

Research Article

Tightly Proper Efficiency in Vector Optimization with Nearly Cone-Subconvexlike Set-Valued Maps

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Received 26 September 2010; Revised 17 December 2010; Accepted 7 January 2011

Academic Editor: Kok Teo

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A scalarization theorem and two Lagrange multiplier theorems are established for tightly proper efficiency in vector optimization involving nearly cone-subconvexlike set-valued maps. A dual is proposed, and some duality results are obtained in terms of tightly properly efficient solutions. A new type of saddle point, which is called tightly proper saddle point of an appropriate set-valued Lagrange map, is introduced and is used to characterize tightly proper efficiency.

1. Introduction

One important problem in vector optimization is to find efficient points of a set. As observed by Kuhn, Tucker and later by Geoffrion, some efficient points exhibit certain abnormal properties. To eliminate such abnormal efficient points, there are many papers to introduce various concepts of proper efficiency; see [1–8]. Particularly, Zaffaroni [9] introduced the concept of tightly proper efficiency and used a special scalar function to characterize the tightly proper efficiency, and obtained some properties of tightly proper efficiency. Zheng [10] extended the concept of superefficiency from normed spaces to locally convex topological vector spaces. Guerraggio et al. [11] and Liu and Song [12] made a survey on a number of definitions of proper efficiency and discussed the relationships among these efficiencies, respectively.

Recently, several authors have turned their interests to vector optimization of set-valued maps, for instance, see [13–18]. Gong [19] discussed set-valued constrained vector optimization problems under the constraint ordering cone with empty interior. Sach [20] discussed the efficiency, weak efficiency and Benson proper efficiency in vector optimization problem involving ic-cone-convexlike set-valued maps. Li [21] extended the concept of Benson proper efficiency to set-valued maps and presented two scalarization theorems

and Lagrange multiplier theorems for set-valued vector optimization problem under cone-subconvexlikeness. Mehra [22], Xia and Qiu [23] discussed the superefficiency in vector optimization problem involving nearly cone-convexlike set-valued maps, nearly cone-subconvexlike set-valued maps, respectively. For other results for proper efficiencies in optimization problems with generalized convexity and generalized constraints, we refer to [24–26] and the references therein.

In this paper, inspired by [10, 21–23], we extend the concept of tight properness from normed linear spaces to locally convex topological vector spaces, and study tightly proper efficiency for vector optimization problem involving nearly cone-subconvexlike set-valued maps and with nonempty interior of constraint cone in the framework of locally convex topological vector spaces.

The paper is organized as follows. Some concepts about tightly proper efficiency, superefficiency and strict efficiency are introduced and a lemma is given in Section 2. In Section 3, the relationships among the concepts of tightly proper efficiency, strict efficiency and superefficiency in local convex topological vector spaces are clarified. In Section 4, the concept of tightly proper efficiency for set-valued vector optimization problem is introduced and a scalarization theorem for tightly proper efficiency in vector optimization problems involving nearly cone-subconvexlike set-valued maps is obtained. In Section 5, we establish two Lagrange multiplier theorems which show that tightly properly efficient solution of the constrained vector optimization problem is equivalent to tightly properly efficient solution of an appropriate unconstrained vector optimization problem. In Section 6, some results on tightly proper duality are given. Finally, a new concept of tightly proper saddle point for set-valued Lagrangian map is introduced and is then utilized to characterize tightly proper efficiency in Section 7. Section 8 contains some remarks and conclusions.

2. Preliminaries

Throughout this paper, let X be a linear space, Y and Z be two real locally convex topological spaces (in brief, LCTS), with topological dual spaces Y^* and Z^* , respectively. For a set $A \subset Y$, $\text{cl } A$, $\text{int } A$, ∂A , and A^c denote the closure, the interior, the boundary, and the complement of A , respectively. Moreover, by B we denote the closed unit ball of Y . A set $C \subset Y$ is said to be a cone if $\lambda c \in C$ for any $c \in C$ and $\lambda \geq 0$. A cone C is said to be convex if $C + C \subset C$, and it is said to be pointed if $C \cap (-C) = \{0\}$. The generated cone of C is defined by

$$\text{cone } C := \{\lambda c \mid \lambda \geq 0, c \in C\}. \quad (2.1)$$

The dual cone of C is defined as

$$C^+ := \{\varphi \in Y^* \mid \varphi(c) \geq 0, \forall c \in C\} \quad (2.2)$$

and the quasi-interior of C^+ is the set

$$C^{+i} := \{\varphi \in Y^* \mid \varphi(c) > 0, \forall c \in C \setminus \{0_Y\}\}. \quad (2.3)$$

Recall that a base of a cone C is a convex subset of C such that

$$0_Y \notin \text{cl } B, \quad C = \text{cone } B. \quad (2.4)$$

Of course, C is pointed whenever C has a base. Furthermore, if C is a nonempty closed convex pointed cone in Y , then $C^{+i} \neq \emptyset$ if and only if C has a base.

Also, in this paper, we assume that, unless indicated otherwise, $C \subset Y$ and $D \subset Z$ are pointed closed convex cones with $\text{int } C \neq \emptyset$ and $\text{int } D \neq \emptyset$, respectively.

Definition 2.1 (see [27]). Let Θ be a base of C . Define

$$\Theta^{\text{st}} := \{\varphi \in Y^* : \exists t > 0 \text{ such that } \varphi(\theta) \geq t, \forall \theta \in \Theta\}. \quad (2.5)$$

Cheng and Fu in [27] discussed the propositions of Θ^{st} , and the following remark also gives some propositions of Θ^{st} .

Remark 2.2 (see [27]). (i) Let $\varphi \in Y^* \setminus \{0_{Y^*}\}$. Then $\varphi \in \Theta^{\text{st}}$ if and only if there exists a neighborhood U of 0_Y such that $\varphi(U - \Theta) < (\leq) 0$.

(ii) If Θ is a bounded base of C , then $\Theta^{\text{st}} = C^{+i}$.

Definition 2.3. A point $\bar{y} \in S \subset Y$ is said to be efficient with respect to C (denoted $\bar{y} \in E(S, C)$) if

$$(S - \bar{y}) \cap -C = \{0_Y\}. \quad (2.6)$$

Remark 2.4 (see [28]). If C is a closed convex pointed cone and $0_Y \in H \subset C$, then $E(S, C) = E(S + H, C)$.

In [10], Zheng generalized two kinds of proper efficiency, namely, Henig proper efficiency and superefficiency, from normed linear spaces to LCTS. And Fu [8] generalized a kind of proper efficiency, namely strict efficiency, from normed linear spaces to LCTS. Let C be an ordering cone with a base Θ . Then $0_Y \notin \text{cl } \Theta$, by the Hahn Banach separation theorem, there are a $f_\Theta \in Y^*$ and an $\alpha > 0$ such that

$$\alpha = \inf\{f_\Theta(\theta) \mid \theta \in \Theta\}. \quad (2.7)$$

Let $U_\Theta = \{y \in Y : |f_\Theta(y)| < \alpha/2\}$. Then U_Θ is a neighborhood of 0_Y and

$$\inf\{f_\Theta(y) : y \in \Theta + U_\Theta\} \geq \frac{\alpha}{2}. \quad (2.8)$$

It is clear that, for each convex neighborhood U of 0_Y with $U \subset U_\Theta$, $\Theta + U$ is convex and $0_Y \notin \text{cl}(\Theta + U)$. Obviously, $S_U(\Theta) := \text{cone}(U + \Theta)$ is convex pointed cone, indeed, $\Theta + U$ is also a base of $S_U(\Theta)$.

