I_1, takes the form

$$\begin{aligned}
I_1 &= x_{P_{r-1}}^{(\alpha)}\frac{(a+\alpha+1)}{(a+\alpha)}\left(\frac{1}{x_{P_{r-1}}^{(\alpha)}}-\frac{1}{x_{P_r-1}^{(\alpha)}}\right)+\frac{x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}(P_r-a)} \\
&= x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)\left(\frac{1}{(a+\alpha)x_{P_{r-1}}^{(\alpha)}}-\frac{1}{(a+\alpha)x_{P_r-1}^{(\alpha)}}+\frac{1}{x_{P_r}^{(\alpha)}(P_r-a)}\right).
\end{aligned} \tag{3.30}$$

The sum

$$\begin{aligned}
-\frac{1}{(a+\alpha)x_{P_r-1}^{(\alpha)}}+\frac{1}{x_{P_r}^{(\alpha)}(P_r-a)} &= -\frac{\Gamma(P_r-a)}{(a+\alpha)\Gamma(P_r+\alpha)}+\frac{\Gamma(P_r-a+1)}{(P_r-a)\Gamma(P_r+\alpha+1)} \\
&= -\frac{1}{(a+\alpha)x_{P_r}^{(\alpha)}}.
\end{aligned} \tag{3.31}$$

Thus,

$$\begin{aligned}\frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}I_1 &= \frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}x_{P_{r-1}}^{(\alpha)}(a+\alpha+1)\left(\frac{1}{(a+\alpha)x_{P_{r-1}}^{(\alpha)}}-\frac{1}{(a+\alpha)x_{P_r}^{(\alpha)}}\right)\\ &= \frac{x_n^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}}.\end{aligned} \tag{3.32}$$

Finally,

$$\begin{aligned}\sum_{k=0}^{n} d_{nk} &= \frac{x_n^{(\alpha)}(a+\alpha+1)}{x_{P_r}^{(\alpha)}}-(a+\alpha)(1+a+\alpha)x_n^{(\alpha)}\left[\sum_{k=P_r+1}^{n-1}\frac{1}{x_k^{(\alpha)}(k-a)}+\frac{1}{x_n^{(\alpha)}(n-a)}\right]-(a+\alpha)\\ &= \frac{(a+\alpha+1)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}}-(a+\alpha)(1+a+\alpha)x_n^{(\alpha)}\left[\frac{1}{a+\alpha}\left(\frac{1}{x_{P_r}^{(\alpha)}}-\frac{1}{x_n^{(\alpha)}}\right)\right]-(a+\alpha)\\ &= \frac{(a+\alpha+1)}{x_{P_r}^{(\alpha)}}\left[x_n^{(\alpha)}-\left(x_n^{(\alpha)}-x_{P_r}^{(\alpha)}\right)\right]-(a+\alpha)\\ &= 1.\end{aligned} \tag{3.33}$$

Clearly, D has null columns. It remains to show that D has finite norm.

For all integers, $n \geq [a]+1$, $x_n^{(\alpha)}$ is positive and $(1/2) \leq q_n^{(\alpha)} \leq 1$. From (3.25),

$$\begin{aligned}\sum_{k=0}^{P_n}\left|d_{P_n,k}\right| &= \left|d_{P_n,P_{n-1}}\right| + \sum_{k=P_{n-1}+1}^{P_n-1}\left|d_{P_n,k}\right| + \left|d_{P_n,P_n}\right|\\ &= \frac{(a+\alpha)}{q_n^{(\alpha)}}\left[1+\frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}}\right]+1+a+\alpha.\end{aligned} \tag{3.34}$$

Since $|x_{P_n}^{(\alpha)}| \geq 2|x_{P_{n-1}}^{(\alpha)}|$, then, $x_{P_{n-1}}^{(\alpha)}/x_{P_n}^{(\alpha)} \leq 1/2$, and the above sum is bounded by $4\alpha+4a+1$. From (3.29),

$$\sum_{k=0}^{n}\left|d_{nk}\right| = \frac{2(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}-x_{P_{r-1}}^{(\alpha)}}+(a+\alpha)+\frac{2(a+\alpha+1)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)}}-(a+\alpha+1). \tag{3.35}$$

