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Research Article

Hypersingular Marcinkiewicz Integrals along Surface with Variable Kernels on Sobolev Space and Hardy-Sobolev Space

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Let $\alpha \geq 0$, the authors introduce in this paper a class of the hypersingular Marcinkiewicz integrals along surface with variable kernels defined by $\mu_{\Omega,\alpha}^{\Phi}(f)(x) = (\int_0^{\infty} |\int_{|y|\leq t}(\Omega(x,y)/|y|^{n-1})f(x-\Phi(|y|)y')dy|^2(dt/t^{3+2\alpha}))^{1/2}$, where $\Omega(x,z) \in L^{\infty}(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ with $q > \max\{1, 2(n-1)/(n+2\alpha)\}$. The authors prove that the operator $\mu_{\Omega,\alpha}^{\Phi}$ is bounded from Sobolev space $L_{\alpha}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ space for $1 < p \leq 2$, and from Hardy-Sobolev space $H_{\alpha}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ space for $n/(n+\alpha) < p \leq 1$. As corollaries of the result, they also prove the $\dot{L}_{\alpha}^2(R^n) - L^2(R^n)$ boundedness of the Littlewood-Paley type operators $\mu_{\Omega,\alpha,S}^{\Phi}$ and $\mu_{\Omega,\alpha,\lambda}^{*,\Phi}$ which relate to the Lusin area integral and the Littlewood-Paley g_{λ}^{*} function.

1. Introduction

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For $x \in \mathbb{R}^n \setminus \{0\}$, let $x' = x/|x|$.

Before stating our theorems, we first introduce some definitions about the variable kernel $\Omega(x,z)$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^{\infty}(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, $q \geq 1$, if $\Omega(x,z)$ satisfies the following two conditions:

(1) $\Omega(x,\lambda z) = \Omega(x,z)$, for any $x, z \in \mathbb{R}^n$ and any $\lambda > 0$;

(2) $\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})} = \sup_{r\geq 0,\ y\in\mathbb{R}^n}(\int_{\mathbb{S}^{n-1}} |\Omega(rz'+y,z')|^q d\sigma(z'))^{1/q} < \infty$.

In 1955, Calderón and Zygmund [1] investigated the L^p boundedness of the singular integrals T_{Ω} with variable kernel. They found that these operators connect closely with the problem about the second-order linear elliptic equations with variable coefficients. In 2002, Tang and Yang [2] gave L^p boundedness of the singular integrals with variable kernels associated to surfaces of the form $\{x = \Phi(|y|)y'\}$, where $y' = y/|y|$ for any $y \in \mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$. That is, they considered the variable Calderón-Zygmund singular integral operator T_Ω^Φ defined by

$$T_\Omega^\Phi(f)(x) = p \cdot v \cdot \int_{\mathbb{R}^n} \frac{\Omega(x,y)}{|y|^n} f(x - \Phi(|y|)y')dy. \tag{1.1}$$

On the other hand, as a related vector-valued singular integral with variable kernel, the Marcinkiewicz singular with rough variable kernel associated with surfaces of the form $\{x = \Phi(|y|)y'\}$ is considered. It is defined by

$$\mu_\Omega^\Phi(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}^\Phi(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{1.2}$$

where

$$F_{\Omega,t}^\Phi(x) = \int_{|y|\leq t} \frac{\Omega(x,y)}{|y|^{n-1}} f(x - \Phi(|y|)y')dy, \tag{1.3}$$

$$\int_{\mathbb{S}^{n-1}} \Omega(x,z')d\sigma(z') = 0. \tag{1.4}$$

If $\Phi(|y|) = |y|$, we put $\mu_\Omega^\Phi = \mu_\Omega$. Historically, the higher dimension Marcinkiewicz integral operator μ_Ω with convolution kernel, that is $\Omega(x,z) = \Omega(z)$, was first defined and studied by Stein [3] in 1958. See also [4–6] for some further works on μ_Ω with convolution kernel. Recently, Xue and Yabuta [7] studied the L^2 boundedness of the operator μ_Ω^Φ with variable kernel.

Theorem 1.1 (see [7]). *Suppose that $\Omega(x,y)$ is positively homogeneous in y of degree 0, and satisfies* (1.4) *and*

(2′) $\sup_{y\in\mathbb{R}^n}(\int_{\mathbb{S}^{n-1}}|\Omega(y,z')|^q d\sigma(z'))^{1/q} < \infty$, *for some $q > 2(n-1)/n$. Let Φ be a positive and monotonic (or negative and monotonic) C^1 function on $(0,\infty)$ and let it satisfy the following conditions:*

(i) $\delta \leq |\Phi(t)/(t\Phi'(t))| \leq M$ *for some* $0 < \delta \leq M < \infty$;

(ii) $\Phi'(t)$ *is monotonic on* $(0,\infty)$.

Then there is a constant C such that $\|\mu_\Omega^\Phi(f)\|_2 \leq C\|f\|_2$, where constant C is independent of f.

