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Research Article

Littlewood-Paley g-Functions and Multipliers for the Laguerre Hypergroup

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Let $L = -(\partial^2/\partial x^2 + (2\alpha + 1/x)(\partial/\partial x) + x^2(\partial^2/\partial t^2))$; $(x,t) \in (0,+\infty) \times \mathbb{R}$, where $\alpha \geq 0$. Then L can generate a hypergroup which is called Laguerre hypergroup, and we denote this hypergroup by $\mathbf{K}$. In this paper, we will consider the Littlewood-Paley g-functions on $\mathbf{K}$ and then we use it to prove the Hölmander multipliers on $\mathbf{K}$.

1. Introduction and Preliminaries

In [1], the authors investigated Littlewood-Paley g-functions for the Laguerre semigroup. Let

$$\mathcal{L}_\alpha = \sum_{i=1}^{d} x_i \frac{\partial^2}{\partial_{x_i^2}} + (\alpha_i + 1 - x_i)\frac{\partial}{\partial_{x_i}}, \tag{1.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $x_i > 0$, then define the following Littlewood-Paley function $\mathcal{G}_\alpha$ by

$$\mathcal{G}_\alpha f(x) = \left(\int_0^\infty \left|t\nabla_\alpha P_t^\alpha f(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \tag{1.2}$$

where $\nabla_\alpha = (\partial_t, \sqrt{x_1}\partial_{x_1}, \dots, \sqrt{x_d}\partial_{x_d})$ and P_t^α is the Poisson semigroup associated to $\mathcal{L}_\alpha$. In [1], the authors prove that $\mathcal{G}_\alpha$ is bounded on $L^p(\mu_\alpha)$ for $1 < p < \infty$. In this paper, we consider the following differential operator

$$L = -\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}\right); \quad (x,t) \in (0,+\infty) \times \mathbb{R}, \tag{1.3}$$

where $\alpha \geq 0$. It is well known that it can generate a hypergroup (cf. [2, 3] or [4]). We will define Littlewood-Paley g-functions associated to L and prove that they are bounded on $L^p(\mathbf{K})$ for $1 < p < \infty$. As an application, we use it to prove the Hömander multiplier theorem on $\mathbf{K}$.

Let $\mathbf{K} = [0, \infty) \times \mathbf{R}$ equipped with the measure

$$dm_\alpha(x,t) = \frac{1}{\pi\Gamma(\alpha+1)} x^{2\alpha+1} dx dt, \quad \alpha \geq 0. \tag{1.4}$$

We denotes by $L^p_\alpha(\mathbf{K})$ the spaces of measurable functions on $\mathbf{K}$ such that $\|f\|_{\alpha,p} < +\infty$, where

$$\begin{aligned} \|f\|_{\alpha,p} &= \left(\int_{\mathbf{K}} |f(x,t)|^p \, dm_\alpha(x,t)\right)^{1/p}, \quad 1 \leq p < \infty, \\ \|f\|_{\alpha,\infty} &= \operatorname{esssup}_{(x,t)\in\mathbf{K}} |f(x,t)|. \end{aligned} \tag{1.5}$$

For $(x,t) \in \mathbf{K}$, the generalized translation operators $T^{(\alpha)}_{(x,t)}$ are defined by

$$\begin{aligned} &T^{(\alpha)}_{(x,t)} f(y,s) \\ &= \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\left(\sqrt{x^2+y^2+2xy\cos\theta}, s+t+xy\sin\theta\right) d\theta, & \text{if } \alpha = 0, \\ \dfrac{\alpha}{\pi}\displaystyle\int_0^{2\pi}\int_0^1 f\left(\sqrt{x^2+y^2+2xyr\cos\theta}, s+t+xyr\sin\theta\right) r\left(1-r^2\right)^{\alpha-1} dr d\theta, & \text{if } \alpha > 0. \end{cases} \end{aligned} \tag{1.6}$$

It is known that $T^{(\alpha)}_{(x,t)}$ satisfies

$$\|T^{(\alpha)}_{(x,t)} f\|_{\alpha,p} \leq \|f\|_{\alpha,p}. \tag{1.7}$$

Let $M_b(\mathbf{K})$ denote the space of bounded Radon measures on $\mathbf{K}$. The convolution on $M_b(\mathbf{K})$ is defined by

$$(\mu * \nu)(f) = \int_{\mathbf{K}\times\mathbf{K}} T^{(\alpha)}_{(x,t)} f(y,s) d\mu(x,t) d\nu(y,s). \tag{1.8}$$

It is easy to see that $\mu * \nu = \nu * \mu$. If $f, g \in L^1_\alpha(\mathbf{K})$ and $\mu = f m_\alpha$, $\nu = g m_\alpha$, then $\mu * \nu = (f * g) m_\alpha$, where $f * g$ is the convolution of functions f and g defined by

$$(f * g)(x,t) = \int_{\mathbf{K}} T^{(\alpha)}_{(x,t)} f(y,s) g(y,-s) dm_\alpha(y,s). \tag{1.9}$$

The following lemma follows from (1.7).

