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Research Article

Lyapunov Stability of Quasilinear Implicit Dynamic Equations on Time Scales

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This paper studies the stability of the solution $x \equiv 0$ for a class of quasilinear implicit dynamic equations on time scales of the form $A_t x^{\Delta} = f(t, x)$. We deal with an index concept to study the solvability and use Lyapunov functions as a tool to approach the stability problem.

1. Introduction

The stability theory of quasilinear differential-algebraic equations (DAEs for short)

$$A_t x'(t) = f\big(t, x'(t), x(t)\big), \quad f(t, 0, 0) = 0 \quad \forall t \in \mathbb{R}, \tag{1.1}$$

with $A_{\cdot}$ being a given $m \times m$-matrix function, has been an intensively discussed field in both theory and practice. This problem can be seen in many real problems, such as in electric circuits, chemical reactions, and vehicle systems. März in [1] has dealt with the question whether the zero-solution of (1.1) is asymptotically stable in the Lyapunov sense with $f(t, x'(t), x(t)) = Bx(t) + g(t, x'(t), x(t))$, with A being constant and small perturbation g.

Together with the theory of DAEs, there has been a great interest in singular difference equation (SDE) (also referred to as descriptor systems, implicit difference equations)

$$A_n x(n+1) = f(n, x(n+1), x(n)), \quad n \in \mathbb{Z}. \tag{1.2}$$

This model appears in many practical areas, such as the Leontiev dynamic model of multisector economy, the Leslie population growth model, and singular discrete optimal control problems. On the other hand, SDEs occur in a natural way of using discretization techniques for solving DAEs and partial differential-algebraic equations, and so forth, which have already attracted much attention from researchers (cf. [2–4]). When $f(n, x(n+1), x(n)) = B_n x(n) + g(n, x(n+1), x(n))$, in [5], the authors considered the solvability of Cauchy problem for (1.2); the question of stability of the zero-solution of (1.2) has been considered in [6] where the nonlinear perturbation $g(n, x(n+1), x(n))$ is small and does not depend on $x(n+1)$.

Further, in recent years, to unify the presentation of continuous and discrete analysis, a new theory was born and is more and more extensively concerned, that is, the theory of the analysis on time scales. The most popular examples of time scales are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Using "language" of time scales, we rewrite (1.1) and (1.2) under a unified form

$$A_t x^{\Delta}(t) = f\left(t, x^{\Delta}(t), x(t)\right), \tag{1.3}$$

with t in time scale $\mathbb{T}$ and Δ being the derivative operator on $\mathbb{T}$. When $\mathbb{T} = \mathbb{R}$, (1.3) is (1.1); if $\mathbb{T} = \mathbb{N}$, we have a similar equation to (1.2) if it is rewritten under the form $A_n(x(n+1) - x(n)) = -A_n x(n) + f(n, x(n+1), x(n));\ n \in \mathbb{N}$.

The purpose of this paper is to answer the question whether results of stability for (1.1) and (1.2) can be extended and unified for the implicit dynamic equations of the form (1.3). The main tool to study the stability of this implicit dynamic equation is a generalized direct Lyapunov method, and the results of this paper can be considered as a generalization of (1.1) and (1.2).

The organization of this paper is as follows. In Section 2, we present shortly some basic notions of the analysis on time scales and give the solvability of Cauchy problem for quasilinear implicit dynamic equations

$$A_t x^{\Delta} = B_t x + f(t, x), \tag{1.4}$$

with small perturbation $f(t, x)$ and for quasilinear implicit dynamic equations of the style

$$A_t x^{\Delta} = f(t, x), \tag{1.5}$$

with the assumption of differentiability for $f(t, x)$. The main results of this paper are established in Section 3 where we deal with the stability of (1.5). The technique we use in this section is somewhat similar to the one in [6–8]. However, we need some improvements because of the complicated structure of every time scale.