Since the condition (2) implies (2′), so the $L^2(\mathbb{R}^n)$ boundedness of μ_Ω^Φ holds if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ with $q > 2(n-1)/n$.

Our aim of this paper is to study the hypersingular Marcinkiewicz integral $\mu_{\Omega,\alpha}^\Phi$ along surfaces with variable kernel Ω, and with index $\alpha \geq 0$, on the homogeneous Sobolev space

$L^p_\alpha(\mathbb{R}^n)$ for $1 < p \le 2$ and the homogeneous Hardy-Sobolev space $H^p_\alpha(\mathbb{R}^n)$ for some $n/(n+\alpha) < p \le 1$. Let $F^\Phi_{\Omega,t}(x)$ be as above, we then define the operators $\mu^\Phi_{\Omega,\alpha}$ by

$$\mu^\Phi_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|F^\Phi_{\Omega,t}(x)\right|^2 \frac{dt}{t^{3+2\alpha}}\right)^{1/2}, \quad \alpha \ge 0. \tag{1.5}$$

Our main results are as follows.

Theorem 1.2. *Suppose that* $\alpha \ge 0$, $\Omega(x,y)$ *satisfies* (1.4) *and* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ *with* $q > \max\{1, 2(n-1)/(n+2\alpha)\}$. *Let* Φ *be a positive and increasing* C^1 *function on* $(0,\infty)$ *and let it satisfy the following conditions:*

(i) $\Phi(t) \simeq t\Phi'(t)$;

(ii) $0 \le \Phi'(t) \le W$ *on* $(0,\infty)$.

Then there is a constant C *such that* $\|\mu^\Phi_{\Omega,\alpha}(f)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2_\alpha(\mathbb{R}^n)}$, *where constant* C *is independent of* f.

Theorem 1.3. *Suppose* $0 < \alpha < n/2$, *and that* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, *with* $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, *and satisfies* (1.4). *Let* Φ *be a positive and increasing* C^1 *function on* $(0,\infty)$ *and let it satisfy the following conditions:*

(i) $\Phi(t) \simeq t\Phi'(t)$;

(ii) $0 < \Phi'(t) \le 1, \Phi(0) = 0$.

Then, for $n/(n+\alpha) < p \le 1$, *there is a constant* C *such that* $\|\mu^\Phi_{\Omega,\alpha}(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{H^p_\alpha(\mathbb{R}^n)}$, *where constant* C *is independent of any* $f \in H^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

Furthermore, our result can be extended to the Littlewood-Paley type operators $\mu^\Phi_{\Omega,\alpha,S}$ and $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$ with variable kernels and index $\alpha \ge 0$, which relate to the Lusin area integral and the Littlewood-Paley g^*_λ function, respectively. Let $F^\Phi_{\Omega,t}(x)$ be as above, we then define the operators $\mu^\Phi_{\Omega,\alpha,S}$ and $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$ for $f \in \mathcal{S}(\mathbb{R}^n)$, respectively by

$$\begin{aligned} \mu^\Phi_{\Omega,\alpha,S}(f)(x) &= \left(\iint_{\Gamma(x)} \left|F^\Phi_{\Omega,t}(y)\right|^2 \frac{dy\,dt}{t^{n+3+2\alpha}}\right)^{1/2}, \\ \mu^{*,\Phi}_{\Omega,\alpha,\lambda}(f)(x) &= \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|F^\Phi_{\Omega,t}(y)\right|^2 \frac{dy\,dt}{t^{n+3+2\alpha}}\right)^{1/2}, \end{aligned} \tag{1.6}$$

with $\lambda > 1$, where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. As an application of Theorem 1.2, we have the following conclusion.

Theorem 1.4. *Under the assumption of Theorem 1.2, then Theorem 1.2 still holds for* $\mu^\Phi_{\Omega,\alpha,S}$ *and* $\mu^{*,\Phi}_{\Omega,\alpha,\lambda}$.

By Theorems 1.2 and 1.3 and applying the interpolation theorem of sublinear operator, we obtain the $L^p_\alpha - L^p$ boundedness of $\mu^\Phi_{\Omega,\alpha}$.

Corollary 1.5. *Suppose* $0 < \alpha < n/2$, *and that* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, *and satisfies* (1.4). *Let* Φ *be given as in Theorem 1.3. Then, for* $1 < p \le 2$, *there exists an absolute positive constant* C *such that*

$$\left\| \mu^{\Phi}_{\Omega,\alpha}(f) \right\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p_\alpha(\mathbb{R}^n)}, \tag{1.7}$$

for all $f \in L^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

Remark 1.6. It is obvious that the conclusions of Theorem 1.2 are the substantial improvements and extensions of Stein's results in [3] about the Marcinkiewicz integral μ_Ω with convolution kernel, and of Ding's results in [8] about the Marcinkiewicz integral μ_Ω with variable kernels.

Remark 1.7. Recently, the authors in [9] proved the boundedness of hypersingular Marcinkiewicz integral with variable kernels on homogeneous Sobolev space $L^p_\alpha(R^n)$ for $1 < p \le 2$ and $0 < \alpha < 1$ without any smoothness on Ω. So Corollary 1.5 extended the results in [9, Theorem 5].