Lemma 1.1. *Let $f \in L^1_\alpha(\mathbf{K})$ and $g \in L^p_\alpha(\mathbf{K})$, $1 \le p \le \infty$. Then*

$$\|f * g\|_{\alpha,p} \le \|f\|_{\alpha,1}\|g\|_{\alpha,p}. \tag{1.10}$$

$(\mathbf{K}, *, i)$ is a hypergroup in the sense of Jewett (cf. [5, 6]), where i denotes the involution defined by $i(x,t) = (x,-t)$. If $\alpha = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbf{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathbf{H}^n$.

The dilations on $\mathbf{K}$ are defined by

$$\delta_r(x,t) = \left(rx, r^2 t\right), \quad r > 0. \tag{1.11}$$

It is clear that the dilations are consistent with the structure of hypergroup. Let

$$f_r(x,t) = r^{-(2\alpha+4)} f\left(\frac{x}{r}, \frac{t}{r^2}\right). \tag{1.12}$$

Then we have

$$\|f_r\|_{\alpha,1} = \|f\|_{\alpha,1}. \tag{1.13}$$

We also introduce a homogeneous norm defined by $\|(x,t)\| = (x^4 + 4t^2)^{1/4}$ (cf. [7]). Then we can defined the ball centered at $(0,0)$ of radius r, that is, the set $B_r = \{(x,t) \in \mathbf{K} : \ \|(x,t)\| < r\}$.

Let $f \in L^1_\alpha(\mathbf{K})$. Set $x = \rho(\cos\theta)^{1/2}$, $t = 1/2\rho^2 \sin\theta$. We get

$$\int_{\mathbf{K}} f(x,t)dm_\alpha(x,t) = \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} \int_0^\infty f\left(\rho(\cos\theta)^{1/2}, \frac{1}{2}\rho^2 \sin\theta\right)\rho^{2\alpha+3}(\cos\theta)^\alpha \ d\rho d\theta. \tag{1.14}$$

If f is radial, that is, there ia a function φ on $[0,\infty)$ such that $f(x,t) = \varphi(\|(x,t)\|)$, then

$$\begin{aligned} \int_{\mathbf{K}} f(x,t)dm_\alpha(x,t) &= \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} (\cos\theta)^\alpha \ d\theta \int_0^\infty \varphi(\rho)\rho^{2\alpha+3} \ d\rho \\ &= \frac{\Gamma((\alpha+1)/2)}{2\sqrt{\pi}\,\Gamma(\alpha+1)\Gamma(\alpha/2+1)} \int_0^\infty \varphi(\rho)\rho^{2\alpha+3} \ d\rho. \end{aligned} \tag{1.15}$$

Specifically,

$$m_\alpha(B_r) = \frac{\Gamma((\alpha+1)/2)}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha/2+1)} r^{2\alpha+4}. \tag{1.16}$$

We consider the partial differential operator

$$L = -\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}\right). \tag{1.17}$$

L is positive and symmetric in $L^2_\alpha(\mathbf{K})$, and is homogeneous of degree 2 with respect to the dilations defined above. When $\alpha = n-1$, L is the radial part of the sublaplacian on the Heisenberg group $\mathbf{H}^n$. We call L the generalized sublaplacian.

Let $L_m^{(\alpha)}$ be the Laguerre polynomial of degree m and order α defined in terms of the generating function by

$$\sum_{m=0}^{\infty} s^m L_m^{(\alpha)}(x) = \frac{1}{(1-s)^{\alpha+1}} \exp\left(-\frac{xs}{1-s}\right). \tag{1.18}$$

For $(\lambda, m) \in \mathbf{R} \times \mathbf{N}$, we put

$$\varphi_{(\lambda,m)}(x,t) = \frac{m!\,\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} e^{i\lambda t} e^{-(1/2)|\lambda|x^2} L_m^{(\alpha)}\left(|\lambda|x^2\right). \tag{1.19}$$

The following proposition summarizes some basic properties of functions $\varphi_{(\lambda,m)}$.