Definition 2.5 (see [8]). Suppose that S is a subset of Y and $B(C)$ denotes the family of all bases of C . \bar{y} is said to be a strictly efficient point with respect to $\Theta \in B(C)$, written as $\bar{y} \in \text{STE}(S, \Theta)$, if there is a convex neighborhood U of 0_Y such that

$$\text{cl cone}(S - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (2.9)$$

\bar{y} is said to be a strictly efficient point with respect to C , written as, $\bar{y} \in \text{STE}(S, C)$ if

$$\bar{y} \in \bigcap_{\Theta \in B(C)} \text{STE}(S, \Theta). \quad (2.10)$$

Remark 2.6. Since $U - \Theta$ is open in Y , thus $\text{cl cone}(S - \bar{y}) \cap (U - \Theta) = \emptyset$ is equivalent to $\text{cone}(S - \bar{y}) \cap (U - \Theta) = \emptyset$.

Definition 2.7. The point $\bar{y} \in S \subset Y$ is called tightly properly efficient with respect to $\Theta \in B(C)$ (denoted $\bar{y} \in \text{TPE}(S, \Theta)$) if there exists a convex cone $K \subset Y$ with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying $(S - \bar{y}) \cap -K = \{0_Y\}$ and there exists a neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta) = \emptyset. \quad (2.11)$$

\bar{y} is said to be a tightly properly efficient point with respect to C , written as, $\bar{y} \in \text{TPE}(S, C)$ if

$$\bar{y} \in \bigcap_{\Theta \in B(C)} \text{TPE}(S, \Theta). \quad (2.12)$$

Now, we give the following example to illustrate Definition 2.7.

Example 2.8. Let $Y = \mathbb{R}^2$, $S = \{(x, y) \in Y \mid -x \leq y \leq 1 \text{ and } x \leq 1\}$. Given C (see Figure 1). Thus, it follows from the direct computation and Definition 2.7 that

$$\text{TPE}(S, C) = \{(x, y) \mid y = -x, -1 \leq x \leq 1\}. \quad (2.13)$$

Remark 2.9. By Definitions 2.7 and 2.3, it is easy to verify that

$$\text{TPE}(S, C) \subseteq E(S, C), \quad (2.14)$$

but, in general, the converse is not valid. The following example illustrates this case.

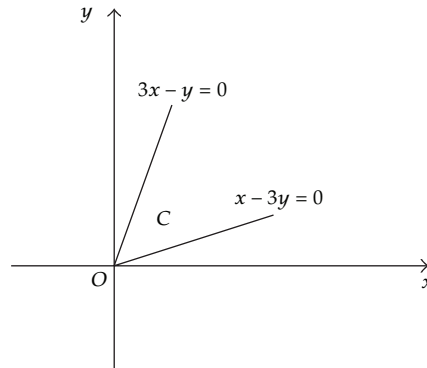


Figure 1: The set C.

Example 2.10. $Y = \mathbb{R}^2$, $S = \{(x, y) \in [0, 1] \times [0, 1] \mid y \geq 1 - \sqrt{1 - (x - 1)^2}\}$, and $C = \mathbb{R}_+^2$. Then, by Definitions 2.3 and 2.7, we get

$$E(S, C) = \left\{ (x, y) \mid y = 1 - \sqrt{1 - (x - 1)^2}, x \in [0, 1] \right\}, \tag{2.15}$$

$$\text{TPE}(S, C) = E(S, C) \setminus \{(0, 1), (1, 0)\},$$

thus, $E(S, C) \not\subseteq \text{TPE}(S, C)$.

Definition 2.11 (see [10]). $\bar{y} \in S$ is called a superefficient point of a subset S of Y with respect to ordering cone C , written as $\bar{y} \in \text{SE}(S, C)$, if, for each neighborhood V of 0_Y , there is neighborhood U of 0_Y such that

$$\text{cl cone}(S - \bar{y}) \cap (U - C) \subset V. \tag{2.16}$$

Definition 2.12 (see [29, 30]). A set-valued map $F : X \rightarrow 2^Y$ is said to be nearly C -subconvexlike on X if $\text{cl cone}(F(X) + C)$ is convex.

Given the two set-valued maps $F : X \rightarrow 2^Y, G : X \rightarrow 2^Z$, let

$$H(x) = (F(x), G(x)), \quad x \in X. \tag{2.17}$$

The product $F \times G$ is called nearly $C \times D$ -subconvexlike on X if H is nearly $C \times D$ -subconvexlike on X . Let $L(Z, Y)$ be the space of continuous linear operators from Z to Y , and let

$$L_+(Z, Y) = \{T \in L(Z, Y) : T(D) \subset C\}. \tag{2.18}$$

Denote by (F, G) the set-valued map from X to $Y \times Z$ defined by

$$(F, G)(x) = F(x) \times G(x). \tag{2.19}$$

If $\varphi \in Y^*$, $T \in L(Z, Y)$, we also define $\varphi F : X \rightarrow 2^R$ and $F + TG : X \rightarrow 2^Y$ by

$$(\varphi F)(x) = \varphi[F(x)], \quad (F + TG)(x) = F(x) + T[G(x)], \quad (2.20)$$

respectively.

Lemma 2.13 (see [23]). *If (F, G) is nearly $C \times D$ -subconvexlike on X , then:*

- (i) *for each $\varphi \in C^+ \setminus \{0_{Y^*}\}$, $(\varphi F, G)$ is nearly $R_+ \times D$ -subconvexlike on X ;*
- (ii) *for each $T \in L_+(Z, Y)$, $F + TG$ is nearly C -subconvexlike on X .*

3. Tightly Proper Efficiency, Strict Efficiency, and Superefficiency

In [11, 12], the authors introduced many concepts of proper efficiency (tightly proper efficiency except) for normed spaces and for topological vector spaces, respectively. Furthermore, they discussed the relationships between superefficiency and other proper efficiencies. If we can get the relationship between tightly proper efficiency and superefficiency, then we can get the relationships between tightly proper efficiency and other proper efficiencies. So, in this section, the aim is to get the equivalent relationships between tightly proper efficiency and superefficiency under suitable assumption by virtue of strict efficiency.