2. Nonlinear Implicit Dynamic Equations on Time Scales

2.1. Some Basic Notations of the Theory of the Analysis on Time Scales

A time scale is a nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ is endowed with the topology inherited from the real numbers with the standard topology. We define the *forward jump operator* and the *backward jump operator* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$

(supplemented by $\inf\emptyset = \sup\mathbb{T}$) and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (supplemented by $\sup\emptyset = \inf\mathbb{T}$). The *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+ \cup \{0\}$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$, and *isolated* if t is right-scattered and left-scattered. For every $a, b \in \mathbb{T}$, by $[a, b]$, we mean the set $\{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. The set $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise, it is $\mathbb{T}$ without this left-scattered maximum. Let f be a function defined on $\mathbb{T}$, valued in $\mathbb{R}^m$. We say that f is *delta differentiable* (or simply: *differentiable*) at $t \in \mathbb{T}^k$ provided there exists a vector $f^\Delta(t) \in \mathbb{R}^m$, called the derivative of f, such that for all $\epsilon > 0$ there is a neighborhood V around t with $\|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leqslant \epsilon|\sigma(t) - s|$ for all $s \in V$. If f is differentiable for every $t \in \mathbb{T}^k$, then f is said to be *differentiable on* $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then delta derivative is $f'(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$, the delta derivative is the forward difference, Δf, from discrete calculus. A function f defined on $\mathbb{T}$ is *rd-continuous* if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. The set of all rd-continuous functions from $\mathbb{T}$ to a Banach space X is denoted by $C_{\mathrm{rd}}(\mathbb{T}, X)$. A matrix function f from $\mathbb{T}$ to $\mathbb{R}^{m\times m}$ is said to be *regressive* if $\det(I + \mu(t)f(t)) \neq 0$ for all $t \in \mathbb{T}^k$, and denote $\mathcal{R}$ the set of regressive functions from $\mathbb{T}$ to $\mathbb{R}^{m\times m}$. Moreover, denote $\mathcal{R}^+$ the set of *positively regressive* functions from $\mathbb{T}$ to $\mathbb{R}$, that is, the set $\{f : \mathbb{T} \to \mathbb{R} : 1 + \mu(t)f(t) > 0\ \forall t \in \mathbb{T}\}$.

Theorem 2.1 (see [9–11]). *Let $t \in \mathbb{T}$ and let A_t be a rd-continuous $m \times m$-matrix function and q_t rd-continuous function. Then, for any $t_0 \in \mathbb{T}^k$, the initial value problem (IVP)*

$$x^\Delta = A_t x + q_t, \qquad x(t_0) = x_0 \tag{2.1}$$

has a unique solution $x(\cdot)$ defined on $t \geqslant t_0$. Further, if A_t is regressive, this solution exists on $t \in \mathbb{T}$.

The solution of the corresponding matrix-valued IVP $X^\Delta = A_t X$, $X(s) = I$ always exists for $t \geqslant s$, even A_t is not regressive. In this case, $\Phi_A(t, s)$ is defined only with $t \geqslant s$ (see [12, 13]) and is called the Cauchy operator of the dynamic equation (2.1). If we suppose further that A_t is regressive, the Cauchy operator $\Phi_A(t, s)$ is defined for all $s, t \in \mathbb{T}$.

We now recall the chain rule for multivariable functions on time scales, this result has been proved in [14]. Let $V : \mathbb{T} \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}^m$ be continuously differentiable. Then $V(\cdot, g(\cdot)) : \mathbb{T} \to \mathbb{R}$ is delta differentiable and there holds

$$\begin{aligned} V^\Delta(t, g(t)) &= V_t^\Delta(t, g(t)) + \int_0^1 \Big\langle V_x'\Big(\sigma(t), g(t) + h\mu(t)g^\Delta(t)\Big), g^\Delta(t)\Big\rangle dh \\ &= V_t^\Delta(t, g(\sigma(t))) + \int_0^1 \Big\langle V_x'\Big(t, g(t) + h\mu(t)g^\Delta(t)\Big), g^\Delta(t)\Big\rangle dh, \end{aligned} \tag{2.2}$$

where V_x' is the derivative (in the second variable of the function $V = V(t, x)$) in normal meaning and $\langle \cdot, \cdot \rangle$ is the scalar product.