Proposition 1.2. *The function $\varphi_{(\lambda,m)}$ satisfies that*

(*a*) $\|\varphi_{(\lambda,m)}\|_{\alpha,\infty} = \varphi_{(\lambda,m)}(0,0) = 1$,

(*b*) $\varphi_{(\lambda,m)}(x,t)\ \varphi_{(\lambda,m)}(y,s) = T^{(\alpha)}_{(x,t)}\varphi_{(\lambda,m)}(y,s)$,

(*c*) $L\varphi_{(\lambda,m)} = |\lambda|(4m+2\alpha+2)\varphi_{(\lambda,m)}$.

Let $f \in L^1_\alpha(\mathbf{K})$, the generalized Fourier transform of f is defined by

$$\widehat{f}(\lambda, m) = \int_{\mathbf{K}} f(x,t)\varphi_{(-\lambda,m)}(x,t)\,dm_\alpha(x,t). \tag{1.20}$$

It is easy to show that

$$\begin{gathered}(f * g)\widehat{\ }(\lambda,m) = \widehat{f}(\lambda,m)\widehat{g}(\lambda,m), \\ \widehat{f}_r(\lambda,m) = \widehat{f}\left(r^2\lambda, m\right).\end{gathered} \tag{1.21}$$

Let $d\gamma_\alpha$ be the positive measure defined on $\mathbf{R} \times \mathbf{N}$ by

$$\int_{\mathbf{R}\times\mathbf{N}} g(\lambda,m)\,d\gamma_\alpha(\lambda,m) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)} \int_{\mathbf{R}} g(\lambda,m)|\lambda|^{\alpha+1} d\lambda. \tag{1.22}$$

Write $L^p_\alpha(\widehat{\mathbf{K}})$ instead of $L^p(\mathbf{R}\times\mathbf{N}, d\gamma_\alpha)$. We have the following Plancherel formula:

$$\|f\|_{\alpha,2} = \|\widehat{f}\|_{L^2_\alpha(\widehat{\mathbf{K}})}, \quad f \in L^1_\alpha(\mathbf{K}) \cap L^2_\alpha(\mathbf{K}). \tag{1.23}$$

Then the generalized Fourier transform can be extended to the tempered distributions. We also have the inverse formula of the generalized Fourier transform.

$$f(x,t) = \int_{\mathbf{R}\times\mathbf{N}} \widehat{f}(\lambda,m)\varphi_{(\lambda,m)}(x,t)d\gamma_{\alpha}(\lambda,m) \tag{1.24}$$

provided $\widehat{f} \in L^1_{\alpha}(\widehat{\mathbf{K}})$.

In the following, we give some basic notes about the heat and Poisson kernel whose proofs can be found in [8]. Let $\{H^s\} = \{e^{-sL}\}$ be the heat semigroup generated by L. There is a unique smooth function $h((x,t),s) = h_s(x,t)$ on $\mathbf{K} \times (0,+\infty)$ such that

$$H^s f(x,t) = f * h_s(x,t). \tag{1.25}$$

We call h_s is the heat kernel associated to L. We have

$$\begin{gathered} h_s(x,t) = \int_{\mathbf{R}} \left(\frac{\lambda}{2\sinh(2\lambda s)}\right)^{\alpha+1} e^{-(1/2)\lambda\coth(2\lambda s)x^2} e^{i\lambda t} d\lambda, \\ h_s(x,t) \leq C s^{-\alpha-2} e^{-(A/s)\|(x,t)\|^2}. \end{gathered} \tag{1.26}$$

Let $\{P^s\} = \{e^{-s\sqrt{L}}\}$ be the Poisson semigroup. There is a unique smooth function $p((x,t),s) = p_s(x,t)$ on $\mathbf{K} \times (0,+\infty)$, which is called the Poisson kernel, such that

$$P^s f(x,t) = f * p_s(x,t). \tag{1.27}$$

The Poisson kernel can be calculated by the subordination. In fact, we have

$$\begin{aligned} &p_s(x,t) \\ &\quad = \frac{4s}{\sqrt{\pi}}\,\Gamma\left(\alpha+\frac{5}{2}\right)\int_0^{\infty}\left(\frac{\lambda}{\sinh\lambda}\right)^{\alpha+1}\left(\left(s^2+x^2\lambda\coth\lambda\right)^2+(2\lambda t)^2\right)^{-(2\alpha+5)/4} \\ &\qquad\qquad \times\cos\left(\left(\alpha+\frac{5}{2}\right)\arctan\left(\frac{2\lambda t}{s^2+x^2\lambda\coth\lambda}\right)\right)d\lambda, \\ &p_s(x,t) \leq C\ s\left(s^2+\|(x,t)\|^2\right)^{-(\alpha+5/2)}. \end{aligned} \tag{1.28}$$

