Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 635604, 9 pages
doi:10.1155/2011/635604

*Research Article*

# On the Growth of Solutions of Some Second-Order Linear Differential Equations

**Feng Peng and Zong-Xuan Chen**

*School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China*

Correspondence should be addressed to Zong-Xuan Chen, chzx@vip.sina.com

Received 10 December 2010; Accepted 9 February 2011

Academic Editor: Alberto Cabada

Copyright © 2011 F. Peng and Z.-X. Chen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the growth of solutions of $f'' + P(z)f' + Q(z)f = 0$, where $P(z)$ and $Q(z)$ are entire functions. When $P(z) = e^{-z}$ and $Q(z) = A_1(z)e^{a_1 z} + A_2(z)e^{a_2 z}$ satisfy some conditions, we prove that every nonzero solution of the above equation has infinite order and hyper-order 1, which improve the previous results.

## 1. Introduction and Results

In this paper, we will assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g., see [1–3]). In addition, we will use the notation $\sigma(f)$ to denote the order of growth of meromorphic function $f(z)$, $\sigma_2(f)$ to denote the hyper-order of $f(z)$ (see [3]). $\sigma_2(f)$ is defined to be

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.1}$$

We consider the second-order linear differential equation

$$f'' + P(z)f' + Q(z)f = 0, \tag{1.2}$$

where $P(z)$ and $Q(z)$ are entire functions of finite order. It is well known that each solution of (1.2) is an entire function, and most solutions of (1.2) have infinite order.

Thus, a natural question is what conditions on $P(z)$ and $Q(z)$ will guarantee that every solution $f(\not\equiv 0)$ of (1.2) has infinite order? Ozawa [4], Gundersen [5], Amemiya and Ozawa [6], and Langley [7] have studied the problem with $P(z) = e^{-z}$ and $Q(z)$ is complex number or polynomial. For the case that $P(z) = e^{-z}$, and $Q(z)$ is transcendental entire function, Gundersen proved the following in [5, Theorem A].

**Theorem A.** *If $Q(z)$ is a transcendental entire function with order $\sigma(Q) \neq 1$, then every solution $f(\not\equiv 0)$ of equation*

$$f'' + e^{-z} f' + Q(z) f = 0 \tag{1.3}$$

*has infinite order.*

Theorem A states that when $\sigma(Q) = 1$, (1.3) may have finite-order solutions. We go deep into the problem: what condition in $Q(z)$ when $\sigma(Q) = 1$ will guarantee every solution $f(\not\equiv 0)$ of (1.3) has infinite order? And more precise estimation for its rate of growth is a very important aspect. Chen investigated the problem and obtain the following in [8, Theorem B and Theorem C].

**Theorem B.** *Let $A_j(z)(\not\equiv 0)(j = 0, 1)$ be entire functions with $\sigma(A_j) < 1$, and let $a, b$ be complex numbers such that $ab \neq 0$ and $a = cb$ $(c > 1)$. then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + A_1 e^{az} f' + A_0 e^{bz} f = 0 \tag{1.4}$$

*has infinite order.*

**Theorem C.** *Let $a, b$ be nonzero complex numbers and $a \neq b$, and let $Q(z)$ be a nonconstant polynomial or $Q(z) = h(z)e^{bz}$ where $h(z)$ is nonzero polynomial, then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + e^{az} f' + Q(z) f = 0 \tag{1.5}$$

*has infinite order and $\sigma_2(f) = 1$.*

For Theorems B and C, many authors, Wang and Lü [9], Huang, Chen, and Li [10], and Cheng and Kang [11] have made some improvement. In this paper, we are concerned with the more general problem, and obtain the following theorem that extend and improve the previous results.