From choice of n, $|x_n^{(\alpha)}| < 2|x_{P_r}^{(\alpha)}|$. Again, using the fact that $|x_{P_r}^{(\alpha)}| \geq 2|x_{P_{r-1}}^{(\alpha)}|$, we have

$$\sum_{k=0}^{n}\left|d_{nk}\right| < \frac{2(a+\alpha)\left(2\left|x_{P_r}^{(\alpha)}\right|\right)}{\left|x_{P_r}^{(\alpha)}\right|-\left|x_{P_r}^{(\alpha)}\right|/2}+4(a+\alpha+1)+2(a+\alpha)+1 = 14(a+\alpha)+5. \tag{3.36}$$

Since there are only a finite number of rows of D with $n < [a] + 1$, D has finite norm and is regular. □

Proof of Case II. If a is a positive integer, $\mu_a^{(\alpha)} = 0$, and $H_\mu^{(\alpha)}$ fails to have a two-sided inverse. However, if we define a new matrix $F = (f_{nk})$ with $f_{a,a} = 1$ and which agrees with $H_\mu^{(\alpha)}$ elsewhere, then F does possess a unique two-sided inverse. Morever, $(F) = (H_\mu^{(\alpha)})$ and, for $k > a$, $f_{nk}^{-1} = (h_{nk}^{(\alpha)})^{-1}$, where the $(h_{nk}^{(\alpha)})^{-1}$ are computed using (3.11) and (3.12).

From (1.5), $x_n^{(\alpha)} = 0$ for $0 \le n < a$. Consequently, $P_0 = 0$, $P_1 = a$ and $P_2 = a + 1$. Now, let $E := AF^{-1} = (e_{nk})$. To prove that E is regular, we are concerned with the behavior of the e_{nk} for all n sufficiently large. We will restrict our attention to $n > a + 1$. Since $f_{nk}^{-1} = (h_{nk}^{(\alpha)})^{-1}$ for all $k > a$, it is clear that $e_{nk} = d_{nk}$ for $k > a$. If we can show that $e_{nk} = 0$ for all $0 \le k \le a$ and $n > a + 1$, then it will follow that E is regular, since D is.

For $n > a + 1$,

$$\begin{gathered}\sum_{j=a}^{n} f_{nj}^{-1} f_{ja} = 0, \\ f_{na}^{-1} f_{aa} = -\sum_{j=a+1}^{n} f_{nj}^{-1} f_{ja} = -\sum_{j=a+1}^{n-1} \left(h_{nj}^{(\alpha)}\right)^{-1} h_{ja}^{(\alpha)} - \left(h_{nn}^{(\alpha)}\right)^{-1} h_{na}^{(\alpha)}.\end{gathered} \tag{3.37}$$

Since $f_{a,a} = 1$ and $(h_{nn}^{(\alpha)})^{-1} h_{na}^{(\alpha)} = -(a+\alpha+1)/(n-a)$, $f_{na}^{-1} = x_n^{(\alpha)}/\Gamma(a+\alpha+1)$. By induction it is showed that $f_{n,a-r}^{-1} = k_r^{(\alpha)}(a) x_n^{(\alpha)}$, where $k_r^{(\alpha)}(a)$ is a function of a.

For $n > a + 1$, $P_{n-1} > P_a \ge P_1 = a \ge r$

$$\begin{aligned} e_{P_n,r} &= \sum_{j=r}^{P_n} a_{P_n,j} f_{jr}^{-1} = b_{n,n-1} f_{P_{n-1},r}^{-1} + b_{n,n} f_{P_n,r}^{-1} \\ &= \frac{1}{q_n^{(\alpha)}} \left(k_{a-r}^{(\alpha)}(a) x_{P_{n-1}}^{(\alpha)} - \frac{x_{P_{n-1}}^{(\alpha)}}{x_{P_n}^{(\alpha)}} \left(x_{P_n}^{(\alpha)} k_{a-r}^{(\alpha)}(a) \right) \right) = 0. \end{aligned} \tag{3.38}$$