Lemma 3.1. *If C has a bounded base Θ , then*

$$\text{TPE}(S, \Theta) = \text{TPE}(S, C). \quad (3.1)$$

Proof. From the definition of $\text{TPE}(S, C)$ and $\text{TPE}(S, \Theta)$, we only need prove that $\text{TPE}(S, \Theta) \subset \text{TPE}(S, C)$ for any $\Theta' \in B(C)$. Indeed, for each $\Theta' \in B(C)$, by the separation theorem, there exists $f \in Y^*$ such that

$$\alpha = \inf\{f(\theta) \mid \theta \in \Theta'\} > 0. \quad (3.2)$$

Hence, $f \in C^{+i}$. Since Θ is bounded, there exists $\lambda > 0$ such that

$$\lambda\Theta \subset \{y \in Y \mid 0 < f(y) < \alpha\}. \quad (3.3)$$

It is clear that $\lambda\Theta \in B(C)$ and $\text{TPE}(S, \Theta) = \text{TPE}(S, \lambda\Theta)$. If there exists $\bar{y} \in \text{TPE}(S, \Theta)$ such that $\bar{y} \notin \text{TPE}(S, \Theta')$, then for any convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying $(S - \bar{y}) \cap (-K) = \{0_Y\}$ and for any neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta') \neq \emptyset. \quad (3.4)$$

It implies that there exists $y \in Y$ such that

$$y \in (-K)^c \cap (U - \Theta'). \quad (3.5)$$

Then there is $u \in U$ and $\theta' \in \Theta'$ such that $y = u - \theta'$, since $\theta' \in \Theta' \in C = \text{cone}(\lambda\Theta)$, then there exists $\mu > 0$ and $\theta \in \lambda\Theta$ such that $\theta' = \mu\theta$. By (3.2) and (3.3), we see that $\mu \geq 1$. Therefore, $u/\mu \in U$ and $y/\mu \in (-K)^C \cap (U - \lambda\Theta)$, it is a contradiction. Therefore, $\text{TPE}(S, \Theta) = \text{TPE}(S, \lambda\Theta) = \text{TPE}(S, \Theta')$ for each $\Theta' \in B(C)$. \square

Proposition 3.2. *If C has a bounded base Θ , then*

$$\text{SE}(S, C) \subseteq \text{TPE}(S, C). \quad (3.6)$$

Proof. By Definition 2.11, for any $\bar{y} \in \text{SE}(S, C)$, there exists a convex neighborhood U of $\{0_Y\}$ with $U \subset U_\Theta$ such that

$$\text{cl cone}(S - \bar{y}) \cap -S_U(\Theta) = \{0_Y\}. \quad (3.7)$$

It is easy to verify that

$$(-S_U(\Theta))^c \cap (U - \Theta) = \emptyset. \quad (3.8)$$

Now, let $K = S_U(\Theta)$ and by Lemma 3.1, we have

$$\bar{y} \in \text{TPE}(S, \Theta) = \text{TPE}(S, C). \quad (3.9)$$

which implies that $\text{SE}(S, C) \subset \text{TPE}(S, C)$. \square

Proposition 3.3. *Let $\Theta \in B(C)$. Then*

$$\text{TPE}(S, \Theta) \subseteq \text{STE}(S, \Theta). \quad (3.10)$$

Proof. For each $\bar{y} \in \text{TPE}(S, \Theta)$, there exists a convex cone $K \subset Y$ with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying

$$(S - \bar{y}) \cap -K = \{0_Y\}, \quad (3.11)$$

and there exists a neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta) = \emptyset. \quad (3.12)$$

Since expression (3.11) can be equivalently expressed as

$$\text{cone}(S - \bar{y}) \cap -K = \{0_Y\}, \quad (3.13)$$

$\text{cone}(S - \bar{y}) \subset [(-K)^c \cup \{0_Y\}]$, and by (3.12), we have

$$\text{cone}(S - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (3.14)$$

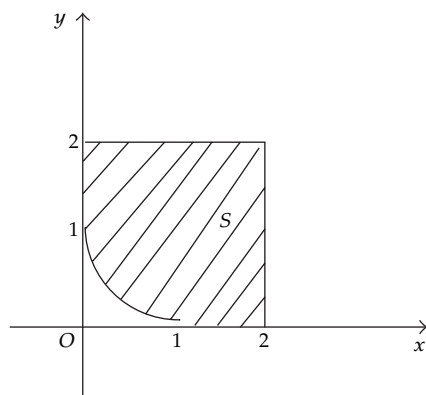


Figure 2: The set S .

Since $U - \Theta$ is open in Y , we get

$$\text{cl cone}(S - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (3.15)$$

It implies that $\bar{y} \in \text{STE}(S, \Theta)$. Therefore this proof is completed. \square

Remark 3.4. If C does not have a bounded base, then the converse of Proposition 3.3 may not hold. The following example illustrates this case.

Example 3.5. Let $Y = \mathbb{R}^2$, $S = \{(x, y) \in [0, 2] \times [0, 2] \mid y \geq 1 - \sqrt{1 - (x - 1)^2} \text{ for } x \in [0, 1]\}$ (see Figure 2) and $C = \{(x, y) \cup \{(0, 0)\} \mid x > 0, y \in \mathbb{R}\}$.

Then, let $\Theta = \{(x, y) \mid x = 1, y \in \mathbb{R}\}$, we have $\Theta \in B(C)$. It follows from the definitions of $\text{STE}(S, \Theta)$ and $\text{TPE}(S, \Theta)$ that

$$\begin{aligned} \text{STE}(S, \Theta) &= \left\{ \left(x, 1 - \sqrt{1 - (x - 1)^2} \right) \mid x \in [0, 1] \right\} \cup \{(0, y) \mid y \in (1, 2]\} \cup \{(x, 0) \mid x \in (1, 2]\}, \\ \text{TPE}(S, \Theta) &= \left\{ \left(x, 1 - \sqrt{1 - (x - 1)^2} \right) \mid x \in (0, 1) \right\}, \end{aligned} \quad (3.16)$$

respectively. Thus, the converse of Proposition 3.3 is not valid.

Proposition 3.6 (see [8]). *If C has a bounded base Θ , then*

$$\text{SE}(S, \Theta) = \text{SE}(S, C) = \text{STE}(S, C) = \text{STE}(S, \Theta). \quad (3.17)$$

From Propositions 3.2, 3.3, and 3.6, we can get immediately the following corollary.

Corollary 3.7. *If C has a bounded base Θ , then*

$$SE(S, C) = TPE(S, C) = STE(S, C). \quad (3.18)$$

Example 3.8. Let $Y = \mathbb{R}^2$, S be given in Example 3.5 and $C = \mathbb{R}_+^2$. Then

$$TPE(S, C) = SE(S, C) = STE(S, C) = \left\{ \left(x, 1 - \sqrt{1 - (x - 1)^2} \right) \mid x \in (0, 1) \right\}. \quad (3.19)$$

Lemma 3.9 (see [23]). *Let $C \subset Y$ be a closed convex pointed cone with a bounded base Θ and $S \subset Y$. Then, $SE(S, C) = SE(S + C, C)$.*

From Corollary 3.7 and Lemma 3.9, we can get the following proposition.