**Theorem 1.1.** *Let $A_j(z)\ (\not\equiv 0)\ (j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, $a_1, a_2$ be complex numbers such that $a_1 a_2 \neq 0$, and let $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + e^{-z} f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0 \tag{1.6}$$

*has infinite order and $\sigma_2(f) = 1$.*

## 2. Remarks and Lemmas for the Proof of Theorem

**Lemma 2.1** (see [12]). *Let $f$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty, H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0\,(i = 1, 2, \ldots, q)$. And let $\varepsilon > 0$ be a given constant. Then,*

(i) *there exists a set $E \subset [-(\pi/2), 3\pi/2)$ with linear measure zero, such that, if $\psi \in [-(\pi/2), 3\pi/2) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$, such that for all $z$ satisfying $\arg z = \psi$ and $|z| \geq R_0$ and for all $(k, j) \in H$, one has*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.1}$$

(ii) *there exists a set $E \subset (1, \infty)$ with finite logarithmic measure, such that for all $z$ satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.2}$$

(iii) *there exists a set $E \subset (0, \infty)$ with finite linear measure, such that for all $z$ satisfying $|z| \notin E$ and for all $(k, j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma+\varepsilon)}. \tag{2.3}$$

**Lemma 2.2** (see [8]). *Suppose that $P(z) = (\alpha + i\beta)z^n + \cdots$ ($\alpha, \beta$ are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z)(\not\equiv 0)$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}, z = re^{i\theta}, \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is $R > 0$, such that for $|z| = r > R$, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P, \theta)r^n\} < \left|g\left(re^{i\theta}\right)\right| < \exp\{(1+\varepsilon)\delta(P, \theta)r^n\}, \tag{2.4}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P, \theta)r^n\} < \left|g\left(re^{i\theta}\right)\right| < \exp\{(1-\varepsilon)\delta(P, \theta)r^n\}, \tag{2.5}$$

*where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.*

Using Lemma 2.2, we can prove Lemma 2.3.

**Lemma 2.3.** *Suppose that* $n \geq 1$ *is a positive entire number. Let* $P_j(z) = a_{jn}z^n + \cdots$ $(j = 1,2)$ *be nonconstant polynomials, where* $a_{jq}$ $(q = 1,2,\ldots,n)$ *are complex numbers and* $a_{1n}a_{2n} \neq 0$*. Set* $z = re^{i\theta}, a_{jn} = |a_{jn}|e^{i\theta_j}, \theta_j \in [-(\pi/2), 3\pi/2), \delta(P_j,\theta) = |a_{jn}|\cos(\theta_j + n\theta)$*, then there is a set* $H_1 \subset [-(\pi/2n), 3\pi/2n)$ *that has linear measure zero. If* $\theta_1 \neq \theta_2$*, then there exists a ray* $\arg z = \theta$*,* $\theta \in (-(\pi/2n), \pi/2n) \setminus (H_1 \cup H_2)$*, such that*

$$\delta(P_1,\theta) > 0, \qquad \delta(P_2,\theta) < 0, \tag{2.6}$$

*or*

$$\delta(P_1,\theta) < 0, \qquad \delta(P_2,\theta) > 0, \tag{2.7}$$

*where* $H_2 = \{\theta : \theta \in [-(\pi/2n), 3\pi/2n), \delta(P_j,\theta) = 0\}$ *is a finite set, which has linear measure zero.*

*Proof.* According to the values of $\theta_1$ and $\theta_2$, we divide our discussion into three cases. □

*Case 1* ($\theta_1 \in (-(\pi/2), \pi/2)$). (a) If $\theta_2 \in (-(\pi/2), \pi/2)$, let $\alpha_1 = \min\{(\pi/2)-\theta_1, \theta_1+\pi/2\}$, $\alpha_2 = \min\{(\pi/2) - \theta_2, \theta_2 + \pi/2\}$, Then there are three cases: (i) $\alpha_1 = \alpha_2$; (ii) $\alpha_1 < \alpha_2$; (iii) $\alpha_1 > \alpha_2$.

(i) $\alpha_1 = \alpha_2$. By $\theta_1 \neq \theta_2$, we know that $\theta_1 = -\theta_2 \neq 0$.

Suppose that $\theta_1 > 0$, then take $\theta = (1/n)((\pi/2) - \theta_1 + t)$, $t$ is any constant in $(0, \theta_1)$.