Throughout this paper, the letter C always remains to denote a positive constant not necessarily the same at each occurrence.

2. The Bounedness on Sobolev Spaces

Before giving the definition of the Sobolev space, let us first recall the Triebel-Lizorkin space.

Fix a radial function $\varphi(x) \in C^\infty$ satisfying $\operatorname{supp}(\varphi) \subseteq \{x : 1/2 < |x| \le 2\}$ and $0 \le \varphi(x) \le 1$, and $\varphi(x) > c > 0$ if $3/5 \le |x| \le 5/3$. Let $\varphi_j(x) = \varphi(2^j x)$. Define the function $\psi_j(x)$ by $\mathcal{F}(\psi_j)(\xi) = \varphi_j(\xi)$, such that $\mathcal{F}(\psi_j * f)(\xi) = \mathcal{F}(f)(\xi)\varphi_j(\xi)$.

For $0 < p,\ q < \infty$, and $\alpha \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha,q}_p$ is the set of all distributions f satisfying

$$\dot{F}^{\alpha,q}_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^{\alpha,q}_p} = \left\| \left(\sum_k \left| 2^{-\alpha k} \psi_k * f \right|^q \right)^{1/q} \right\|_p < \infty \right\}. \tag{2.1}$$

For $p \ge 1$, the homogeneous Sobolev spaces $L^p_\alpha(\mathbb{R}^n)$ is defined by $L^p_\alpha(\mathbb{R}^n) = \dot{F}^{\alpha,2}_p(\mathbb{R}^n)$, namely $\|f\|_{L^p_\alpha} = \|f\|_{\dot{F}^{\alpha,2}_p}$. From [10] we know that for any $f \in L^2_\alpha(\mathbb{R}^n)$

$$\|f\|_{L^2_\alpha(\mathbb{R}^n)} \cong \left(\int_{\mathbb{R}^n} \left| \mathcal{F}(f)(\xi) \right|^2 |\xi|^{2\alpha} d\xi \right)^{1/2}, \tag{2.2}$$

and if α is a nonnegative integer, then for any $f \in L^p_\alpha(\mathbb{R}^n)$

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} \cong \sum_{|\tau|=\alpha} \|D^\tau f\|_{L^p(\mathbb{R}^n)}. \tag{2.3}$$

For $0 < p \leq 1$, we define the homogeneous Hardy-Sobolev space $H^p_\alpha(\mathbb{R}^n)$ by $H^p_\alpha(\mathbb{R}^n) = \dot{F}^{\alpha,2}_p(\mathbb{R}^n)$. It is well known that $H^p(\mathbb{R}^n) = \dot{F}^{0,2}_p(\mathbb{R}^n)$ for $0 < p \leq 1$, one can refer [10] for the details.

Next, let us give the main lemmas we will use in proving theorems.

Lemma 2.1 (see [11]). *Suppose that $n \geq 2$ and $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ has the form $f(x) = f_0(|x|)P(x)$ where $P(x)$ is a solid spherical harmonic polynomial of degree m. Then the Fourier transform of f has the form $\mathcal{F}(f)(x) = F_0(|x|)P(x)$, where*

$$F_0(r) = 2\pi i^{-m} r^{-((n+2m-2)/2)} \int_0^\infty f_0(s) J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2} ds, \tag{2.4}$$

and $r = |\xi|$, $J_m(s)$ is the Bessel function.

Lemma 2.2 (see [12]). *For $\lambda = (n-2)/2$, and $-\lambda \leq \alpha \leq 1$, there exists $C > 0$ such that for any $h \geq 0$ and $m = 1, 2, \ldots$,*

$$\left| \int_0^h \frac{J_{m+\lambda}(t)}{t^{\lambda+\alpha}} dt \right| \leq \frac{C}{m^{\lambda+\alpha}}. \tag{2.5}$$

Lemma 2.3. *Let $\alpha \geq 0$, $\lambda = (n-2)/2$, Φ is a C^1 function on $(0, \infty)$ and let it satisfy the conditions (i) and (ii) in Theorem 1.2.*

Denote $g_\alpha(f)(x) = \left(\int_0^{+\infty} |N_\varepsilon f(x)|^2 (d\varepsilon/\varepsilon^{1+2\alpha})\right)^{1/2}$, if

$$\mathcal{F}(N_\varepsilon f)(\xi) = \int_0^{\Phi(\varepsilon)|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}} dt \cdot \mathcal{F}(f)(\xi). \tag{2.6}$$

Then there exists a constant C independent of m, such that $\|g_\alpha(f)\|_{L^2} \leq C/m^{\lambda+1+\alpha} \|f\|_{L^2_\alpha}$ for every integer $m \in \mathbb{N}, m > \alpha$.