For $n > a + 1$, $n \ne P_i$ for any integer i, $0 \le r \le a$, and s the integer such that $P_s < n < P_{s+1}$,

$$\begin{aligned} e_{n,r} &= \sum_{j=r}^{n} a_{n,j} f_{jr}^{-1} = a_{n,P_{s-1}} f_{P_{s-1},r}^{-1} + a_{n,P_s} f_{P_s,r}^{-1} + a_{n,n} f_{n,r}^{-1} \\ &= \frac{x_n^{(\alpha)}}{x_{P_s}^{(\alpha)} - x_{P_{s-1}}^{(\alpha)}} \left(k_{a-r}^{(\alpha)}(a) x_{P_{s-1}}^{(\alpha)} - k_{a-r}^{(\alpha)}(a) x_{P_s}^{(\alpha)} \right) + k_{a-r}^{(\alpha)}(a) x_n^{(\alpha)} = 0. \end{aligned} \tag{3.39}$$

□

Proof of Case III. If a is complex, then none of the $\mu_n^{(\alpha)}$ vanish, and we may use the matrix D of Case I. It will be sufficient to show that D has finite norm. From (3.25),

$$\sum_{k=0}^{P_n} |d_{P_n,k}| = \left| \frac{-(a+\alpha)}{q_n^{(\alpha)}} \right| + \left| \frac{(a+\alpha)(P_n+\alpha+1) x_{P_{n-1}}^{(\alpha)}}{q_n^{(\alpha)} (P_n - a) x_{P_n}^{(\alpha)}} \right| + \sum_{k=P_{n-1}+2}^{P_n-1} |d_{P_n,k}| + |d_{P_n,P_{n-1}+1}|. \tag{3.40}$$

Again, $|x_{P_n}^{(\alpha)}| \geq 2|x_{P_{n-1}}^{(\alpha)}|$. It can be shown that $1/2 \leq |q_n^{(\alpha)}| \leq 3/2$. Since

$$|d_{P_n,P_{n-1}+1}| = \left|\frac{(a+\alpha)(a+\alpha+1)}{q_n^{(\alpha)}(P_{n-1}+1+\alpha)}\right|, \tag{3.41}$$

the first two and last terms of (3.40), are clearly bounded in n.

For $P_{n-1}+1 < k < P_n$, using (3.16),

$$\begin{aligned}
|d_{P_n,k}| &= \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)\Gamma(k-a)}{\Gamma(P_{n-1}+1-a)\Gamma(k+1+\alpha)q_n^{(\alpha)}}\right| \\
&= \left|\frac{(a+\alpha)(a+\alpha+1)(k-a-1)\cdots(P_{n-1}-a+1)}{\Gamma(k+1+\alpha)q_n^{(\alpha)}}\right|\Gamma(P_{n-1}+1+\alpha) \\
&\leq \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}}\right| \\
&\quad\cdot\left|\frac{|P_{n-1}+1-a|(|P_{n-1}+1-a|+1)\cdots(|P_{n-1}+1-a|+k-P_{n-1}-2)}{\Gamma(k+1+\alpha)}\right| \\
&= \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}\Gamma(|P_{n-1}+1-a|)}\right|\frac{\Gamma(k+w)}{\Gamma(k+1+\alpha)},
\end{aligned} \tag{3.42}$$

where $w = |P_{n-1}+1-a| - P_{n-1} - 1$. $w < 0$ for all n sufficiently large. From Lemma 2.1, we can write

$$\begin{aligned}
\sum_{k=P_{n-1}+2}^{P_n-1}|d_{P_n,k}| &\leq \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}\Gamma(|P_{n-1}+1-a|)}\right|\left[\frac{\Gamma(x+w)}{(w-\alpha)\Gamma(x+\alpha)}\right]_{P_{n-1}+2}^{P_n} \\
&< \left|\frac{(a+\alpha)(a+\alpha+1)\Gamma(P_{n-1}+1+\alpha)}{q_n^{(\alpha)}\Gamma(|P_{n-1}+1-a|)}\right|\frac{\Gamma(P_{n-1}+2+w)}{(\alpha-w)\Gamma(P_{n-1}+2+\alpha)},
\end{aligned} \tag{3.43}$$

and the sum is uniformly bounded in n, since $-w$ is bounded away from zero.