We refer to [12, 15] for more information on the analysis on time scales.

2.2. Linear Equations with Small Nonlinear Perturbation

Let $\mathbb{T}$ be a time scale. We consider a class of nonlinear equations of the form

$$A_t x^\Delta = B_t x + f(t,x). \tag{2.3}$$

The homogeneous linear implicit dynamic equations (LIDEs) associated to (2.3) are

$$A_t x^\Delta = B_t x, \tag{2.4}$$

where $A_\cdot, B_\cdot \in C_{\rm rd}(\mathbb{T}^k, \mathbb{R}^{m\times m})$ and $f(t,x)$ is rd-continuous in $(t,x) \in \mathbb{T}\times\mathbb{R}^m$. In the case where the matrices A_t are invertible for every $t \in \mathbb{T}$, we can multiply both sides of (2.3) by A_t^{-1} to obtain an ordinary dynamic equation

$$x^\Delta = A_t^{-1}B_t x + A_t^{-1} f(t,x), \quad t \in \mathbb{T}, \tag{2.5}$$

which has been well studied. If there is at least a t such that A_t is singular, we cannot solve explicitly the leading term x^Δ. In fact, we are concerned with a so-called ill-posed problem where the solutions of Cauchy problem may exist only on a submanifold or even they do not exist. One of the ways to solve this equation is to impose some further assumptions stated under the form of indices of the equation.

We introduce the so-called index-1 of (2.4). Suppose that rank $A_t = r$ for all $t \in \mathbb{T}$ and let $T_t \in GL(\mathbb{R}^m)$ such that $T_t|_{\ker A_t}$ is an isomorphism between $\ker A_t$ and $\ker A_{\rho(t)}$; $T_\cdot \in C_{\rm rd}(\mathbb{T}^k, \mathbb{R}^{m\times m})$. Let Q_t be a projector onto $\ker A_t$ satisfying $Q_\cdot \in C_{\rm rd}(\mathbb{T}^k, \mathbb{R}^{m\times m})$. We can find such operators T_t and Q_t by the following way: let matrix A_t possess a singular value decomposition

$$A_t = U_t \Sigma_t V_t^\top, \tag{2.6}$$

where U_t, V_t are orthogonal matrices and Σ_t is a diagonal matrix with singular values $\sigma_t^1 \geqslant \sigma_t^2 \geqslant \cdots \geqslant \sigma_t^r > 0$ on its main diagonal. Since $A_\cdot \in C_{\rm rd}(\mathbb{T}^k, \mathbb{R}^{m\times m})$, on the above decomposition of A_t, we can choose the matrix V_t to be in $C_{\rm rd}(\mathbb{T}^k, \mathbb{R}^{m\times m})$ (see [16]). Hence, by putting $Q_t = V_t \operatorname{diag}(O, I_{m-r})V_t^\top$ and $T_t = V_{\rho(t)}V_t^{-1}$, we obtain Q_t and V_t as the requirement.

Let

$$S_t = \{x \in \mathbb{R}^m, B_t x \in \operatorname{im} A_t\}, \tag{2.7}$$

and $P_t := I - Q_t$.

Under these notations, we have the following Lemma.

Lemma 2.2. *The following assertions are equivalent*

(i) $\ker A_{\rho(t)} \cap S_t = \{0\}$;

(ii) *the matrix* $G_t = A_t - B_t T_t Q_t$ *is nonsingular;*

(iii) $\mathbb{R}^m = \ker A_{\rho(t)} \oplus S_t$, *for all* $t \in \mathbb{T}$.