Proof. Let $\eta(|x|) = \int_0^{|x|} (J_{m+\lambda}(t)/t^{\lambda+1}) dt$, then we have

$$\begin{aligned}
\|g_\alpha(f)\|_2^2 &= \int_{\mathbb{R}^n} \int_0^{+\infty} |N_\varepsilon f(x)|^2 \frac{d\varepsilon}{\varepsilon^{1+2\alpha}} dx \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} \left|\eta(\Phi(\varepsilon)|\xi|) \mathcal{F}(f)(\xi)\right|^2 d\xi \frac{d\varepsilon}{\varepsilon^{1+2\alpha}} \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} \left|\eta\left(\Phi\left(\frac{\beta}{|\xi|}\right)|\xi|\right) \mathcal{F}(f)(\xi)\right|^2 |\xi|^{2\alpha} d\xi \frac{d\beta}{\beta^{1+2\alpha}} \\
&= \int_{\mathbb{R}^n} \int_0^{+\infty} \left|\eta\left(\Phi\left(\frac{\beta}{|\xi|}\right)|\xi|\right)\right|^2 \frac{d\beta}{\beta^{1+2\alpha}} |\mathcal{F}(f)(\xi)|^2 |\xi|^{2\alpha} d\xi.
\end{aligned} \tag{2.7}$$

So it suffices to show $\int_0^{+\infty} \eta(\Phi(\beta/|\xi|)|\xi|)^2 (d\beta/\beta^{1+2\alpha}) \leq (C/m^{\lambda+1+\alpha})^2$.

Decompose this integral into two parts $\int_0^{+\infty} = \int_0^{m/2} + \int_{m/2}^{+\infty} =: I_1 + I_2$.

For I_2, by using Lemma 2.2 and $\Phi(t) \simeq t\Phi'(t)$, we can get

$$\begin{aligned} I_2 &= \int_{m/2}^{+\infty} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\ &\leq \frac{C}{m^{2\lambda+2}} \int_{m/2}^{+\infty} \frac{d\beta}{\beta^{1+2\alpha}} \\ &\leq \frac{C}{m^{2\lambda+2+2\alpha}}. \end{aligned} \tag{2.8}$$

For the other part I_1, applying Stirling's formula, we have

$$\sqrt{2\pi} x^{x-1/2} e^{-x} \leq \Gamma(x) \leq 2\sqrt{2\pi} x^{x-1/2} e^{-x}. \tag{2.9}$$

Also in [13], the authors proved the following inequality

$$|J_\nu(t)| \leq \frac{(t/2)^\nu}{\Gamma(\nu+1)}. \tag{2.10}$$

So by (2.9) and (2.10), $0 \leq \alpha < [\alpha] + 1 \leq m$, and noting that $\Phi(t) \leq Wt$, we have

$$\begin{aligned} I_1 &= \int_0^{m/2} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\ &\leq \int_0^{m/2} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{|J_{m+\lambda}(t)|}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\ &\leq \frac{1}{2^{2m+2\lambda}\Gamma^2(m+\lambda+1)} \int_0^{m/2} \left(\int_0^{\Phi(\beta/|\xi|)|\xi|} \frac{t^{m+\lambda}}{t^{\lambda+1}} dt \right)^2 \frac{d\beta}{\beta^{1+2\alpha}} \\ &\leq \frac{1}{2^{2m+2\lambda}\Gamma^2(m+\lambda+1)} \int_0^{m/2} \left(\Phi'\left(\frac{\beta}{|\xi|} \right) \right)^{2m} \frac{d\beta}{\beta^{1+2\alpha}} \\ &\leq \frac{e^{2m+2\lambda+2}}{2\pi 2^{2m+2\lambda}(m+\lambda+1)^{2m+2\lambda+1}} \int_0^{m/2} \left(\Phi'\left(\frac{\beta}{|\xi|} \right)^{2m} \right) \frac{d\beta}{\beta^{1+2\alpha-2m}} \\ &\leq C \frac{e^{2m+2\lambda+2}}{2\pi 2^{2m+2\lambda}(m+\lambda+1)^{2m+2\lambda+1}} \int_0^{m/2} \frac{d\beta}{\beta^{1+2\alpha-2m}} \\ &\leq C \left(\frac{e}{4} \right)^{2m+2\lambda+2} \frac{1}{m^{2\alpha+2\lambda+2}} \\ &\leq \frac{C}{m^{2\alpha+2\lambda+2}}. \end{aligned} \tag{2.11}$$

So far we can deduce the desired conclusion of Lemma 2.3. □

Proof of Theorem 1.2. The basic idea of proof can go back to [14], for recently papers, one see [8, 15]. By the same argument as in [1], let $\{Y_{m,j}\}$ $(m \ge 1,\ j = 1, 2, \dots, D_m)$ denote the complete system of normalized surface spherical harmonics. See [14] for instance, we can decompose $\Omega(x, y')$ as following:

$$\Omega(x, y') = \sum_{m=1}^{+\infty} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(y') \text{ is a finite sum.} \tag{2.12}$$

Denote

$$a_m(x) = \left(\sum_{j=1}^{D_m} \left| a_{m,j}(x) \right|^2 \right)^{1/2}, \qquad b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}, \tag{2.13}$$

then we get

$$\sum_{j=1}^{D_m} b_{m,j}^2(x) = 1, \qquad \Omega(x, y') = \sum_{m=1}^{+\infty} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) Y_{m,j}(y'). \tag{2.14}$$