The heat maximal function M_H is defined by

$$M_H f(x,t) = \sup_{s>0}\left|H^s f(x,t)\right| = \sup_{s>0}\left|(f * h_s)(x,t)\right|. \tag{1.29}$$

The Poisson maximal function M_P is defined by

$$M_P f(x,t) = \sup_{s>0}\left|P^s f(x,t)\right| = \sup_{s>0}\left|(f * p_s)(x,t)\right|. \tag{1.30}$$

The Hardy-Littlewood maximal function is defined by

$$M_B f(x,t) = \sup_{r>0} \frac{1}{m_\alpha(B_r)} \int_{B_r} T^{(\alpha)}_{(x,t)}(|f|)(y,s)\,dm_\alpha(y,s) = \sup_{r>0}(|f| * b_r)(x,t), \tag{1.31}$$

where $b(x,t) = (1/(m_\alpha(B_1)))\chi_{B_1}(x,t)$.

The following proposition is the main result of [8].

Proposition 1.3. *M_B M_P and M_B are operators on $\mathbf{K}$ of weak type $(1,1)$ and strong type (p,p) for $1 < p \leq \infty$.*

The paper is organized as follows. In the second section, we prove that Littlewood-Paley g-functions are bounded operators on $L^p_\alpha(\mathbf{K})$. As an application, we prove the Hörmander multiplier theorem on $\mathbf{K}$ in the last section.

Throughout the paper, we will use C to denote the positive constant, which is not necessarily same at each occurrence.

2. Littlewood-Paley g-Function on K

Let $k \in \mathbb{N}$, then we define the following $\mathcal{G}$-function and g^*_λ-function

$$\begin{aligned} g_k(f)^2(x,t) &= \int_0^\infty \left|\partial_s^k P^s f(x,t)\right|^2 s^{2k-1}\,ds, \\ g^*_k(f)^2(x,t) &= \int_0^\infty \left(\int_{\mathbf{K}} s^{-(\alpha+1)}\left(1+s^{-2}\|(y,r)\|^4\right)^{-k}\left|\partial_s P^s T^{(\alpha)}_{(y,r)} f(x,t)\right|^2 dm_\alpha(y,r)\right) ds. \end{aligned} \tag{2.1}$$

Then, we can prove

Theorem 2.1. (a) *For $k \in \mathbb{N}$ and $f \in L^2(\mathbf{K})$, there exists $C_k > 0$ such that*

$$\|g_k(f)\|_{\alpha,2} = C_k \|f\|_{\alpha,2}. \tag{2.2}$$

(b) *For $1 < p < \infty$ and $f \in L^p(\mathbf{K})$, there exist positive constants C_1 and C_2, such that*

$$C_1\|f\|_{\alpha,p} \leq \|g_k(f)\|_{\alpha,p} \leq C_2\|f\|_{\alpha,p}. \tag{2.3}$$

(c) *If $k > (\alpha+2)/2$ and $f \in L^p(\mathbf{K})$, $p > 2$, then there exists a constant $C > 0$ such that*

$$\|g^*_k(f)\|_{\alpha,p} \leq C\|f\|_{\alpha,p}. \tag{2.4}$$

Proof. (a) When $k \in \mathbb{N}$, by the Plancherel theorem for the Fourier transform on $\mathbf{K}$,

$$\begin{aligned}\|g_k(f)\|_{\alpha,2}^2 &= \int_{\mathbf{K}}\left(\int_0^\infty \left|\partial_s^k P^s f(x,t)\right|^2 s^{2k-1}ds\right)dm_\alpha(x,t)\\ &= \int_0^\infty\left(\int_{\mathbf{R}\times\mathbf{N}}\left|\left(\partial_s^k P^s f\right)^\frown(\lambda,m)\right|^2 d\gamma_\alpha(\lambda,m)\right)s^{2k-1}ds\\ &= \int_0^\infty\left(\int_{\mathbf{R}}\sum_{m=0}^\infty\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}\left|\left(\partial_s^k P^s f\right)^\frown(\lambda,m)\right|^2|\lambda|^{\alpha+1}d\lambda\right)s^{2k-1}ds.\end{aligned} \tag{2.5}$$