Proof. (i)⇒(ii) Let $t \in \mathbb{T}$ and $x \in \mathbb{R}^m$ such that $(A_t - B_tT_tQ_t)x = 0 \Leftrightarrow B_t(T_tQ_tx) = Ax$. This equation implies $T_tQ_tx \in S_t$. Since $\ker A_{\rho(t)} \cap S_t = \{0\}$ and $T_tQ_tx \in \ker A_{\rho(t)}$, it follows that $T_tQ_tx = 0$. Hence, $Q_tx = 0$ which implies $A_tx = 0$. This means that $x \in \ker A_t$. Thus, $x = Q_tx = 0$, that is, the matrix $G_t = A_t - B_tT_tQ_t$ is nonsingular.

(ii)⇒(iii) It is obvious that $x = (I + T_tQ_tG_t^{-1}B_t)x - T_tQ_tG_t^{-1}B_tx$. We see that $T_tQ_tG_t^{-1}B_tx \in \ker A_{\rho(t)}$ and $B_t(I + T_tQ_tG_t^{-1}B_t)x = B_tx - (A_t - B_tT_tQ_t)G_t^{-1}B_tx + A_tG_t^{-1}B_tx = A_tG_t^{-1}B_tx \in \operatorname{im}A_t$. Thus, $(I + T_tQ_tG_t^{-1}B_t)x \in S_t$ and we have $\mathbb{R}^m = S_t + \ker A_{\rho(t)}$.

Let $x \in \ker A_{\rho(t)} \cap S_t$, that is, $x \in S_t$ and $x \in \ker A_{\rho(t)}$. Since $x \in S_t$, there is a $z \in \mathbb{R}^m$ such that $B_tx = A_tz = A_tP_tz$ and since $x \in \ker A_{\rho(t)}$, $T_t^{-1}x \in \ker A_t$. Therefore, $T_t^{-1}x = Q_tT_t^{-1}x$. Hence, $(A_t - B_tT_tQ_t)T_t^{-1}x = -(A_t - B_tT_tQ_t)P_tz$ which follows that $T_t^{-1}x = -P_tz$. Thus, $T_t^{-1}x = 0$ and then $x = 0$. So, we have that (iii). (iii)⇒(i) is obvious.

Lemma 2.2 is proved. □

Lemma 2.3. *Suppose that the matrix G_t is nonsingular. Then, there hold the following assertions:*

$$(1)\ P_t = G_t^{-1}A_t, \tag{2.8}$$

$$(2)\ Q_t = -G_t^{-1}B_tT_tQ_t, \tag{2.9}$$

$$(3)\ \widetilde{Q}_t := -T_tQ_tG_t^{-1}B_t \textit{ is the projector onto } \ker A_{\rho(t)} \textit{ along } S_t, \tag{2.10}$$

$$(4)\ (a)\ P_tG_t{}^{-1}B_t = P_tG_t^{-1}B_tP_{\rho(t)}, \tag{2.11}$$

$$(b)\ Q_tG_t{}^{-1}B_t = Q_tG_t^{-1}B_tP_{\rho(t)} - T_t{}^{-1}Q_{\rho(t)}, \tag{2.12}$$

$$(5)\ T_tQ_tG_t^{-1} \textit{ does not depend on the choice of } T_t \textit{ and } Q_t. \tag{2.13}$$

Proof. (1) Noting that $G_tP_t = (A_t - B_tT_tQ_t)P_t = A_tP_t = A_t$, we get (2.8).

(2) From $B_tT_tQ_t = A_t - G_t$, it follows $G_t^{-1}B_tT_tQ_t = P_t - I = -Q_t$. Thus, we have (2.9).

(3) $\widetilde{Q}_t^2 = T_tQ_tG_t^{-1}B_tT_tQ_tG_t^{-1}B_t \overset{(2.9)}{=} -T_tQ_tQ_tG_t^{-1}B_t = -T_tQ_tG_t^{-1}B_t = \widetilde{Q}_t$ and $A_{\rho(t)}\widetilde{Q}_t = -A_{\rho(t)}T_tQ_tG_t^{-1}B_t = 0$. This means that $\widetilde{Q}_t$ is a projector onto $\ker A_{\rho(t)}$. From the proof of (iii), Lemma 2.2, we see that $\widetilde{Q}_t$ is the projector onto $\ker A_{\rho(t)}$ along S_t.