Proposition 3.10. *If C has a bounded base Θ and S is a nonempty subset of Y , then $TPE(S, C) = TPE(S + C, C)$.*

4. Tightly Proper Efficiency and Scalarization

Let $D \subset Z$ be a closed convex pointed cone. We consider the following vector optimization problem with set-valued maps

$$\begin{aligned} & C\text{-min } F(x), \\ & \text{s.t. } G(x) \cap (-D) \neq \emptyset, \quad x \in X, \end{aligned} \quad (\text{VP})$$

where $F : X \rightarrow 2^Y, G : X \rightarrow 2^Z$ are set-valued maps with nonempty values. Let $A = \{x \in X : G(x) \cap (-D) \neq \emptyset\}$ be the set of all feasible solutions of (VP).

Definition 4.1. $\bar{x} \in A$ is said to be a tightly properly efficient solution of (VP), if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in TPE(F(A), C)$.

We call (\bar{x}, \bar{y}) is a tightly properly efficient minimizer of (VP). The set of all tightly properly efficient solutions of (VP) is denoted by $TPE(\text{VP})$.

In association with the vector optimization problem (VP) of set-valued maps, we consider the following scalar optimization problem with set-valued map F :

$$\begin{aligned} & \min \quad \varphi(F(x)), \\ & \text{s.t. } \quad x \in A, \end{aligned} \quad (\text{SP}_\varphi)$$

where $\varphi \in Y^* \setminus \{0_{Y^*}\}$. The set of all optimal solutions of (SP_φ) is denoted by $M(\text{SP}_\varphi)$, that is,

$$M(\text{SP}_\varphi) = \{\bar{x} \in A : \exists \bar{y} \in F(\bar{x}) \text{ such that } \varphi(\bar{y}) \leq \varphi(y), \forall y \in F(A)\}. \quad (4.1)$$

The fundamental results characterize tightly properly efficient solution of (VP) in terms of the solutions of (SP_φ) are given below.

Theorem 4.2. *Let the cone C have a bounded base Θ . Let $\bar{x} \in A$, $\bar{y} \in F(\bar{x})$, and $F - \bar{y}$ be nearly C -subconvexlike on A . Then $\bar{y} \in \text{TPE}(F(A), C)$ if and only if there exists $\varphi \in C^{+i}$ such that $\varphi((F(A) - \bar{y})) \geq 0$.*

Proof. Necessity. Let $\bar{y} \in \text{TPE}(F(A), C)$. Then, by Lemma 3.1 and Proposition 3.10, we have $\bar{y} \in \text{TPE}(F(A) + C, \Theta)$. Hence, there exists a convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying $(F(A) + C - \bar{y}) \cap (-K) = \{0_Y\}$ and there exists a convex neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta) = \emptyset. \quad (4.2)$$

From the above expression and $(F(A) + C - \bar{y}) \cap (-K) = \{0_Y\}$, we have

$$\text{cone}(F(A) + C - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (4.3)$$

Since $U - \Theta$ is open in Y , we have

$$\text{cl cone}(F(A) + C - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (4.4)$$

By the assumption that $F - \bar{y}$ is nearly C -subconvexlike on A , thus $\text{cl cone}(F(A) + C - \bar{y})$ is convex set. By the Hahn-Banach separation theorem, there exists $\varphi \in Y^* \setminus \{0_{Y^*}\}$ such that

$$\varphi(\text{cl cone}(F(A) + C - \bar{y})) > \varphi(U - \Theta). \quad (4.5)$$

It is easy to see that

$$\varphi(\text{cone}(F(A) + C - \bar{y})) \geq 0, \quad \varphi(U - \Theta) < 0. \quad (4.6)$$

Hence, we obtain

$$\varphi(F(A) - \bar{y}) \geq 0. \quad (4.7)$$

Furthermore, according to Remark 2.2, we have $\varphi \in C^{+i}$.

Sufficiency. Suppose that there exists $\varphi \in C^{+i}$ such that $\varphi(F(A) - \bar{y}) \geq 0$. Since C has a bounded base Θ , thus by Remark 2.2(ii), we know that $\varphi \in \Theta^{\text{st}}$. And by Remark 2.2(i), we can take a convex neighborhood U of 0_Y such that

$$\varphi(U - \Theta) < 0. \quad (4.8)$$

By $\varphi(F(A) - \bar{y}) \geq 0$, we have

$$\varphi(\text{cl cone}(F(A) - \bar{y})) \geq 0. \quad (4.9)$$

From the above expression and (4.8), we get

$$\text{cl cone}(F(A) - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (4.10)$$

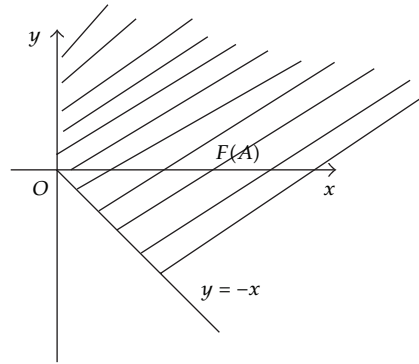


Figure 3: The set $F(A)$.

Therefore, $\bar{y} \in \text{STE}(S, \Theta)$. Noting that C has a bounded base Θ and by Lemma 3.1, we have $\bar{y} \in \text{TPE}(S, C)$. □

Now, we give the following example to illustrate Theorem 4.2.

Example 4.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $Z = \mathbb{R}$. Given $C = \mathbb{R}_+^2$, $D = \mathbb{R}_+$. Let

$$\begin{aligned} F(x) &= \{(x, y) \mid y \geq -x\} \quad \text{for any } x \in X, \\ G(x) &= [-x, -x + 1] \quad \text{for any } x \in X. \end{aligned} \tag{4.11}$$

Thus, feasible set of (VP)

$$A = \{x \in X \mid G(x) \cap (-\mathbb{R}_+) \neq \emptyset\} = [0, +\infty). \tag{4.12}$$

By Definition 4.1, we get

$$\text{TPE}(F(A), C) = \{(x, y) \mid y = -x, x > 0\}. \tag{4.13}$$

For any point $(\bar{x}, \bar{y}) \in \text{TPE}(F(A), C)$, there exists $\varphi \in C^{+i}$ such that

$$\varphi(F(A) - (\bar{x}, \bar{y})) \geq 0. \tag{4.14}$$

Indeed, for any $(x, y) \in F(A) - (\bar{x}, \bar{y})$, we consider the following three cases.

Case 1. If (x, y) is in the first quadrant, then for any $\varphi \in C^{+i}$ such that $\varphi((x, y)) \geq 0$.

Case 2. If (x, y) is in the second quadrant, then there exists $k \leq 0$ such that $y = kx$. Let $\varphi = (t_1, t_2)$ such that

$$t_1 > 0, t_2 > 0, \quad 0 \leq t_1 \leq -kt_2. \tag{4.15}$$

Then, we have

$$t_1x + t_2y = t_1x + t_2kx = (t_1 + kt_2)x \geq 0. \quad (4.16)$$

Case 3. If (x, y) in the fourth quadrant, then there exists $k \leq 0$ such that $y = kx$. Let $\varphi = (t_1, t_2)$ such that

$$t_1 > 0, \quad t_2 > 0, \quad t_1 \geq -kt_2. \quad (4.17)$$

Then, we have

$$t_1x + t_2y = t_1x + t_2kx = (t_1 + kt_2)x \geq 0. \quad (4.18)$$

Therefore, it follows from Cases 1, 2, and 3 that there exists $\varphi \in C^{+i}$ such that $\varphi(F(A) - (\bar{x}, \bar{y})) \geq 0$.