Since

$$\left(\partial_s^k P^s f\right)^\frown(\lambda,m) = \widehat{f}(\lambda,m)\left(-\sqrt{(4m+2\alpha+2)|\lambda|}\right)^k e^{-s\sqrt{(4m+2\alpha+2)|\lambda|}}, \tag{2.6}$$

we get

$$\begin{aligned}&\|g_k(f)\|_{\alpha,2}^2\\ &= \int_0^\infty\left(\int_{\mathbf{R}}\sum_{m=0}^\infty\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}\left|\widehat{f}(\lambda,m)\right|^2((4m+2\alpha+2)|\lambda|)^k e^{-2s\sqrt{(4m+2\alpha+2)|\lambda|}}|\lambda|^{\alpha+1}d\lambda\right)s^{2k-1}ds.\end{aligned} \tag{2.7}$$

By

$$\int_0^\infty e^{-2s\sqrt{(4m+2\alpha+2)|\lambda|}}s^{2k-1}ds = C_k((4m+2\alpha+2)|\lambda|)^{-k}, \tag{2.8}$$

we have

$$\|g_k(f)\|_{\alpha,2}^2 = C_k\int_{\mathbf{R}}\sum_{m=0}^\infty\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}\left|\widehat{f}(\lambda,m)\right|^2|\lambda|^{\alpha+1}d\lambda = C_k\|f\|_{\alpha,2}^2. \tag{2.9}$$

Therefore

$$\|g_k(f)\|_{\alpha,2} = C_k\|f\|_{\alpha,2}. \tag{2.10}$$

(b) As $\{P^s\}$ is a contraction semigroup (cf. Proposition 5.1 in [3]), we can get $\|g_k(f)\|_{\alpha,p} \le C_2\|f\|_{\alpha,p}$ (cf. [9]). For the reverse, we can prove by polarization to the identity and (a) (cf. [10]).

(c) We first prove

$$\int_{\mathbf{K}} g_k^*(f)^2(x,t)\varphi(x,t)dm_\alpha(x,t) \le C\int_{\mathbf{K}} g_1(f)^2(x,t)M_B\varphi(x,t)dm_\alpha(x,t), \tag{2.11}$$

where $0 \le \varphi \in L_\alpha^q(\mathbf{K})$ and $\|\varphi\|_{\alpha,q} \le 1$, $1/q + 2/p = 1$.

Since $k > (\alpha+2)/2$, we know

$$\int_{\mathbf{K}} (1 + \|(y,r)\|^4)^{-k} dm_\alpha(y,r) < \infty. \tag{2.12}$$

By Proposition 1.3,

$$\begin{aligned}
&\int_{\mathbf{K}} g_k^*(f)^2(x,t)\psi(x,t)dm_\alpha(x,t)\\
&= \int_{\mathbf{K}} \left(\int_0^\infty \int_{\mathbf{K}} s^{-(\alpha+1)} \left(1 + s^{-2}\|(y,r)\|^4\right)^{-k} \left|\partial_s P^s T_{(y,r)}^{(\alpha)} f(x,t)\right|^2 dm_\alpha(y,r)ds \right) \psi(x,t)dm_\alpha(x,t)\\
&= \int_0^\infty \int_{\mathbf{K}} s^{-(\alpha+1)} \left|\partial_s P^s f(y,r)\right|^2 \left(\int_{\mathbf{K}} T_{(x,t)}^{(\alpha)} \left(1 + s^{-2}\|(y,r)\|^4\right)^{-k} \psi(x,t)dm_\alpha(x,t) \right) dm_\alpha(y,r)ds\\
&\leq C \int_{\mathbf{K}} g_1(f)^2(y,r) M_B\psi(y,r)dm_\alpha(y,r)\\
&\leq C\|g_1(f)\|_{\alpha,p}^2 \|M_B\psi\|_{\alpha,q} \leq C\|f\|_{\alpha,p}^2.
\end{aligned} \tag{2.13}$$

Therefore $\|g_k^*(f)\|_{\alpha,p} \leq C\|f\|_{\alpha,p}$. This gives the proof of Theorem 2.1. □

We can also consider the Littlewood-Paley g-function that is defined by the heat semigroup as follows: let $k \in \mathbb{N}$, we define

$$\begin{aligned}
\mathcal{G}_k^H(f)^2(x,t) &= \int_0^\infty \left|\partial_s^k H^s f(x,t)\right|^2 s^{2k-1} ds,\\
\mathcal{G}_k^{H,*}(f)^2(x,t) &= \int_0^\infty \left(\int_{\mathbf{K}} s^{-(\alpha+1)} \left(1 + s^{-2}\|(y,r)\|^4\right)^{-k} \left|\partial_s H^s T_{(y,r)}^{(\alpha)} f(x,t)\right|^2 dm_\alpha(y,r) \right) ds.
\end{aligned} \tag{2.14}$$