Since $H_1 \cup H_2$ has linear measure zero, there exists $t \in (0,\theta_1)$ such that $\theta = (1/n)((\pi/2) - \theta_1 + t) \in (0, \pi/2n) \setminus (H_1 \cup H_2)$. Thus $n\theta = (\pi/2) - \theta_1 + t \in (0, \pi/2)$. By $\theta_1 = -\theta_2$ and $\theta_1 > 0$ that is $\theta_1 \in (0, \pi/2)$, we have

$$\theta_1 + n\theta = \frac{\pi}{2} + t \in \left(\frac{\pi}{2}, \pi\right), \qquad \theta_2 + n\theta = \frac{\pi}{2} - 2\theta_1 + t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.8}$$

Therefore,

$$\delta(P_1,\theta) = |a_{1n}|\cos(\theta_1 + n\theta) < 0, \qquad \delta(P_2,\theta) = |a_{2n}|\cos(\theta_2 + n\theta) > 0. \tag{2.9}$$

When $\theta_1 < 0$, then $\theta_2 > 0$, we can prove it by using similar argument action as in the above proof.

(ii) $\alpha_1 < \alpha_2$, then $\theta_1 \neq 0$. Suppose that $\theta_1 > 0$, then $\theta_1 > \theta_2$, $0 < \theta_1 - \theta_2 < \pi$. Let $\theta_0 = \min\{\theta_1, \theta_1 - \theta_2\}$, and take $\theta = (1/n)((\pi/2) - \theta_1 + t)$, and $t$ is any constant in $(0, \theta_0)$.

Since $H_1 \cup H_2$ has a linear measure zero, there exists $t \in (0, \theta_0)$ such that $\theta = (1/n)((\pi/2) - \theta_1 + t) \in (0, \pi/2n) \setminus (H_1 \cup H_2)$,

$$\theta_1 + n\theta = \frac{\pi}{2} + t \in \left(\frac{\pi}{2}, \pi\right), \qquad \theta_2 + n\theta = \frac{\pi}{2} - (\theta_1 - \theta_2) + t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.10}$$

Therefore

$$\delta(P_1,\theta) = |a_{1n}|\cos(\theta_1 + n\theta) < 0, \quad \delta(P_2,\theta) = |a_{2n}|\cos(\theta_2 + n\theta) > 0. \tag{2.11}$$

Suppose that $\theta_1 < 0$, then $\theta_1 < \theta_2,\ 0 < \theta_2 - \theta_1 < \pi$. Let $\theta_0 = \min\{-\theta_1, \theta_2 - \theta_1\}$, and take $\theta = (1/n)(-(\pi/2) - \theta_1 - t)$, and $t$ is any constant in $(0, \theta_0)$.

Since $H_1 \cup H_2$ has linear measure zero, there exists $t \in (0, \theta_0)$ such that $\theta = (1/n)(-(\pi/2) - \theta_1 - t) \in (-\pi/2n, 0) \setminus (H_1 \cup H_2)$,

$$\theta_1 + n\theta = -\frac{\pi}{2} - t \in \left(-\pi, -\frac{\pi}{2}\right), \qquad \theta_2 + n\theta = -\frac{\pi}{2} + (\theta_2 - \theta_1) - t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.12}$$

Therefore,

$$\delta(P_1, \theta) = |a_{1n}| \cos(\theta_1 + n\theta) < 0, \qquad \delta(P_2, \theta) = |a_{2n}| \cos(\theta_2 + n\theta) > 0. \tag{2.13}$$

(iii) $\alpha_1 > \alpha_2$, then $\theta_2 \neq 0$. Using similar method as in proof of (ii), we know that there exists $\theta \in (-(\pi/2n), \pi/2n) \setminus (H_1 \cup H_2)$ such that $\delta(P_1, \theta) > 0$, $\delta(P_2, \theta) < 0$.

(b) When $\theta_2 \in (\pi/2, 3\pi/2)$, we can prove it by using the same argument action as in (a).

(c) When $\theta_2 \in \{\pi/2, -(\pi/2)\}$, we just prove the case that $\theta_2 = \pi/2$ (when $\theta_2 = -(\pi/2)$, we can prove it by using the same reasoning).

Let $\theta_0 = \min\{\pi/2, (\pi/2) - \theta_1\}$, take $\theta = t/n$, $t$ is any constant in $(0, \theta_0)$.