Then, applying Hölder inequality twice, we have for any $0 < \varepsilon < 1$ that

$$\begin{aligned}
\left| \mu_{\Omega,\alpha}^{\Phi} f(x) \right|^2 &= \int_0^{\infty} \left| \int_{|y| \le t} \sum_{m=1}^{+\infty} b_{m,j}(x) \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\
&\le \left(\sum_{m=1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \\
&\quad \times \int_0^{+\infty} \left| \int_{|y| \le t} \sum_{j=1}^{D_m} b_{m,j}(x) \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\
&\le \left(\sum_{m=1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \int_0^{+\infty} \left(\sum_{j=1}^{D_m} b_{m,j}^2(x) \right) \\
&\quad \times \sum_{j=1}^{D_m} \left| \int_{|y| \le t} \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\
&= \left(\sum_{m=1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \\
&\quad \times \int_0^{+\infty} \sum_{j=1}^{D_m} \left| \int_{|y| \le t} \frac{Y_{m,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|) y') dy \right|^2 \frac{dt}{t^{3+2\alpha}}.
\end{aligned} \tag{2.15}$$

By [14, page 230, equation (4.4)], we can observe that the series in the first parenthesis on the right-hand side of the inequality above, for each x fixed, is equal to $\|\Omega(x,\cdot)\|^2_{L^2_{-\gamma}(\mathbb{S}^{n-1})}$, where $L^2_{-\gamma}(\mathbb{S}^{n-1})$ is the Sobolev space on $\mathbb{S}^{n-1}$ with $\gamma = \varepsilon((1/2)+\alpha)$ for any $0<\varepsilon<1$. So if we take ε sufficiently close to 1, then by the Sobolev imbedding theorem $L^q \subset L^2_{-\gamma}$, we have

$$\left(\sum_m a_m^2(x) m^{-\varepsilon(1+2\alpha)}\right)^{1/2} \le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})} := C\|\Omega\| \tag{2.16}$$

with $q > \max\{1, 2(n-1)/(n+2\alpha)\}$.

By Fourier transform and (2.16), we get

$$\begin{aligned}\left\|\mu^{\Phi}_{\Omega,\alpha}(f)\right\|_2^2 &\le C\|\Omega\|^2 \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)} \int_0^{+\infty}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\left|\int_{|y|\le t}\frac{Y_{m,j}(y')}{|y|^{n-1}} f(x-\Phi(|y|)y')dy\right|^2 dx\frac{dt}{t^{3+2\alpha}} \\ &\le C\|\Omega\|^2 \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)}\sum_{j=1}^{D_m}\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\mathcal{F}\left(\int_{|y|\le t}\frac{Y_{m,j}(y')}{|y|^{n-1}} f(\cdot-\Phi(|y|)y')dy\right)(\xi)\right|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &=: C\|\Omega\|^2 \sum_{m=1}^{+\infty} m^{\varepsilon(1+2\alpha)}\sum_{j=1}^{D_m}\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2.\end{aligned} \tag{2.17}$$

For $\mu^{\Phi}_{\Omega,j,\alpha}(f)$, we have

$$\begin{aligned}\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2 &= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{|y|\le t}\int_{\mathbb{R}^n}\frac{Y_{m,j}(y')}{|y|^{n-1}} f(x-\Phi(|y|)y')e^{-2\pi i x\cdot\xi}dx\,dy\right|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{|y|\le t}\frac{Y_{m,j}(y')}{|y|^{n-1}} e^{-2\pi i\Phi(|y|)y'\cdot\xi}\right. \\ &\qquad \left.\times\int_{\mathbb{R}^n} f(x-\Phi(|y|)y')e^{-2\pi i(x-\Phi(|y|)y')\cdot\xi}dx\,dy\right|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\int_{|y|\le t}\frac{Y_{m,j}(y')}{|y|^{n-1}} e^{-2\pi i\Phi(|y|)y'\cdot\xi}dy\right|^2 |\mathcal{F}(f)(\xi)|^2 d\xi\frac{dt}{t^{3+2\alpha}} \\ &= \int_{\mathbb{R}^n}\int_0^{+\infty}\left|\frac{1}{t^{1+\alpha}}\int_{|y|\le t}\frac{Y_{m,j}(y')}{|y|^{n-1}} e^{-2\pi i\Phi(|y|)y'\cdot\xi}dy\right|^2 \frac{dt}{t}|\mathcal{F}(f)(\xi)|^2 d\xi.\end{aligned} \tag{2.18}$$