Similar to the proof of Theorem 2.1, we can prove

Theorem 2.2. *(a) For $k \in \mathbb{N}$ and $f \in L^2(\mathbf{K})$, there exists $C_k > 0$ such that*

$$\|\mathcal{G}_k^H(f)\|_{\alpha,2} = C_k\|f\|_{\alpha,2}. \tag{2.15}$$

(b) For $1 < p < \infty$ and $f \in L^p(\mathbf{K})$, there exist constants C_1 and C_2, such that

$$C_1\|f\|_{\alpha,p} \leq \|\mathcal{G}_k^H(f)\|_{\alpha,p} \leq C_2\|f\|_{\alpha,p}. \tag{2.16}$$

(c) If $k > (\alpha+2)/2$ and $f \in L^p(\mathbf{K})$, $p > 2$, then $\|\mathcal{G}_k^{H,*}(f)\|_{\alpha,p} \leq C\|f\|_{\alpha,p}$.

By Theorem 2.2, we can get (cf. [10])

Corollary 2.3. *Let $k \in \mathbb{N}$ and $f \in L^2(\mathbf{K})$, if $\mathcal{G}_k^H(f) \in L^p(\mathbf{K})$, $1 < p < \infty$, then $f \in L^p(\mathbf{K})$ and there exists $C > 0$ such that*

$$C\|f\|_{\alpha,p} \leq \|\mathcal{G}_k^H(f)\|_{\alpha,p}. \tag{2.17}$$

3. Hörmander Multiplier Theorem on K

In this section, we prove the Hörmander multiplier theorem on **K**. The main tool we use is the Littlewood-Paley theory that we have proved.

We first introduce some notations. Assume Ψ is a function defined on $\mathbf{R} \times \mathbf{N}$, then let $\Delta_-\Psi(\lambda, 0) = \Psi(\lambda, 0)$ and for $m \geq 1$,

$$\begin{aligned} \Delta_-\Psi(\lambda, m) &= \Psi(\lambda, m) - \Psi(\lambda, m-1), \\ \Delta_+\Psi(\lambda, m) &= \Psi(\lambda, m+1) - \Psi(\lambda, m). \end{aligned} \tag{3.1}$$

Then we define the following differential operators:

$$\begin{aligned} \Lambda_1\Psi(\lambda, m) &= \frac{1}{|\lambda|}(m\Delta_-\Psi(\lambda, m) + (\alpha+1)\Delta_+\Psi(\lambda, m)), \\ \Lambda_2\Psi(\lambda, m) &= \frac{-1}{2\lambda}((\alpha+m+1)\Delta_+\Psi(\lambda, m) + m\Delta_-\Psi(\lambda, m)). \end{aligned} \tag{3.2}$$

We have the following lemma.

Lemma 3.1. *Let $g(\lambda, m) = ((4m+2\alpha+2)|\lambda|)e^{-(4m+2\alpha+2)|\lambda|s}h(\lambda, m)$, where $k \in \mathbb{N}$, $h(\lambda, m)$ is a $([(\alpha+1)/2]+1)$ times differentiable function on $\mathbb{R}^2$ and satisfies*

$$\left|\left(\Lambda_1 + 2\left(\Lambda_2 + \frac{\partial}{\partial\lambda}\right)\right)^j h(\lambda, m)\right| \leq C_j((4m+2\alpha+2)|\lambda|)^{-j} \tag{3.3}$$

for $j = 0, 1, 2, \ldots, [(\alpha+1)/2]+1$. Then one has

$$\left|\left(\Lambda_1 + 2\left(\Lambda_2 + \frac{\partial}{\partial\lambda}\right)\right)g(\lambda, m)\right| \leq C\max\left\{\frac{1}{|\lambda|s}, 1 + \frac{m}{|\lambda|s}\right\}e^{-\epsilon(4m+2\alpha+2)|\lambda|s}, \tag{3.4}$$

where $0 < \epsilon < 1$ and $s > 0$.