Since $H_1 \cup H_2$ has a linear measure zero, there exists $t \in (0, \theta_0)$, such that $\theta = t/n \in (0, \pi/2n) \setminus (H_1 \cup H_2)$. Then

$$\theta_2 + n\theta = \theta_2 + t \in \left(\frac{\pi}{2}, \pi\right). \tag{2.14}$$

When $\theta_1 \in (-(\pi/2), 0),\ t \in (0, \pi/2)$, thus, $-\pi/2 < \theta_1 + n\theta = \theta_1 + t < \pi/2$.

When $\theta_1 \in [0, \pi/2)$, $t \in (0, (\pi/2) - \theta_1)$, thus, $0 < \theta_1 + n\theta = \theta_1 + t < \theta_1 + (\pi/2) - \theta_1 = \pi/2$.

Therefore

$$\begin{gathered}\theta_1 + n\theta = \theta_1 + t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \delta(P_1, \theta) = |a_{1n}| \cos(\theta_1 + n\theta) > 0, \qquad \delta(P_2, \theta) = |a_{2n}| \cos(\theta_2 + n\theta) < 0.\end{gathered} \tag{2.15}$$

*Case 2.* When $\theta_1 \in (\pi/2, 3\pi/2)$, or $\theta_1 \in \{\pi/2, -(\pi/2)\}$ and $\theta_2 \notin \{\pi/2, -\pi/2\}$, using a proof similar to Case 1, we can get the conclusion.

*Case 3* ($\theta_1 \in \{\pi/2, -\pi/2\}$ and $\theta_2 \in \{\pi/2, -\pi/2\}$). By $\theta_1 \neq \theta_2$, there are only two cases: $\theta_1 = \pi/2,\ \theta_2 = -\pi/2$; or $\theta_1 = -\pi/2, \theta_2 = \pi/2$.

If $\theta_1 = \pi/2,\ \theta_2 = -\pi/2$. Take $\theta = t/n$, and $t$ is any constant in $(0, \pi/2)$.

Since $H_1 \cup H_2$ has linear measure zero, there exists $t \in (0, \pi/2)$ such that $\theta = t/n \in (0, \pi/2n) \setminus (H_1 \cup H_2)$. Using a proof similar to Case 1(c), we can prove it.

When $\theta_1 = -\pi/2,\ \theta_2 = \pi/2$, we can prove it by using the same reasoning

*Remark 2.4.* Using the similar reasoning of Lemma 2.3, we can obtain that, in Lemma 2.3, if $\theta \in (-\pi/2n, \pi/2n) \setminus (H_1 \cup H_2)$ is replaced by $\theta \in (\pi/2n, 3\pi/2n) \setminus (H_1 \cup H_2)$, then it has the same result.

**Lemma 2.5** (see [8]). *Let $A, B$ be entire functions with finite order. If $f(z)$ is a solution of the equation*

$$f'' + Af' + Bf = 0 \tag{2.16}$$

*then $\sigma_2(f) \le \max\{\sigma(A), \sigma(B)\}$.*

**Lemma 2.6** (see [12]). *Let $f$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant, Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on $\alpha$ and $i, j$ ( $0 \le i < j \le 2$), such that for all $z$ satisfying $|z| = r \notin [0,1] \cup E$,*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B\left(\frac{T(\alpha r, f)}{r}\log^{\alpha} r \log T(\alpha r, f)\right)^{j-i}. \tag{2.17}$$

*Remark 2.7.* In Lemma 2.6, when $\alpha = 2$, $i = 0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B(T(2r, f)\log T(2r, f))^{j} \le B[T(2r, f)]^{j+1}, \quad j = 1, 2. \tag{2.18}$$

**Lemma 2.8** (see [13]). *Suppose that $g : (0, +\infty) \to R$ and $h : (0, +\infty) \to R$ are nondecreasing functions, such that $g(r) \le h(r)$, $r \notin E$, where $E$ is a set with at most finite measure, then for any constant $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

## 3. Proof of Theorem 1.1

Suppose that $f(\not\equiv 0)$ is a solution of (1.6), then, $f$ is an entire function.