For the integral on the right hand side of the above inequality, by changing of variable, we can get

$$\begin{aligned}
&\frac{1}{t^{1+\alpha}}\int_{|y|\le t}\frac{\Upsilon_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i\Phi(|y|)y'\cdot\xi}dy\\
&\quad=\frac{1}{t^{1+\alpha}}\int_0^t\int_{\mathbb{S}^{n-1}}\Upsilon_{m,j}(y')e^{-2\pi i\Phi(s)y'\cdot\xi}dy'ds\\
&\quad=\frac{1}{t^{1+\alpha}}\int_0^{\Phi(t)}\int_{\mathbb{S}^{n-1}}\Upsilon_{m,j}(y')e^{-2\pi i\gamma y'\cdot\xi}\left(\Phi^{-1}(\gamma)\right)'dy'd\gamma\\
&\quad=\frac{1}{t^{1+\alpha}}\int_{|y|\le\Phi(t)}\frac{\Upsilon_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i y\cdot\xi}\left(\Phi^{-1}(|y|)\right)'dy.
\end{aligned}\tag{2.19}$$

So we have

$$\left\|\mu^{\Phi}_{\Omega,j,\alpha}(f)\right\|_2^2=\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\frac{1}{t^{1+\alpha}}\int_{|y|\le\Phi(t)}\frac{\Upsilon_{m,j}(y')}{|y|^{n-1}}e^{-2\pi i y\cdot\xi}\left(\Phi^{-1}(|y|)\right)'dy\right|^2\frac{dt}{t}|\mathcal{F}(f)(\xi)|^2d\xi.\tag{2.20}$$

Put $P_{m,j}(x)=\Upsilon_{m,j}(x')|x|^m$ and $\varphi^{\Phi,m,j}_{t,\alpha}(x)=P_{m,j}(x)\cdot|x|^{-n-m+1}\chi_{|x|\le\Phi(t)}(x)(\Phi^{-1}(|x|))'t^{-1-\alpha}$, we can deduce from Lemma 2.1 that

$$\mathcal{F}\left(\varphi^{\Phi,m,j}_{t,\alpha}\right)(\xi)=P_{m,j}(|\xi|)\cdot F_0(|\xi|)=\Upsilon_{m,j}(\xi')\cdot|\xi|^mF_0(|\xi|),\tag{2.21}$$

where

$$\begin{aligned}
F_0(r)&=2\pi i^{-m}r^{-(n/2)-m+1}\int_0^{\Phi(t)}t^{-1-\alpha}s^{-n-m+1}\left(\Phi^{-1}(s)\right)'J_{(n/2)+m-1}(2\pi rs)s^{(n/2)+m}ds\\
&=2\pi i^{-m}r^{-(n/2)-m+1}t^{-1-\alpha}\int_0^{\Phi(t)}s^{-(n/2)+1}J_{(n/2)+m-1}(2\pi rs)d\left(\Phi^{-1}(s)\right)\\
&=2\pi i^{-m}r^{-(n/2)-m+1}t^{-1-\alpha}\int_0^t\frac{J_{(n/2)+m-1}(2\pi r\Phi(\beta))}{(\Phi(\beta))^{(n/2)-1}}d\beta\\
&=(2\pi)^{(n/2)}i^{-m}r^{-m}\frac{t^{-\alpha}}{t}\int_0^t\frac{J_{(n/2)+m-1}(2\pi r\Phi(\beta))}{(2\pi r\Phi(\beta))^{(n/2)-1}}d\beta.
\end{aligned}\tag{2.22}$$

Hence, we have

$$
\begin{aligned}
&\left\|\mu_{\Omega,j,\alpha}^{\Phi}(f)\right\|_2^2 \\
&= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\varphi_{t,\alpha}^{\Phi,m,j} * f(x)\right|^2 dx\frac{dt}{t} \\
&= \int_0^{+\infty}\int_{\mathbb{R}^n}\left|\mathcal{F}\left(\varphi_{t,\alpha}^{\Phi,m,j} * f\right)(\xi)\right|^2 d\xi\frac{dt}{t} \\
&\le \int_{\mathbb{R}^n}\int_0^{+\infty}\left|Y_{m,j}(\xi')|\xi|^m i^{-m}|\xi|^{-m}(2\pi)^{n/2}\frac{t^{-\alpha}}{t}\int_0^t \frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\right|^2\frac{dt}{t}|\mathcal{F}(f)(\xi)|^2 d\xi \\
&\le C\int_{\mathbb{R}^n}\int_0^{+\infty}\left|Y_{m,j}(\xi')\frac{1}{t}\int_0^t \frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\right|^2\frac{dt}{t^{1+2\alpha}}|\mathcal{F}(f)(\xi)|^2 d\xi.
\end{aligned}
\tag{2.23}
$$

By [14], we know that $\sum_{j=1}^{D_m}|Y_{m,j}(z')|^2 \cong m^{n-2}$.
So we can get

$$
\sum_{j=1}^{D_m}\left\|\mu_{\Omega,j,\alpha}^{\Phi}(f)\right\|_2^2 \le Cm^{n-2}\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\frac{1}{t}\int_0^t \frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta\right|^2\frac{dt}{t^{1+2\alpha}}|\mathcal{F}(f)(\xi)|^2 d\xi.
\tag{2.24}
$$