(4) Since $T_t^{-1}Q_{\rho(t)}x \in \ker A_t$ for any x,

$$P_tG_t^{-1}B_tQ_{\rho(t)} = P_tG_t^{-1}B_tT_tT_t^{-1}Q_{\rho(t)} = -P_tG_t^{-1}(A_t - B_tT_tQ_t)Q_tT_t^{-1}Q_{\rho(t)} = 0. \tag{2.14}$$

Therefore, $P_tG_t^{-1}B_t = P_tG_t^{-1}B_tP_{\rho(t)}$ so we have (2.11). Finally,

$$\begin{aligned} Q_tG_t^{-1}B_t &= Q_tG_t^{-1}B_tP_{\rho(t)} + Q_tG_t^{-1}B_tT_tQ_tT_t^{-1}Q_{\rho(t)} \\ &= Q_tG_t^{-1}B_tP_{\rho(t)} - Q_tG_t^{-1}(A_t - B_tT_tQ_t)Q_tT_t^{-1}Q_{\rho(t)} \\ &= Q_tG_t^{-1}B_tP_{\rho(t)} - Q_tT_t^{-1}Q_{\rho(t)} = Q_tG_t^{-1}B_tP_{\rho(t)} - T_t^{-1}Q_{\rho(t)}. \end{aligned} \tag{2.15}$$

Thus, we get (2.12).

(5) Let T_t' be another linear transformation from $\mathbb{R}^m$ onto $\mathbb{R}^m$ satisfying $T_t'|_{\ker A_t}$ to be an isomorphism from $\ker A_t$ onto $\ker A_{\rho(t)}$ and Q_t' a projector onto $\ker A_t$. Denote $G_t' = A_t - B_t T_t' Q_t'$. It is easy to see that

$$T_t Q_t G_t^{-1} G_t' = T_t Q_t G_t^{-1}\left(A_t - B_t T_t' Q_t'\right) = T_t Q_t P_t - T_t Q_t G_t^{-1} B_t T_t' Q_t' = T_t Q_t T_t' Q_t' = T_t' Q_t'. \tag{2.16}$$

Therefore, $T_t Q_t G_t^{-1} = T_t' Q_t' G_t^{'-1}$. The proof of Lemma 2.3 is complete. □

Definition 2.4. The LIDE (2.4) is said to be *index-1* if for all $t \in \mathbb{T}$, the following conditions hold:

(i) $\operatorname{rank} A_t = r = \text{constant}$ $(1 \leqslant r \leqslant m-1)$,

(ii) $\ker A_{\rho(t)} \cap S_t = \{0\}$.

Now, we add the following assumptions.

Hypothesis 2.5. (1) The homogeneous LIDE (2.4) is of index-1.

(2) $f(t,x)$ is rd-continuous and satisfies the Lipschitz condition,

$$\left\| f(t,w) - f\left(t,w'\right)\right\| \leqslant L_t \left\|w - w'\right\|, \quad \forall w, w' \in \mathbb{R}^m, \tag{2.17}$$

where

$$\gamma_t := L_t \left\| T_t Q_t G_t^{-1} \right\| < 1 \quad \forall t \in \mathbb{T}^k. \tag{2.18}$$

Remark 2.6. By the item (2.13) of Lemma 2.3, the condition (2.18) is independent from the choice of T_t and Q_t.