*First Step*

We prove that $\sigma(f) = \infty$. Suppose, to the contrary, that $\sigma(f) = \sigma < \infty$. By Lemma 2.1, for any given $\varepsilon$ $(0 < \varepsilon < (|a_2| - |a_1|)/(|a_2| + |a_1|))$, there exists a set $E_1 \subset [-(\pi/2), 3\pi/2)$ of linear measure zero, such that if $\theta \in [-\pi/2, 3\pi/2) \setminus E_1$, then, there is a constant $R_0 = R_0(\theta) > 1$, such that for all $z$ satisfying $\arg z = \theta$ and $|z| \ge R_0$, we have

$$\left|\frac{f''(z)}{f(z)}\right| \le |z|^{2(\sigma-1+\varepsilon)}, \qquad \left|\frac{f'(z)}{f(z)}\right| \le |z|^{\sigma-1+\varepsilon}. \tag{3.1}$$

Let $z = \mathrm{re}^{i\theta}$, $a_1 = |a_1|e^{i\theta_1}$, $a_2 = |a_2|e^{i\theta_2}$, $\theta_1, \theta_2 \in [-(\pi/2), 3\pi/2)$.

*Case 1* ($\arg a_1 \ne \pi$, which is $\theta_1 \ne \pi$). (i) Suppose that $\theta_1 \ne \theta_2$. By Lemmas 2.2 and 2.3, for the above $\varepsilon$, there is a ray $\arg z = \theta$, such that $\theta \in (-(\pi/2), \pi/2) \setminus (E_1 \cup H_1 \cup H_2)$ (where $H_1$ and $H_2$ are defined as in Lemma 2.3, and $E_1 \cup H_1 \cup H_2$ is of the linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0, \qquad \delta(a_2 z, \theta) < 0, \tag{3.2}$$

or

$$\delta(a_1z,\theta) < 0, \qquad \delta(a_2z,\theta) > 0. \tag{3.3}$$

When $\delta(a_1z,\theta) > 0, \delta(a_2z,\theta) < 0$, for sufficiently large $r$, we have

$$|A_1e^{a_1z}| \geq \exp\{(1-\varepsilon)\delta(a_1z,\theta)r\}, \qquad |A_2e^{a_2z}| \leq \exp\{(1-\varepsilon)\delta(a_2z,\theta)r\} \leq 1. \tag{3.4}$$

Hence

$$|A_1e^{a_1z} + A_2e^{a_2z}| \geq |A_1e^{a_1z}| - |A_2e^{a_2z}| \geq \exp\{(1-\varepsilon)\delta(a_1z,\theta)r\} - 1. \tag{3.5}$$

By (1.6), we obtain

$$\left|\frac{f''(z)}{f(z)}\right| + \left|e^{-z}\right|\left|\frac{f'(z)}{f(z)}\right| \geq |A_1e^{a_1z} + A_2e^{a_2z}|. \tag{3.6}$$

Since $\theta \in (-(\pi/2), \pi/2)$, we know that $\cos\theta > 0$, then $e^{-r\cos\theta} < 1$. Substituting (3.1) and (3.5) into (3.6), we get

$$\begin{gathered} r^{2(\sigma-1+\varepsilon)} + e^{-r\cos\theta}r^{\sigma-1+\varepsilon} \geq \exp\{(1-\varepsilon)\delta(a_1z,\theta)r\} - 1, \\ 2r^{2(\sigma-1+\varepsilon)} \geq \exp\{(1-\varepsilon)\delta(a_1z,\theta)r\} - 1. \end{gathered} \tag{3.7}$$

By $\delta(a_1z,\theta) > 0$, we know that (3.7) is a contradiction.

When $\delta(a_1z,\theta) < 0,\ \delta(a_2z,\theta) > 0$, using a proof similar to the above, we can also get a contradiction.

(ii) Suppose that $\theta_1 = \theta_2$. By Lemma 2.2, for the above $\varepsilon$, there is a ray $\arg z = \theta$ such that $\theta \in (-(\pi/2), \pi/2) \setminus (E_1 \cup H_1 \cup H_2)$ and $\delta(a_1z,\theta) > 0$. Since $|a_1| \leq |a_2|,\ a_1 \neq a_2$, and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2z,\theta) > \delta(a_1z,\theta) > 0$. For sufficiently large $r$, we have

$$|A_1e^{a_1z}| \leq \exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}, \qquad |A_2e^{a_2z}| \geq \exp\{(1-\varepsilon)\delta(a_2z,\theta)r\}. \tag{3.8}$$

Hence,

$$\begin{aligned} |A_1e^{a_1z} + A_2e^{a_2z}| &\geq |A_2e^{a_2z}| - |A_1e^{a_1z}| \geq \exp\{(1-\varepsilon)\delta(a_2z,\theta)r\} - \exp\{(1+\varepsilon)\delta(a_1z,\theta)r\} \\ &\geq M_1\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}, \end{aligned} \tag{3.9}$$

where $M_1 = \exp\{[(1-\varepsilon)\delta(a_2z,\theta) - (1+\varepsilon)\delta(a_1z,\theta)]r\} - 1$.