Set $\lambda = (n/2)-1, \rho = 2\pi|\xi|\Phi(\beta)$ and note that $\Phi(t) \simeq t\Phi'(t)$, we can deduce that

$$
\begin{aligned}
U &:= \frac{1}{t}\int_0^t \frac{J_{(n/2)+m-1}(2\pi|\xi|\Phi(\beta))}{(2\pi|\xi|\Phi(\beta))^{(n/2)-1}}d\beta \\
&= \frac{1}{t}\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda}}\frac{1}{2\pi|\xi|\Phi'(\Phi^{-1}(\rho/2\pi|\xi|))}d\rho \\
&= \frac{1}{t}\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}\Phi^{-1}\left(\frac{\rho}{2\pi|\xi|}\right)d\rho.
\end{aligned}
\tag{2.25}
$$

Noting that $\Phi(t)$ is increasing, by using the second mean-value theorem, we get, for some $0 \le \eta < 2\pi|\xi|\Phi(t)$,

$$
\begin{aligned}
|U| &\le \left|\frac{1}{t}\Phi^{-1}(\Phi(t))\int_{\eta}^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}d\rho\right| \\
&\le \left|\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}d\rho\right|.
\end{aligned}
\tag{2.26}
$$

From (2.26), it follows that

$$\sum_{j=1}^{D_m}\left\|\mu_{m,j,\alpha}^{\Phi}(f)\right\|_2^2 \leq Cm^{n-2}\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\int_0^{2\pi|\xi|\Phi(t)}\frac{J_{m+\lambda}(\rho)}{\rho^{\lambda+1}}d\rho\cdot\mathcal{F}(f)(\xi)\right|^2\frac{dt}{t^{1+2\alpha}}d\xi. \tag{2.27}$$

Thus using Lemma 2.3, we can deduce the desired conclusion of Theorem 1.2. □

Proof of Theorem 1.4. First, we know that $\mu_{\Omega,\alpha,S}^{\Phi}(f)(x)\leq 2^{\lambda n}\mu_{\Omega,\alpha,\lambda}^{*,\Phi}(f)(x)$. On the other hand,

$$\begin{aligned}
&\left\|\mu_{\Omega,\alpha,\lambda}^{*,\Phi}(f)\right\|_2^2\\
&\quad=\int_{\mathbb{R}^n}\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t}\int_{|z|\leq t}\frac{\Omega(x,z)}{|z|^{n-1}}f\left(x-\Phi(|z|)z'\right)dz\right|^2\frac{dzdt}{t^{n+1+2\alpha}}dx\\
&\quad=\int_0^{\infty}\int_{\mathbb{R}^n}\left(\frac{1}{t^n}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}dx\right)\left|\frac{1}{t}\int_{|z|\leq t}\frac{\Omega(x,z)}{|z|^{n-1}}f\left(x-\Phi(|z|)z'\right)dz\right|^2\frac{dzdt}{t^{1+2\alpha}}\\
&\quad\leq C\left\|\mu_{\Omega,\alpha}^{\Phi}(f)\right\|_2^2.
\end{aligned} \tag{2.28}$$

Thus, using Theorem 1.2, we can finish Theorem 1.4. □

3. The Bounedness on Hardy-Sobolev Spaces

In order to prove the boundedness for operator $\mu_{\Omega,\alpha}^{\Phi}$ on Hardy-Sobolev spaces and prove Theorem 1.3, we first introduce a new kind of atomic decomposition for Hardy-Sobolev space as following which will be used next.

Definition 3.1 (see [16]). For $\alpha\geq 0$, the function $a(x)$ is called a $(p,2,\alpha)$ atom if it satisfies the following three conditions:

(1) $\operatorname{supp}(a)\subset B$ with a ball $B\subset\mathbb{R}^n$;

(2) $\|a\|_{L_\alpha^2}\leq|B|^{(1/2)-(1/p)}$;

(3) $\int_{\mathbb{R}^n}a(x)P(x)=0$, for any polynomial $P(x)$ of degree $\leq N=[n((1/p)-1)\alpha]$.

By [16], we have that every $f\in H_\alpha^p(\mathbb{R}^n)$ can be written as a sum of $(p,2,\alpha)$ atoms $a_j(x)$, that is,

$$f=\sum_j\lambda_j a_j. \tag{3.1}$$

Proof of Theorem 1.3. Similar to the argument of Lemma 3.3 in [17] and using above atomic decomposition, it suffices to show that

$$\left\|\mu_{\Omega,\alpha}^{\Phi}(a)\right\|_{L^p}^p \leq C, \tag{3.2}$$

with the constant C independent of any $(p,2,\alpha)$ atom a.