From Theorem 4.2, we can get immediately the following corollary.

Corollary 4.4. *Let the cone C have a bounded base Θ . For any $y_0 \in F(A)$ if $F - y_0$ is nearly C -subconvexlike on A . Then*

$$\text{TPE}(\text{VP}) = \bigcup_{\varphi \in C^{+i}} M(\text{SP}_\varphi). \quad (4.19)$$

5. Tightly Proper Efficiency and the Lagrange Multipliers

In this section, we establish two Lagrange multiplier theorems which show that tightly properly efficient solution of the constrained vector optimization problem (VP), is equivalent to tightly properly efficient solution of an appropriate unconstrained vector optimization problem.

Definition 5.1 (see [17]). Let $D \subset Z$ be a closed convex pointed cone with $\text{int} D \neq \emptyset$. We say that (VP) satisfies the generalized Slater constraint qualification, if there exists $x' \in X$ such that

$$G(x') \cap (-\text{int} D) \neq \emptyset. \quad (5.1)$$

Theorem 5.2. *Let C have a bounded base Θ and $\text{int} D \neq \emptyset$. Let $\bar{x} \in A$, $\bar{y} \in F(\bar{x})$ and $(F - \bar{y}, G)$ is nearly $C \times D$ -subconvexlike on X . Furthermore, let (VP) satisfies the generalized Slater constraint qualification. If $\bar{x} \in \text{TPE}(\text{VP})$ and $\bar{y} \in \text{TPE}(F(A), C)$, then there exists $T \in L_+(Z, Y)$ such that*

$$\begin{aligned} 0_Y &\in T(G(\bar{x}) \cap (-D)), \\ \bar{y} &\in \text{TPE}((F + TG)(X), C). \end{aligned} \quad (5.2)$$

Proof. Since C has bounded base Θ , by Lemma 2.13, we have $\bar{y} \in \text{TPE}(F(A), \Theta)$. Thus, there is a convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying

$$(F(A) - \bar{y}) \cap (-K) = \{0_Y\}, \quad (5.3)$$

and there exists an absolutely convex open neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta) = \emptyset. \quad (5.4)$$

Since (5.3) is equivalent to $\text{cone}(F(A) + C - \bar{y}) \cap (-K) = \{0_Y\}$, and from (5.4) we see that

$$\text{cone}(F(A) + C - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (5.5)$$

Moreover, for any $x \in X \setminus A$, we have $G(x) \cap (-D) = \emptyset$. Therefore,

$$\text{cone}[(F - \bar{y}, G)(X) + (C, D)] \cap (U - \Theta, -\text{int } D) = \emptyset. \quad (5.6)$$

Since $(U - \Theta, -\text{int } D)$ is open in $Y \times Z$, thus, we get

$$\text{cl cone}[(F - \bar{y}, G)(X) + (C, D)] \cap (U - \Theta, -\text{int } D) = \emptyset. \quad (5.7)$$

By the assumption that $(F - \bar{y}, G)$ is nearly $C \times D$ -subconvexlike on X , we have

$$\text{cl cone}[(F - \bar{y}, G)(X) + (C, D)] \quad (5.8)$$

is convex. Hence, it follows from the Hahn-Banach separation theorem that there exists $(\varphi, \psi) \in (Y^*, Z^*) \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that

$$\varphi[\text{cone}(F(x) - \bar{y} + C)] + \psi[\text{cone}(G(x) + D)] > \varphi(U - \Theta) + \psi(-\text{int } D), \quad \forall x \in X. \quad (5.9)$$

Thus, we obtain

$$\varphi(F(x) - \bar{y}) + \psi(G(x)) \geq 0, \quad \forall x \in X, \quad (5.10)$$

$$\varphi(U + \Theta) + \psi(\text{int } D) > 0. \quad (5.11)$$

Since D is a cone, we get

$$\varphi(U + \Theta) \geq 0, \quad (5.12)$$

$$\psi(\text{int } D) \geq 0. \quad (5.13)$$

Since $\bar{x} \in A$, $G(\bar{x}) \cap (-D) \neq \emptyset$. Choose $\bar{z} \in G(\bar{x}) \cap (-D)$. By (5.13), we know that $\varphi \in D^+$, thus

$$\varphi(\bar{z}) \leq 0. \quad (5.14)$$

Letting $x = \bar{x}$ and noting that $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$ in (5.10), we get

$$\varphi(\bar{z}) \geq 0. \quad (5.15)$$

Thus, $\varphi(\bar{z}) = 0$, which implies

$$0 \in \varphi[G(\bar{x}) \cap (-D)]. \quad (5.16)$$

Now, we claim that $\varphi \neq 0_{Y^*}$. If this is not the case, then

$$\varphi \in D^+ \setminus \{0_{Z^*}\}. \quad (5.17)$$

By the generalized Slater constraint qualification, then there exists $x' \in X$ such that

$$G(x') \cap (-\text{int } D) \neq \emptyset, \quad (5.18)$$

and so there exists $z' \in G(x')$ such that $z' \in -\text{int } D$. Hence, $\varphi(z') < 0$. But substituting $\varphi = 0_{Y^*}$ into (5.10), and by taking $x = x'$, and $z' \in G(x')$ in (5.10), we have

$$\varphi(z') \geq 0. \quad (5.19)$$

This contradiction shows that $\varphi \neq 0_{Y^*}$. Therefore $\varphi \in Y^* \setminus \{0_{Y^*}\}$. From (5.12) and Remark 2.2, we have $\varphi \in \Theta^{\text{st}}$. And since Θ is a bounded base of C , so $\varphi \in C^{+i}$. Hence, we can choose $c \in C \setminus \{0_Y\}$ such that $\varphi(c) = 1$ and define the operator $T : Z \rightarrow Y$ by

$$T(z) = \varphi(z)c, \quad \forall z \in Z. \quad (5.20)$$

Clearly, $T \in L_+(Z, Y)$ and by (5.16), we see that

$$0_Y \in T[G(\bar{x}) \cap (-D)]. \quad (5.21)$$

Therefore,

$$\bar{y} \in F(\bar{x}) \subset F(\bar{x}) + TG(x). \quad (5.22)$$

From (5.10) and (5.20), we obtain

$$\begin{aligned} \varphi(F(x) + TG(x) - \bar{y}) &= \varphi(F(x) - \bar{y}) + \varphi(G(x))\varphi(c) \\ &= \varphi(F(x) - \bar{y}) + \varphi(G(x)) \geq 0, \quad \forall x \in X. \end{aligned} \quad (5.23)$$