Since $0 < \varepsilon < (|a_2| - |a_1|)/(|a_2| + |a_1|)$, we see that $(1-\varepsilon)\delta(a_2z,\theta) - (1+\varepsilon)\delta(a_1z,\theta) > 0$, then $\exp\{[(1-\varepsilon)\delta(a_2z,\theta) - (1+\varepsilon)\delta(a_1z,\theta)]r\} > 1, M_1 > 0$.

Since $\theta \in (-(\pi/2), \pi/2)$, we know that $\cos\theta > 0$, then $e^{-r\cos\theta} < 1$. Substituting (3.1) and (3.9) into (3.6), we obtain

$$r^{2(\sigma-1+\varepsilon)} + e^{-r\cos\theta} r^{\sigma-1+\varepsilon} \geq M_1 \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\},$$
$$2r^{2(\sigma-1+\varepsilon)} \geq M_1 \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\}. \tag{3.10}$$

Since $\delta(a_1 z, \theta) > 0$, we know that (3.10) is a contradiction.

*Case 2* ($a_1 < -1$, which is $\theta_1 = \pi$). (i) Suppose that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.2, for the above $\varepsilon$, there is a ray $\arg z = \theta$ such that $\theta \in (-(\pi/2), \pi/2) \setminus (E_1 \cup H_1 \cup H_2)$ and $\delta(a_2 z, \theta) > 0$. Because $\cos\theta > 0$, $\delta(a_1 z, \theta) = |a_1|\cos(\theta_1 + \theta) = -|a_1| \cos\theta < 0$. For sufficiently large $r$, we have

$$|A_1 e^{a_1 z}| \leq \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} \leq 1, \qquad |A_2 e^{a_2 z}| \geq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\}. \tag{3.11}$$

Hence

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| \geq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} - 1. \tag{3.12}$$

Using the same reasoning as in Case 1(i), we can get a contradiction.

(ii) Suppose that $\theta_1 = \theta_2 = \pi$. By Lemma 2.2, for the above $\varepsilon$, there is a ray $\arg z = \theta$ such that $\theta \in (\pi/2, 3\pi/2) \setminus (E_1 \cup H_1 \cup H_2)$, then $\cos\theta < 0$, $\delta(a_1 z, \theta) = -|a_1|\cos\theta > 0$, $\delta(a_2 z, \theta) = -|a_2|\cos\theta > 0$, Since $|a_1| \leq |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$. Thus, $\delta(a_1 z, \theta) < \delta(a_2 z, \theta)$, for sufficiently large $r$, we get that (3.8) and (3.9) hold.

Since $a_1 < -1$, $\cos\theta < 0$, then $\delta(a_1 z, \theta) = -|a_1|\cos\theta > -\cos\theta > 0$.

Using the same reasoning as in Case 1(ii), we can get a contradiction.

Concluding the above proof, we obtain $\sigma(f) = \infty$.

*Second Step*

We prove that $\sigma_2(f) = 1$.

By Lemma 2.5 and $\max\{\sigma(e^{-z}), \sigma(A_1 e^{a_1 z} + A_2 e^{a_2 z})\} = 1$, then $\sigma_2(f) \leq 1$.

By Lemma 2.6 and Remark 2.7, we know that there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all $z$ satisfying $|z| = r \notin [0,1] \cup E_2$, we get that (2.18) holds.

For Cases 1 and 2(i) in first step, we have proved that there is a ray $\arg z = \theta$ satisfying $\theta \in (-\pi/2, \pi/2) \setminus (E_1 \cup H_1 \cup H_2)$, for sufficiently large $r$, we get that (3.5) or (3.9) or (3.12) hold, that is,

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq \exp\{h_1 r\}, \tag{3.13}$$

where $h_1 > 0$ is a constant.