We assume further that we can choose the projector function Q_t onto $\ker A_t$ such that $Q_{\rho(t)} = Q_t$ for all right-dense and left-scattered t; $Q_{\rho(t)}$ is differentiable at every $t \in \mathbb{T}^k$ and $(Q_{\rho(t)})^\Delta$ is rd-continuous. For each $t \in \mathbb{T}^k$, we have $(P_{\rho(t)}x(t))^\Delta = P_{\rho(\sigma(t))}x^\Delta(t) + (P_{\rho(t)})^\Delta x(t)$. Therefore,

$$A_t x^\Delta(t) = A_t P_t x^\Delta(t) = A_t\Big(\left(P_{\rho(t)}x(t)\right)^\Delta - \left(P_{\rho(t)}\right)^\Delta x(t)\Big), \tag{2.19}$$

and the implicit equation (2.3) can be rewritten as

$$A_t\left(P_{\rho(t)}x\right)^\Delta = \Big(A_t\left(P_{\rho(t)}\right)^\Delta + B_t\Big)x + f(t,x), \quad t \in \mathbb{T}^k. \tag{2.20}$$

Thus, we should look for solutions of (2.3) from the space C_N^1:

$$C_N^1\Big(\mathbb{T}^k, \mathbb{R}^m\Big) = \Big\{x(\cdot) \in C_{\mathrm{rd}}\Big(\mathbb{T}^k, \mathbb{R}^m\Big) : P_{\rho(t)}x(t) \text{ is differentiable at every } t \in \mathbb{T}^k\Big\}. \tag{2.21}$$

Note that C_N^1 does not depend on the choice of the projector function since the relations $P_t\overline{P}_t = \overline{P}_t$ and $\overline{P}_t P_t = P_t$ are true for each two projectors P_t and $\overline{P}_t$ along the space $\ker A_t$.

We now describe shortly the decomposition technique for (2.3) as follows.

Since (2.3) has index-1 and by virtue of Lemma 2.2, we see that the matrices G_t are nonsingular for all $t \in \mathbb{T}^k$. Multiplying (2.3) by $P_tG_t^{-1}$ and $Q_tG_t^{-1}$, respectively, it yields

$$\begin{aligned} P_t x^{\Delta} &= P_t G_t^{-1} B_t x + P_t G_t^{-1} f(t,x), \\ 0 &= Q_t G_t^{-1} B_t x + Q_t G_t^{-1} f(t,x). \end{aligned} \tag{2.22}$$

Therefore, by using the results of Lemma 2.3, we get

$$\begin{aligned} \left(P_{\rho(t)}x\right)^{\Delta} &= \left(P_{\rho(t)}\right)^{\Delta}\left(I + T_tQ_tG_t^{-1}B_t\right)P_{\rho(t)}x + P_tG_t^{-1}B_tP_{\rho(t)}x \\ &\quad + \left(\left(P_{\rho(t)}\right)^{\Delta}T_tQ_tG_t^{-1} + P_tG_t^{-1}\right)f(t,x), \\ Q_{\rho(t)}x &= T_tQ_tG_t^{-1}B_tP_{\rho(t)}x + T_tQ_tG_t^{-1}f(t,x). \end{aligned} \tag{2.23}$$

By denoting $u = P_{\rho(t)}x, v = Q_{\rho(t)}x$, (2.23) becomes a dynamic equation on time scale

$$u^{\Delta} = \left(P_{\rho(t)}\right)^{\Delta}\left(I + T_tQ_tG_t^{-1}B_t\right)u + P_tG_t^{-1}B_tu + \left(\left(P_{\rho(t)}\right)^{\Delta}T_tQ_tG_t^{-1} + P_tG_t^{-1}\right)f(t,u+v), \tag{2.24}$$

and an algebraic relation

$$v = T_tQ_tG_t^{-1}B_tu + T_tQ_tG_t^{-1}f(t,u+v). \tag{2.25}$$

For fixed $u \in \mathbb{R}^m$ and $t \in \mathbb{T}^k$, we consider a mapping $C_t : \operatorname{im} Q_{\rho(t)} \to \operatorname{im} Q_{\rho(t)}$ given by