If $n \neq P_i$, for any i, then from (3.29),

$$\begin{aligned}
\sum_{k=0}^{n}|d_{nk}| &= \left|\frac{-(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}}\right| + \sum_{k=P_{r-1}+1}^{P_r-1}\left|\frac{(a+\alpha)(a+\alpha+1)x_{P_{r-1}}^{(\alpha)}x_n^{(\alpha)}}{\left(x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}\right)x_k^{(\alpha)}(k-a)}\right| \\
&\quad + \left|\frac{(a+\alpha)x_n^{(\alpha)}}{x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}}\right| + \left|\frac{(a+\alpha)(a+\alpha+1)x_{P_{r-1}}^{(\alpha)}x_n^{(\alpha)}}{\left(x_{P_r}^{(\alpha)} - x_{P_{r-1}}^{(\alpha)}\right)x_{P_r}^{(\alpha)}(P_r-a)}\right| \\
&\quad + \sum_{k=P_r+1}^{n-1}\left|\frac{(a+\alpha)(a+\alpha+1)x_n^{(\alpha)}}{x_k^{(\alpha)}(k-a)}\right| + \left|\frac{-(a+\alpha)(n+1+\alpha)}{n-a}\right|.
\end{aligned} \tag{3.44}$$

Terms 1, 3, 4, and 6 of (3.44) are clearly bounded in n. Recalling that $q_r^{(\alpha)} = 1 - x_{P_{r-1}}^{(\alpha)} / x_{P_r}^{(\alpha)}$, the first summation may be written in the form

$$\left| \frac{(a+\alpha)(a+\alpha+1)x_n^{(\alpha)}}{q_r^{(\alpha)} x_{P_r}^{(\alpha)}} \right| \sum_{k=P_{r-1}+1}^{P_r - 1} \left| \frac{x_{P_{r-1}}^{(\alpha)}}{x_k^{(\alpha)}(k-a)} \right|. \tag{3.45}$$

The summation is identical with the one in (3.40), and the above expression is uniformly bounded, since $|x_n^{(\alpha)}| < 2|x_{P_r}^{(\alpha)}|$. Using an argument similar to the one used in establishing (3.40), the second summation of (3.44) can be shown to be uniformly bounded. □

□

Acknowledgment

The first author acknowledges support from the Scientific and Technical Research Council of Turkey in the preparation of this paper, the authors wish to thank the referee for his careful reading of the manuscript and for his helpful suggestions.

References

[1] K. Endl, "Abstracts of short communications and scientific program," *International Congress of Mathematicians*, vol. 73, p. 46, 1958.
[2] K. Endl, "Untersuchungen über Momentenprobleme bei Verfahren vom Hausdorffschen Typus," *Mathematische Annalen*, vol. 139, pp. 403–432, 1960.
[3] A. Jakimovski, "The product of summability methods; new classes of transformations and their properties," Tech. Note, Contract no. AF61 (052)-187, 1959.
[4] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, UK, 1949.
[5] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, vol. 31 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 1957.
[6] E. Hille and J. D. Tamarkin, "On moment functions," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 19, pp. 902–908, 1933.
[7] E. Hille and J. D. Tamarkin, "Questions of relative inclusion in the domain of Hausdorff means," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 19, pp. 573–577, 1933.
[8] L. L. Silvermann and J. D. Tamarkin, "On the generalization of Abel's theorem for certain definitions of summability," *Mathematische Zeitschrift*, vol. 29, pp. 161–170, 1928.
[9] B. E. Rhoades, "Size of convergence domains for known Hausdorff prime matrices," *Journal of Mathematical Analysis and Applications*, vol. 19, pp. 457–468, 1967.
[10] K. Zeller, "Merkwürdigkeiten bei Matrixverfahren; Einfolgenverfahren," *Archiv für Mathematische*, vol. 4, pp. 1–5, 1953.
[11] M. R. Parameswaran, "Remark on the structure of the summability field of a Hausdorff matrix," *Proceedings of the National Institute of Sciences of India. Part A*, vol. 27, pp. 175–177, 1961.
[12] H. T. Davis, *The Summation of Series*, The Principia Press of Trinity University, San Antonio, Texas, USA, 1962.