Since $(F - \bar{y}, G)$ is nearly $C \times D$ -subconvexlike on X , by Lemma 2.13, we have $F + TG - \bar{y}$ is nearly C -subconvexlike on X . From (5.22), Theorem 4.2 and the above expression, we have

$$\bar{y} \in \text{TPE}((F + TG)(X), C). \quad (5.24)$$

Therefore, the proof is completed. \square

Theorem 5.3. *Let $C \subset Y$ be a closed convex pointed cone with a bounded base Θ , $\bar{x} \in A$ and $\bar{y} \in F(\bar{x})$. If there exists $T \in L_+(Z, Y)$ such that $0_Y \in T(G(\bar{x}) \cap (-D))$ and $\bar{y} \in \text{TPE}((F + TG)(X), C)$, then $\bar{x} \in \text{TPE}(\text{VP})$ and $\bar{y} \in \text{TPE}(F(A), C)$.*

Proof. Since C has a bounded base, and $\bar{y} \in \text{TPE}((F + TG)(X), C)$, we have $\bar{y} \in \text{TPE}((F + TG)(X) + C, C)$. Thus, there exists a convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying

$$[(F + TG)(X) + C - \bar{y}] \cap (-K) = \{0_Y\}, \quad (5.25)$$

and there exists a convex neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta) = \emptyset. \quad (5.26)$$

By $0_Y \in T(G(\bar{x}) \cap (-D))$, we have

$$F(A) + TG(A) + C \supset F(A). \quad (5.27)$$

Thus,

$$\begin{aligned} (F(A) - \bar{y}) \cap (-K) &= \{0_Y\}, \\ (-K)^c \cap (U - \Theta) &= \emptyset. \end{aligned} \quad (5.28)$$

Therefore, by the definition of $\text{TPE}(F(A), C)$ and $\text{TPE}(\text{VP})$, we get $\bar{x} \in \text{TPE}(\text{VP})$ and $\bar{y} \in \text{TPE}(F(A), C)$, respectively. \square

6. Tightly Proper Efficiency and Duality

Definition 6.1. The set-valued Lagrangian map $L : X \times L_+(Z, Y) \rightarrow 2^Y$ for problem (VP) is defined by

$$L(x, T) = F(x) + TG(x), \quad \forall x \in X, \forall T \in L_+(Z, Y). \quad (6.1)$$

Definition 6.2. The set-valued map $\Phi : L_+(Z, Y) \rightarrow 2^Y$, defined by

$$\Phi(T) = \text{TPE}(L(X, T), C), \quad T \in L_+(Z, Y). \quad (6.2)$$

is called a tightly properly dual map for (VP). We now associate the following Lagrange dual problem with (VP):

$$\text{C-max } \bigcup_{T \in L_+(Z, Y)} \Phi(T). \quad (\text{VD})$$

Definition 6.3. A point $y_0 \in \bigcup_{T \in L_+(Z, Y)} \Phi(T)$ is said to be an efficient point of (VD) if

$$y - y_0 \notin C \setminus \{0_Y\}, \quad \forall y \in \bigcup_{T \in L_+(Z, Y)} \Phi(T). \quad (6.3)$$

We now can establish the following dual theorems.

Theorem 6.4 (weak duality). *If $\bar{x} \in A$ and $y_0 \in \bigcup_{T \in L_+(Z, Y)} \Phi(T)$. Then*

$$[y_0 - F(\bar{x})] \cap (C \setminus \{0_Y\}) = \emptyset. \quad (6.4)$$

Proof. One has

$$y_0 \in \bigcup_{T \in L_+(Z, Y)} \Phi(T). \quad (6.5)$$

Then, there exists $\bar{T} \in L_+(Z, Y)$ such that

$$\begin{aligned} y_0 &\in \Phi(\bar{T}) \\ &= \text{TPE} \left[\bigcup_{x \in X} (F(x) + \bar{T}G(x)), C \right] \\ &\subseteq \min \left[\bigcup_{x \in X} (F(x) + \bar{T}G(x)), C \right]. \end{aligned} \quad (6.6)$$

Hence,

$$(y_0 - F(x) - \bar{T}G(x)) \cap (C \setminus \{0_Y\}) = \emptyset. \quad (6.7)$$

Particularly,

$$y_0 - y - \bar{T}(z) \notin C \setminus \{0_Y\}, \quad y \in F(x), \quad z \in G(\bar{x}). \quad (6.8)$$

Noting that

$$\begin{aligned} & \bar{x} \in A \\ & \implies G(\bar{x}) \cap (-D) \neq \emptyset \\ & \implies \exists \bar{z} \in G(\bar{x}) \text{ s.t. } -\bar{z} \in D \\ & \implies -\bar{T}(\bar{z}) \in C, \end{aligned} \tag{6.9}$$

and taking $z = \bar{z}$ in (6.8), we have

$$y_0 - y - \bar{T}(\bar{z}) \notin C \setminus \{0_Y\}, \quad \forall y \in F(\bar{x}). \tag{6.10}$$

Hence, from $-\bar{T}(\bar{z}) \in C$ and $C + C \setminus \{0_Y\} \subset C \setminus \{0_Y\}$, we get

$$y_0 - y \notin C \setminus \{0_Y\}, \quad \forall y \in F(\bar{x}). \tag{6.11}$$

This completes the proof. □

Theorem 6.5 (strong duality). *Let C be a closed convex pointed cone with a bounded base Θ in Y and D be a closed convex pointed cone with $\text{int}D \neq \emptyset$ in Z . Let $\bar{x} \in A$, $\bar{y} \in F(\bar{x})$, $(F - \bar{y}, G)$ be nearly $C \times D$ -subconvexlike on X . Furthermore, let (VP) satisfy the generalized Slater constraint qualification. Then, $\bar{x} \in \text{TPE}(\text{VP})$ and $\bar{y} \in \text{TPE}(F(A), C)$ if and only if \bar{y} is an efficient point of (VD).*

Proof. Let $\bar{x} \in \text{TPE}(\text{VP})$ and $\bar{y} \in \text{TPE}(F(A), C)$, then according to Theorem 5.2, there exists $T \in L_+(Z, Y)$ such that $0_Y \in T(G(\bar{x}) \cap -D)$ and $\bar{y} \in \text{TPE}((T + FG)(X), C)$. Hence

$$\bar{y} \in \text{TPE} \left[\bigcup_{x \in X} (F(x) + TG(x)), C \right] = \Phi(T) \subset \bigcup_{T \in L_+(Z, Y)} \Phi(T). \tag{6.12}$$

By Theorem 6.4, we know that \bar{y} is an efficient point of (VD).