$$C_t(v) := T_tQ_tG_t^{-1}B_tu + T_tQ_tG_t^{-1}f(t,u+v). \tag{2.26}$$

We see that

$$\left\|C_t(v) - C_t(v')\right\| = \left\|T_tQ_tG_t^{-1}\right\|\left\|f(t,u+v) - f(t,u+v')\right\| \leqslant \gamma_t\left\|v - v'\right\|, \tag{2.27}$$

for any $v, v' \in \operatorname{im} Q_{\rho(t)}$. Since $\gamma_t < 1$, C_t is a contractive mapping. Hence, by the fixed point theorem, there exists a mapping $g_t : \operatorname{im} P_{\rho(t)} \to \operatorname{im} Q_{\rho(t)}$ satisfying

$$g_t(u) = T_tQ_tG_t^{-1}B_tu + T_tQ_tG_t^{-1}f\left(t,u + g_t(u)\right), \tag{2.28}$$

and it is easy to see that $g_t(u)$ is rd-continuous in t.

Moreover,

$$\begin{aligned} \left\|g_t(u) - g_t(u')\right\| &\leqslant \left\|T_tQ_tG_t^{-1}B_t\right\|\left\|u - u'\right\| + \left\|T_tQ_tG_t^{-1}\right\|\left\|f\left(t,u+g_t(u)\right) - f\left(t,u'+g_t(u')\right)\right\| \\ &\leqslant \left\|T_tQ_tG_t^{-1}B_t\right\|\left\|u - u'\right\| + L_t\left\|T_tQ_tG_t^{-1}\right\|\left(\left\|u - u'\right\| + \left\|g_t(u) - g_t(u')\right\|\right). \end{aligned} \tag{2.29}$$

This deduces

$$\left\|g_t(u)-g_t(u')\right\| \leqslant \gamma_t(1-\gamma_t)^{-1}L_t^{-1}(L_t+\|B_t\|)\left\|u-u'\right\|. \tag{2.30}$$

Thus, g_t is Lipschitz continuous with the Lipschitz constant $\delta_t := \gamma_t(1-\gamma_t)^{-1}L_t^{-1}(L_t+\|B_t\|)$. Substituting g_t into (2.24), we obtain

$$u^{\Delta} = \left(P_{\rho(t)}\right)^{\Delta}\left(I+T_tQ_tG_t^{-1}B_t\right)u + P_tG_t^{-1}B_tu + \left(\left(P_{\rho(t)}\right)^{\Delta}T_tQ_tG_t^{-1} + P_tG_t^{-1}\right)f\left(t,u+g_t(u)\right). \tag{2.31}$$

It is easy to see that the right-hand side of (2.31) satisfies the Lipschitz condition with the Lipschitz constant

$$\omega_t = \left\|\left(P_{\rho(t)}\right)^{\Delta}\left(I+T_tQ_tG_t^{-1}B_t\right)+P_tG_t^{-1}B_t\right\| + L_t(1+\delta_t)\left\|\left(P_{\rho(t)}\right)^{\Delta}T_tQ_tG_t^{-1}+P_tG_t^{-1}\right\|. \tag{2.32}$$

Applying the global existence theorem (see [12]), we see that (2.31), with the initial condition $u(t_0)=P_{\rho(t_0)}x_0$ has a unique solution $u(t)=u(t;t_0,x_0)$, $(t\geqslant t_0)$.

Thus, we get the following theorem.