Proof. Without loss of the generality, we can assume that $\lambda > 0$. when $m = 0$, we have

$$\Lambda_1 + 2\left(\Lambda_2 + \frac{\partial}{\partial\lambda}\right) = 2\frac{\partial}{\partial\lambda}. \tag{3.5}$$

It is easy to calculate

$$\left|\frac{\partial}{\partial\lambda}g(\lambda,0)\right| \le C\frac{1}{\lambda s}e^{-\epsilon(4m+2\alpha+2)\lambda s}. \tag{3.6}$$

When $m \ge 1$, we have

$$\Lambda_1 + 2\left(\Lambda_2 + \frac{\partial}{\partial\lambda}\right) = 2\left(\frac{\partial}{\partial\lambda} - \frac{m}{\lambda}\Delta_{-1}\right). \tag{3.7}$$

Since

$$\begin{aligned}\left(\frac{\partial}{\partial\lambda} - \frac{m}{\lambda}\Delta_{-1}\right)g(\lambda,m) = {} & ((4m+2\alpha+2)|\lambda|)e^{-(4m+2\alpha+2)|\lambda|s}\left(\frac{\partial}{\partial\lambda} - \frac{m}{\lambda}\Delta_{-1}\right)h(\lambda,m) \\ & + \frac{\partial}{\partial\lambda}\left\{((4m+2\alpha+2)|\lambda|)e^{-(4m+2\alpha+2)|\lambda|s}\right\}h(\lambda,m) \\ & - \frac{m}{\lambda}\Delta_{-1}f(m)g(m-1),\end{aligned} \tag{3.8}$$

we get

$$\left|\left(\frac{\partial}{\partial\lambda} - \frac{m}{\lambda}\Delta_{-1}\right)g(\lambda,m)\right| \le C\left(1 + \frac{m}{\lambda s}\right)e^{-\epsilon(4m+2\alpha+2)\lambda s}. \tag{3.9}$$

Then Lemma 3.1 is proved. □

Then we can prove Hörmander multiplier theorem on the Laguerre hypergroup $\mathbf{K}$.

Theorem 3.2. *Let $h(\lambda,m)$ be a $([(\alpha+1)/2]+1)$ times differentiable function on $\mathbb{R}^2$ and satisfies*

$$\left|\left(\Lambda_1 + 2\left(\Lambda_2 + \frac{\partial}{\partial\lambda}\right)\right)^j h(\lambda,m)\right| \le C_j((4m+2\alpha+2)|\lambda|)^{-j} \tag{3.10}$$

for $j = 0,1,2,\ldots,[(\alpha+1)/2]+1$ and T is an operator which is defined by $\widehat{Tf}(\lambda,m) = h(\lambda,m)\widehat{f}(\lambda,m)$, then T is bounded on $L^p_\alpha(\mathbf{K})$, where $1 < p < \infty$.

Proof. We just prove the theorem for $2 < p < \infty$, for $1 < p < 2$; we can get the result by the dual theorem. By Theorem 2.2, Corollary 2.3 and the note that $Tf \in L^2(\mathbf{K})$, it is sufficient to prove the following:

$$\mathcal{G}_2^H(Tf)(x,t) \le C\mathcal{G}_1^{H,*}(f)(x,t), \quad (x,t) \in \mathbf{K}. \tag{3.11}$$

Let $u_s = H^s f$ and $U^s = H^s(Tf)$, then we can get

$$U^{s+t} = G_t * u_s(x,t), \tag{3.12}$$

where $\widehat{G}_t(\lambda,m) = e^{-2(2m+\alpha+1)|\lambda|t}h(\lambda,m)$.

Differentiating (3.12) with respect to t and s, then assuming that $t = s$, we can get

$$\partial_s^2 H^{2s}(Tf) = F_s * \partial_s H^s f, \tag{3.13}$$

where

$$\widehat{F}_s(\lambda,m) = -((4m+2\alpha+2)|\lambda|)e^{-(4m+2\alpha+2)|\lambda|s}h(\lambda,m). \tag{3.14}$$

Therefore

$$\left|\partial_s^2 H^{2s}(Tf)(x,t)\right| \le \int_{\mathbf{K}} F_s(y,r)\left|T_{(x,t)}^{(\alpha)}\partial_s H^s f(y,r)\right| dm_\alpha(y,r). \tag{3.15}$$

By the Cauchy-Schwartz inequality,

$$\left|\partial_s^2 H^{2s}(Tf)(x,t)\right|^2 \le A(s)\int_{\mathbf{K}}\left(1+s^{-2}\|(y,r)\|^4\right)^{-1}\left|T_{(x,t)}^{(\alpha)}\partial_s H^s f(y,r)\right|^2 dm_\alpha(y,r), \tag{3.16}$$

where

$$A(s) = \int_{\mathbf{K}}\left(1+s^{-2}\|(x,t)\|^4\right)|F_s(x,t)|^2 dm_\alpha(x,t). \tag{3.17}$$