Assume $\operatorname{supp}(a) \subset B(0,R)$. We first note that

$$\begin{aligned}\left\|\mu_{\Omega,\alpha}^{\Phi}(a)\right\|_{L^p}^p &\leq \int_{|x|\leq 8R}\left|\mu_{\Omega,\alpha}^{\Phi}(a)(x)\right|^p dx + \int_{|x|>8R}\left|\mu_{\Omega,\alpha}^{\Phi}(a)(x)\right|^p dx \\ &=: U_1 + U_2.\end{aligned} \tag{3.3}$$

For U_1, using Theorem 1.2, it is not difficult to deduce that

$$\begin{aligned}U_1 &\leq C\left\|\mu_{\Omega,\alpha}^{\Phi}(a)\right\|_{L^2}^p R^{n(1-(p/2))} \leq C\|a\|_{L^2_\alpha}^p R^{n(1-(p/2))} \\ &\leq CR^{n((p/2)-1)} R^{n(1-(p/2))} \leq C.\end{aligned} \tag{3.4}$$

For U_2, we first consider the case $n/(n+\alpha) < p < 1$, according to [15, Lemma 5.5], for $0 < \alpha < n/2$ and $(p,2,\alpha)$ atom a with support $B = B(0,R)$, one has

$$\int_B |a(x)|dx \leq CR^{n-(n/p)+\alpha}. \tag{3.5}$$

Using Minkowski inequality and Hölder inequality for integrals, and (3.5), we can get

$$\begin{aligned}U_2 &= \int_{|x|>8R}\left|\mu_{\Omega,\alpha}^{\Phi}(a)(x)\right|^p dx \\ &= \int_{|x|>8R}\left(\int_0^{+\infty}\left|\int_{|y|\leq t}\frac{\Omega(x,y)}{|y|^{n-1}}a(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t^{3+2\alpha}}\right)^{p/2} dx \\ &\leq \int_{|x|>8R}\left|\int_{\mathbb{R}^n}\frac{|\Omega(x,y)|}{|y|^{n+\alpha}}|a(x-\Phi(|y|)y')|dy\right|^p dx.\end{aligned} \tag{3.6}$$

For the integral on the right hand side of the above inequality, by changing of variable and noting that $0 < \Phi'(t) \leq 1, \Phi(0) = 0$, we can get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|\Omega(x,y)|}{|y|^{n+\alpha}} |a(x-\Phi(|y|)y')| dy \\
&= \int_{\mathbb{S}^{n-1}} \int_0^R \frac{|\Omega(x,y')|}{r^{1+\alpha}} |a(x-\Phi(r)y')| dr\, dy' \\
&= \int_{\mathbb{S}^{n-1}} \int_0^{\Phi(R)} \frac{|\Omega(x,y')|}{(\Phi^{-1}(\gamma))^{1+\alpha}} |a(x-\gamma y')| \frac{1}{\Phi'(\Phi^{-1}(\gamma))} d\gamma\, dy' \\
&= \int_{\mathbb{S}^{n-1}} \int_0^{\Phi(R)} \frac{|\Omega(x,y')|}{(\Phi^{-1}(\gamma))^{1+\alpha}} |a(x-\gamma y')| \frac{\Phi^{-1}(\gamma)}{\gamma} d\gamma\, dy' \\
&= \int_{\mathbb{S}^{n-1}} \int_0^{\Phi(R)} \frac{|\Omega(x,y')|}{(\Phi^{-1}(\gamma))^{\alpha}\gamma} |a(x-\gamma y')| d\gamma\, dy' \\
&= \int_{|y|\leq\Phi(R)} \frac{|\Omega(x,y)|}{|y|^n (\Phi^{-1}(|y|))^{\alpha}} |a(x-y)| dy \\
&= \int_{|x-y|\leq\Phi(R)} \frac{|\Omega(x,x-y)|}{|x-y|^n (\Phi^{-1}(|x-y|))^{\alpha}} |a(y)| dy \\
&\leq \int_{|x-y|\leq\Phi(R)} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} |a(y)| dy.
\end{aligned} \tag{3.7}$$

By (3.7), we can get

$$\begin{aligned}
U_2 &\leq \sum_{j=3}^{+\infty} \int_{2^jR<|x|<2^{j+1}R} \left| \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} |a(y)| dy \right|^p dx \\
&\leq \sum_{j=3}^{+\infty} \left(2^jR\right)^{n(1-p)} \left(\int_{2^jR<|x|<2^{j+1}R} \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} |a(y)| dy\, dx \right)^p \\
&\leq \sum_{j=3}^{+\infty} \left(2^jR\right)^{n(1-p)} \left(\int_B |a(y)| \int_{2^jR<|x|<2^{j+1}R} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} dx\, dy \right)^p \\
&\leq C\|\Omega\|^p_{L^\infty\times L^1} \left(\int_B |a(y)| dy \right)^p \cdot \sum_{j=3}^{+\infty} \left(2^jR\right)^{-\alpha p} \left(2^jR\right)^{n(1-p)}.
\end{aligned} \tag{3.8}$$

Thus by (3.5) and the condition $p > n/(n+\alpha)$,

$$U_2 \leq C\|\Omega\|^p_{L^\infty\times L^1} \sum_{j=3}^{+\infty} 2^{j(n-np-\alpha p)} \leq C. \tag{3.9}$$

As for $p = 1$, similar to the argument of $n/(n+\alpha) < p < 1$, we can easily get $U_2 \leq C$. So far the proof of Theorem 1.3 has been finished. □

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