Conversely, Since \bar{y} is an efficient point of (VD), then $\bar{y} \in \bigcup_{T \in L_+(Z, Y)} \Phi(T)$. Hence, there exists $T \in L_+(Z, Y)$ such that

$$\bar{y} \in \Phi(T) = \text{TPE}((F + TG)(X), C). \tag{6.13}$$

Since C has a bounded base Θ , by Lemma 3.1 and Proposition 3.10, we have

$$\begin{aligned} \bar{y} & \in \text{TPE}((F + TG)(X), C) \\ & = \text{TPE}((F + TG)(X) + C, C) \\ & = \text{TPE}((F + TG)(X) + C, \Theta). \end{aligned} \tag{6.14}$$

Hence, there exists a convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying $((F+TG)(X)+C-\bar{y}) \cap (-K)$ and there exists an absolutely open convex neighborhood U of 0_Y such that

$$(-K)^c \cap (U - \Theta) = \emptyset. \quad (6.15)$$

Hence, we have

$$\text{cone}((F+TG)(X) + C - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (6.16)$$

Since, $U - \Theta$ is open subset of Y , we have

$$\text{cl cone}((F+TG)(X) + C - \bar{y}) \cap (U - \Theta) = \emptyset. \quad (6.17)$$

Since $(F - \bar{y}, G)$ is nearly $C \times D$ -subconvexlike on X , by Lemma 2.13, we have $F + TG - \bar{y}$ is nearly C -subconvexlike on X , which implies that

$$\text{cl cone}((F+TG)(X) + C - \bar{y}) \quad (6.18)$$

is convex. From (6.17) and by the Hahn-Banach separation theorem, there exists $\varphi \in Y^* \setminus \{0_{Y^*}\}$ such that

$$\varphi(\text{cl cone}(F(A) + C - \bar{y})) > \varphi(U - \Theta). \quad (6.19)$$

From this, we have

$$\varphi(\text{cone}(F(A) + C - \bar{y})) \geq 0, \quad (6.20)$$

$$\varphi(U - \Theta) < 0. \quad (6.21)$$

From (6.21), we know that $\varphi \in \Theta^{\text{st}}$. And by Θ is bounded base of C , it implies that C^{+i} . For any $x \in A$, there exists $z_x \in G(x) \cap (-D)$. Since $T \in L_+(Z, Y)$, we have $-T(z_x) \in C$ and hence $\varphi(T(z_x)) \leq 0$. From this and (6.20), we have

$$\varphi(y - \bar{y}) \geq \varphi(y + T(z_x) - \bar{y}) \geq 0, \quad \forall x \in A, y \in F(x), \quad (6.22)$$

that is $\varphi(F(A) - \bar{y}) \geq 0$. By Theorem 4.2, we have $\bar{x} \in \text{TPE}(\text{VP})$ and $\bar{y} \in \text{TPE}(F(A), C)$. \square

7. Tightly Proper Efficiency and Tightly Proper Saddle Point

We now introduce a new concept of tightly proper saddle point for a set-valued Lagrange map $L(X, T)$ and use it to characterize tightly proper efficiency.

Definition 7.1. Let $\bar{y} \in S \subset Y$, C is a closed convex pointed cone of Y and $\Theta \in B(C)$. $\bar{y} \in \text{TPM}(S, \Theta)$ if there exists a convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying $(S - \bar{y}) \cap K = \{0_Y\}$ and there is a convex neighborhood U of 0_Y such that

$$K^c \cap (U + \Theta) = \emptyset. \tag{7.1}$$

\bar{y} is said to be a tightly properly efficient point with respect to C , written as, $\bar{y} \in \text{TPM}(S, C)$ if

$$\bar{y} \in \bigcap_{\Theta \in B(C)} \text{TPM}(S, \Theta). \tag{7.2}$$

It is easy to find that $\bar{y} \in \text{TPM}(S, C)$ if and only if $-\bar{y} \in \text{TPE}(-S, C)$, and if C is bounded, then we also have $\text{TPM}(S, C) = \text{TPM}(S, \Theta)$.

Definition 7.2. A pair $(\bar{x}, \bar{T}) \in X \times L_+(Z, Y)$ is said to be a tightly proper saddle point of Lagrangian map L if

$$L(\bar{x}, \bar{T}) \cap \text{TPE} \left[\bigcup_{x \in X} L(x, \bar{T}), C \right] \cap \text{TPM} \left[\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T), C \right] \neq \emptyset. \tag{7.3}$$

We first present an important equivalent characterization for a tightly proper saddle point of the Lagrange map L .

Lemma 7.3. $(\bar{x}, \bar{T}) \in X \times L_+(Z, Y)$ is said to be a tight proper saddle point of Lagrange map L if only if there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

- (i) $\bar{y} \in \text{TPE}[\bigcup_{x \in X} L(x, \bar{T}), C] \cap \text{TPM}[\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T), C]$,
- (ii) $\bar{T}(\bar{z}) = 0_Y$.

Proof. Necessity. Since (\bar{x}, \bar{T}) is a tightly proper saddle point of L , by Definition 7.2 there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

$$\bar{y} + \bar{T}(\bar{z}) \in \text{TPE} \left[\bigcup_{x \in X} L(x, \bar{T}), C \right], \tag{7.4}$$

$$\bar{y} + \bar{T}(\bar{z}) \in \text{TPM} \left[\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T), C \right]. \tag{7.5}$$

From (7.5) and the definition of $\text{TPM}(S, C)$, then there exists a convex cone K with $C \setminus \{0_Y\} \subset \text{int } K$ satisfying

$$\left(\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T) - C - (\bar{y} + \bar{T}(\bar{z})) \right) \cap K = \{0_Y\}, \tag{7.6}$$

and there is a convex neighborhood U of 0_Y such that

$$K^c \cap (U + \Theta) = \emptyset. \quad (7.7)$$

Since, for every $T \in L_+(Z, Y)$,

$$\begin{aligned} T(\bar{z}) - \bar{T}(\bar{z}) &= [\bar{y} + T(\bar{z})] - [\bar{y} + \bar{T}(\bar{z})] \in F(\bar{x}) + T[G(\bar{x})] - [\bar{y} + \bar{T}(\bar{z})] \\ &= L(\bar{x}, T) - [\bar{y} + \bar{T}(\bar{z})]. \end{aligned} \quad (7.8)$$

We have

$$\{T(\bar{z}) : T \in L_+(Z, Y)\} - C - \bar{T}(\bar{z}) \subseteq \bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T) - C - [\bar{y} + \bar{T}(\bar{z})]. \quad (7.9)$$

Thus, from (7.6), we have

$$K \cap \left[\bigcup_{T \in L_+(Z, Y)} \{T(\bar{z})\} - C - \bar{T}(\bar{z}) \right] = \{0_Y\}. \quad (7.10)$$

Let $f : L(Z, Y) \rightarrow Y$ be defined by

$$f(T) = -T(\bar{z}). \quad (7.11)$$

Then, (7.10) can be written as

$$(-K) \cap \left[f(L_+(Z, Y)) + C - f(\bar{T}) \right] = \{0_Y\} \quad (7.12)$$

By (7.7) and the above expression show that $\bar{T} \in L_+(Z, Y)$ is a tightly properly efficient point of the vector optimization problem

$$\begin{aligned} &C\text{-min } f(T) \\ &\text{s.t. } T \in L_+(Z, Y) \end{aligned} \quad (7.13)$$

Since f is a linear map, of course, $-f$ is nearly C -subconvexlike on $L_+(Z, Y)$. Hence, by Theorem 4.2, there exists $\varphi \in C^{+i}$ such that

$$\varphi[-\bar{T}(\bar{z})] = \varphi[f(\bar{T})] \leq \varphi[f(T)] = \varphi[-T(\bar{z})], \quad \forall T \in L_+(Z, Y). \quad (7.14)$$