Since $\theta \in (-(\pi/2), \pi/2) \setminus (E_1 \cup H_1 \cup H_2)$, then $\cos\theta > 0$, $e^{-r\cos\theta} < 1$. By (2.18), (3.6), and (3.13), we obtain

$$\exp\{h_1 r\} \leq B[T(2r, f)]^3 + e^{-r\cos\theta} B[T(2r, f)]^2 \leq 2B[T(2r, f)]^3. \tag{3.14}$$

By $h_1 > 0$, (3.14) and Lemma 2.8, we know that there exists $r_0$, when $r > r_0$, we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$.

For Case 2(ii) in first step, we have proved that there is a ray $\arg z = \theta$ satisfying $\theta \in (\pi/2, 3\pi/2) \setminus (E_1 \cup H_1 \cup H_2)$, for sufficiently large $r$, we get (3.9) hold, and we also get that $\cos\theta < 0, \delta(a_1 z, \theta) > -\cos\theta > 0$.

By (2.18), (3.6), and (3.9), we obtain

$$\begin{gathered} M_1 \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \leq B\left[T(2r, f)\right]^3 + e^{-r\cos\theta} B\left[T(2r, f)\right]^2, \\ M_1 \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \leq 2e^{-r\cos\theta} B\left[T(2r, f)\right]^3. \end{gathered} \tag{3.15}$$

By $\delta(a_1 z, \theta) > -\cos\theta > 0, M_1 > 0$ and (3.15) and Lemma 2.8, we know that there exists $r_0$, when $r > r_0$, we have $\sigma_2(f) \geq 1$, then $\sigma_2(f) = 1$.

Concluding the above proof, we obtain $\sigma_2(f) = 1$.

Theorem 1.1 is thus proved.

## Acknowledgment

This project was supported by the National Natural Science Foundation of China (no. 10871076).

## References

[1] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.

[2] L. Yang, *Value Distribution Theory*, Science Press, Beijing, China, 1993.

[3] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, vol. 557 of *Mathematics and Its Applications*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 2003.

[4] M. Ozawa, "On a solution of $w'' + e^{-z}w' + (az+b)w = 0$," *Kodai Mathematical Journal*, vol. 3, no. 2, pp. 295–309, 1980.

[5] G. G. Gundersen, "On the question of whether $f'' + e^{-z}f' + B(z)f = 0$ can admit a solution $f \not\equiv 0$ of finite order," *Proceedings of the Royal Society of Edinburgh A*, vol. 102, no. 1-2, pp. 9–17, 1986.

[6] I. Amemiya and M. Ozawa, "Non-existence of finite order solutions of $w'' + e^{-z}w' + Q(z)w = 0$," *Hokkaido Mathematical Journal*, vol. 10, pp. 1–17, 1981.

[7] J. K. Langley, "On complex oscillation and a problem of Ozawa," *Kodai Mathematical Journal*, vol. 9, no. 3, pp. 430–439, 1986.

[8] Z. X. Chen, "The growth of solutions of $f'' + e^{-z}f' + Q(z)f = 0$ where the order $(Q) = 1$," *Science in China A*, vol. 45, no. 3, pp. 290–300, 2002.

[9] J. Wang and W. R. Lü, "The fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients," *Acta Mathematicae Applicatae Sinica*, vol. 27, no. 1, pp. 72–80, 2004 (Chinese).

[10] Z.-B. Huang, Z.-X. Chen, and Q. Li, "Subnormal solutions of second-order nonhomogeneous linear differential equations with periodic coefficients," *Journal of Inequalities and Applications*, vol. 2009, Article ID 416273, pp. 1–12, 2009.

[11] T. Cheng and Y.-M. Kang, "The growth of the solution of a certain linear differential equation," *Journal of Fudan University: Natural Science*, vol. 45, no. 5, pp. 611–618, 2006.

[12] G. G. Gundersen, "Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates," *Journal of the London Mathematical Society*, vol. 37, no. 1, pp. 88–104, 1988.

[13] W. K. Hayman, "The local growth of power series: a survey of the Wiman-Valiron method," *Canadian Mathematical Bulletin*, vol. 17, no. 3, pp. 317–358, 1974.