In the following, we prove

$$A(s) \le Cs^{-\alpha-3}. \tag{3.18}$$

We write

$$\begin{aligned} A(s) &= \int_{\|(x,t)\|\le\sqrt{s}}\left(1+s^{-2}\|(x,t)\|^4\right)|F_s(x,t)|^2 dm_\alpha(x,t) \\ &\quad + \int_{\|(x,t)\|>\sqrt{s}}\left(1+s^{-2}\|(x,t)\|^4\right)|F_s(x,t)|^2 dm_\alpha(x,t) \\ &= A_1(s) + A_2(s). \end{aligned} \tag{3.19}$$

For $A_1(s)$, we can easily get

$$\begin{aligned}
A_1(s) &\le C\int_{\mathbf{K}} |F_s(x,t)|^2 dm_\alpha(x,t) = C\int_{\mathbf{R}\times\mathbf{N}} \left|\widehat{F}_s(\lambda,m)\right|^2 d\gamma_\alpha(\lambda,m)\\
&= C\int_{\mathbf{R}\times\mathbf{N}} ((4m+2\alpha+2)|\lambda|)^2 e^{-(8m+4\alpha+4)|\lambda|s} h^2(\lambda,m) d\gamma_\alpha(\lambda,m)\\
&\le C\int_{\mathbf{R}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)} ((4m+2\alpha+2)|\lambda|)^2 e^{-(8m+4\alpha+4)|\lambda|s} |\lambda|^{\alpha+1} d\lambda\\
&= Cs^{-\alpha-4}\int_{\mathbf{R}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)} ((4m+2\alpha+2)|\lambda|)^2 e^{-(8m+4\alpha+4)|\lambda|} |\lambda|^{\alpha+1} d\lambda\\
&\le Cs^{-\alpha-4} \sum_{m=0}^{\infty} (4m+2\alpha+2)^{-2} \le Cs^{-\alpha-4}.
\end{aligned} \tag{3.20}$$

For $A_2(s)$, we have

$$\begin{aligned}
A_2(s) &\le Cs^{-2}\int_{\mathbf{K}} \left(4t^2+x^4\right)|F_s(x,t)|^2 dm_\alpha(x,t)\\
&= Cs^{-2}\int_{\mathbf{K}} \left|\left(2it-|x|^2\right)F_s(x,t)\right|^2 dm_\alpha(x,t)\\
&= Cs^{-2}\int_{\mathbf{R}\times\mathbf{N}} \left|\left(\Lambda_1+2\left(\Lambda_2+\frac{\partial}{\partial\lambda}\right)\right)\widehat{F}_s(\lambda,m)\right|^2 d\gamma_\alpha(\lambda,m).
\end{aligned} \tag{3.21}$$

By Lemma 3.1,

$$\left(\Lambda_1+2\left(\Lambda_2+\frac{\partial}{\partial\lambda}\right)\right)\widehat{F}_s(\lambda,m)\Bigg| \le C\max\left\{\frac{1}{|\lambda|s}, 1+\frac{m}{|\lambda|s}\right\} e^{-\epsilon(4m+2\alpha+2)|\lambda|s}, \tag{3.22}$$

where $0<\epsilon<1$.

So

$$\begin{aligned}
A_2(s) &\le Cs^{-2}\int_{\mathbf{R}\times\mathbf{N}} e^{-\epsilon(8m+4\alpha+4)|\lambda|s} d\gamma_\alpha(\lambda,m)\\
&= Cs^{-\alpha-4}\int_{\mathbf{R}\times\mathbf{N}} e^{-\epsilon(8m+4\alpha+4)|\lambda|} d\gamma_\alpha(\lambda,m)\\
&\le Cs^{-\alpha-4}.
\end{aligned} \tag{3.23}$$

Therefore (3.18) holds. Then

$$\left|\partial_s^2 H^{2s}(Tf)(x,t)\right|^2 \le Cs^{-\alpha-4}\int_{\mathbf{K}} \left(1+s^{-2}\|(y,r)\|^4\right)^{-1} \left|T_{(x,t)}^{(\alpha)}\partial_s H^s f(y,r)\right|^2 dm_\alpha(y,r). \tag{3.24}$$

Integrating the both sides of the above inequality with $s^3 ds$, we have

$$\mathcal{G}_2^H(x,t) \leq C\mathcal{G}_1^{H,*}(f)(x,t). \tag{3.25}$$

Then Theorem 3.2 is proved. □

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