Now, we claim that

$$-\bar{z} \in D. \quad (7.15)$$

If this is not true, then since D is a closed convex cone set, by the strong separation theorem in topological vector space, there exists $\mu \in Z^* \setminus \{0_{Z^*}\}$ such that

$$\mu(-\bar{z}) < \mu(\lambda d), \quad \forall d \in D, \forall \lambda > 0. \quad (7.16)$$

In the above expression, taking $d = 0_z \in D$ gets

$$\mu(\bar{z}) > 0, \quad (7.17)$$

while letting $\lambda \rightarrow +\infty$ leads to

$$\mu(d) \geq 0, \quad \forall d \in D. \quad (7.18)$$

Hence,

$$\mu \in D^+ \setminus \{0_{Z^*}\}. \quad (7.19)$$

Let $c^* \in \text{int } C$ be fixed, and define $T^* : Z \rightarrow Y$ as

$$T^*(z) = \left[\frac{\mu(z)}{\mu(\bar{z})} \right] c^* + \bar{T}(z). \quad (7.20)$$

It is evident that $T^* \in L(Z, Y)$ and that

$$T^*(d) = \left[\frac{\mu(d)}{\mu(\bar{z})} \right] c^* + \bar{T}(d) \in C + C \subset C, \quad \forall d \in D. \quad (7.21)$$

Hence, $T^* \in L_+(Z, Y)$. And taking $z = \bar{z}$ in (7.20), we obtain

$$T^*(\bar{z}) - \bar{T}(\bar{z}) = c^*. \quad (7.22)$$

Hence,

$$\varphi[T^*(\bar{z})] - \varphi[\bar{T}(\bar{z})] = \varphi(c^*) > 0, \quad (7.23)$$

which contradicts (7.14). Therefore,

$$-\bar{z} \in D. \quad (7.24)$$

Thus, $-\bar{T}(\bar{z}) \in C$, and since $\bar{T} \in L_+(Z, Y)$. If $\bar{T}(\bar{z}) \neq 0_Y$, then

$$-\bar{T}(\bar{z}) \in C \setminus \{0_Y\}, \quad (7.25)$$

hence $\varphi[\bar{T}(\bar{z})] < 0$, by $\varphi \in C^i$. But, taking $T = 0 \in L_+(Z, Y)$ in (7.14) leads to

$$\varphi(\bar{T}(\bar{z})) \geq 0. \quad (7.26)$$

This contradiction shows that $\bar{T}(\bar{z}) = 0_Y$, that is, condition (ii) holds.

Therefore, by (7.4) and (7.5), we know

$$\bar{y} \in \text{TPE} \left[\bigcup_{x \in X} L(x, \bar{T}), C \right] \cap \text{TPM} \left[\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T), C \right], \quad (7.27)$$

that is condition (i) holds.

Sufficiency. From $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, and condition (ii), we get

$$\bar{y} = \bar{y} + \bar{T}(\bar{z}) \in F(\bar{x}) + \bar{T}[G(\bar{x})] = L(\bar{x}, \bar{T}). \quad (7.28)$$

And by condition (i), we obtain

$$\bar{y} \in L(\bar{x}, \bar{T}) \cap \text{TPE} \left[\bigcup_{x \in X} L(x, \bar{T}), C \right] \cap \text{TPM} \left[\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T), C \right]. \quad (7.29)$$

Therefore, (\bar{x}, \bar{T}) is a tightly proper saddle point of L , and the proof is completed. \square

The following saddle-point theorem allows us to express a tightly properly efficient solution of (VP) as a tightly proper saddle of the set-valued Lagrange map L .

Theorem 7.4. *Let F be nearly C -convexlike on A . If for any point $y_0 \in Y$ such that $(F - y_0, G)$ is nearly $(C \times D)$ -convexlike on X , and (VP) satisfy generalized Slater constraint qualification.*

- (i) *If (\bar{x}, \bar{T}) is a tightly proper saddle point of L , then \bar{x} is a tightly properly efficient solution of (VP).*
- (ii) *If (\bar{x}, \bar{y}) be a tightly properly efficient minimizer of (VP), $\bar{y} \in \text{TPM}[\bigcup_{T \in L_+(Z, Y)} L(\bar{x}, T), C]$. Then there exists $\bar{T} \in L_+(Z, Y)$ such that (\bar{x}, \bar{T}) is a tightly proper saddle point of Lagrange map L .*

Proof. (i) By the necessity of Lemma 7.3, we have

$$0_Y \in \bar{T}[G(\bar{x})], \quad (7.30)$$

and there exists $\bar{y} \in F(\bar{x})$ such that (\bar{x}, \bar{y}) is a tightly properly efficient minimizer of the problem

$$\begin{aligned} & C\text{-min } F(x) + \bar{T}[G(x)] \\ & \text{s.t. } x \in X. \end{aligned} \quad (\text{UVP})$$

According to Theorem 5.3, (\bar{x}, \bar{y}) is a tightly properly efficient minimizer of (VP). Therefore, \bar{x} is a tightly properly efficient solution of (VP).

(ii) From the assumption, and by Theorem 5.2, there exists $\bar{T} \in L_+(Z, Y)$ such that

$$\begin{aligned} \bar{y} &\in \text{TPE} \left[\bigcup_{x \in X} L(x, \bar{T}), C \right], \\ 0_Y &\in T[G(\bar{x}) \cap (-D)]. \end{aligned} \quad (7.31)$$

Therefore there exists $\bar{z} \in G(\bar{x})$ such that $\bar{T}(\bar{z}) = 0_Y$. Hence, from Lemma 7.3, it follows that (\bar{x}, \bar{T}) is a tightly proper saddle point of Lagrange map L . \square

8. Conclusions

In this paper, we have extended the concept of tightly proper efficiency from normed linear spaces to locally convex topological vector spaces and got the equivalent relations among tightly proper efficiency, strict efficiency and superefficiency. We have also obtained a scalarization theorem and two Lagrange multiplier theorems for tightly proper efficiency in vector optimization involving nearly cone-subconvexlike set-valued maps. Then, we have introduced a Lagrange dual problem and got some duality results in terms of tightly properly efficient solutions. To characterize tightly proper efficiency, we have also introduced a new type of saddle point, which is called the tightly proper saddle point of an appropriate set-valued Lagrange map, and obtained its necessary and sufficient optimality conditions. Simultaneously, we have also given some examples to illustrate these concepts and results. On the other hand, by using the results of the Section 3 in this paper, we know that the above results hold for superefficiency and strict efficiency in vector optimization involving nearly cone-convexlike set-valued maps and, by virtue of [12, Theorem 3.11], all the above results also hold for positive proper efficiency, Hurwicz proper efficiency, global Henig proper efficiency and global Borwein proper efficiency in vector optimization with set-valued maps under the conditions that the set-valued F and G is closed convex and the ordering cone $C \subset Y$ has a weakly compact base.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which helped to improve the paper. This research was partially supported by the National Natural Science Foundation of China (Grant no. 10871216) and the Fundamental Research Funds for the Central Universities (project no. CDJXS11102212).

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