Theorem 2.7. *Let Hypothesis 2.5 and the assumptions on the projector Q_t be satisfied. Then,* (2.3) *with the initial condition*

$$P_{\rho(t_0)}\left(x(t_0)-x_0\right)=0 \tag{2.33}$$

has a unique solution. This solution is expressed by

$$x(t)=x(t;t_0,x_0)=u(t;t_0,x_0)+g_t\left(u(t;t_0,x_0)\right), \quad t\geqslant t_0,\ t\in\mathbb{T}^k, \tag{2.34}$$

where $u(t)=u(t;t_0,x_0)$ is the solution of (2.31) *with $u(t_0)=P_{\rho(t_0)}x_0$.*

We now describe the solution space of the implicit dynamic equation (2.3). Denote

$$\begin{gathered} Ł_t=\left\{x\in\mathbb{R}^m : Q_{\rho(t)}x=T_tQ_tG_t^{-1}B_tP_{\rho(t)}x+T_tQ_tG_t^{-1}f(t,x)\right\}, \\ \Omega_t=\left\{x\in\mathbb{R}^m : B_tx+f(t,x)\in \operatorname{im} A_t\right\}. \end{gathered} \tag{2.35}$$

Lemma 2.8. *There hold the following statements:*

(i) $Ł_t=\Omega_t$,

(ii) *If $f(t,0)=0$ for all $t\in\mathbb{T}$ then $\Omega_t\cap\ker A_{\rho(t)}=\{0\}$.*

Proof. (i) Let $y\in Ł_t$, that is, $Q_{\rho(t)}y=T_tQ_tG_t^{-1}B_tP_{\rho(t)}y+T_tQ_tG_t^{-1}f(t,y)$. We have

$$y=P_{\rho(t)}y+Q_{\rho(t)}y=\left(I+T_tQ_tG_t^{-1}B_t\right)P_{\rho(t)}y+T_tQ_tG_t^{-1}f(t,y). \tag{2.36}$$

Hence,

$$\begin{aligned} B_t y + f(t,y) &= B_t\left(I + T_t Q_t G_t^{-1} B_t\right) P_{\rho(t)} y + \left(I + B_t T_t Q_t G_t^{-1}\right) f(t,y) \\ &= \left(I + B_t T_t Q_t G_t^{-1}\right) B_t P_{\rho(t)} y + \left(I + B_t T_t Q_t G_t^{-1}\right) f(t,y) \\ &= \left(I + B_t T_t Q_t G_t^{-1}\right)\left(B_t P_{\rho(t)} y + f(t,y)\right). \end{aligned} \tag{2.37}$$

From

$$I + B_t T_t Q_t G_t^{-1} = I + (A_t - G_t) G_t^{-1} = A_t G_t^{-1}, \tag{2.38}$$

it yields

$$B_t y + f(t,y) = A_t G_t^{-1}\left(B_t P_{\rho(t)} y + f(t,y)\right) \in \operatorname{im} A_t \Longrightarrow y \in \Omega_t. \tag{2.39}$$

Conversely, suppose that $y \in \Omega_t$, that is, there exists $z \in \mathbb{R}^m$ such that $B_t y + f(t,y) = A_t z$. We have to prove

$$Q_{\rho(t)} y = T_t Q_t G_t^{-1} B_t P_{\rho(t)} y + T_t Q_t G_t^{-1} f(t,y), \tag{2.40}$$

or equivalently,

$$y = T_t Q_t G_t^{-1} f(t,y) + T_t Q_t G_t^{-1} B_t P_{\rho(t)} y + P_{\rho(t)} y. \tag{2.41}$$

Indeed,

$$\begin{aligned} &T_t Q_t G_t^{-1} f(t,y) + T_t Q_t G_t^{-1} B_t P_{\rho(t)} y + P_{\rho(t)} y \\ &\quad = T_t Q_t G_t^{-1} f(t,y) + T_t Q_t G_t^{-1} B_t y - T_t Q_t G_t^{-1} B_t Q_{\rho(t)} y + P_{\rho(t)} y \\ &\quad = T_t Q_t G_t^{-1}\left(f(t,y) + B_t y\right) - T_t Q_t G_t^{-1} B_t Q_{\rho(t)} y + P_{\rho(t)} y \\ &\quad = T_t Q_t G_t^{-1} A_t z - T_t Q_t G_t^{-1} B_t Q_{\rho(t)} y + P_{\rho(t)} y \\ &\quad = T_t Q_t P_t z - T_t Q_t G_t^{-1} B_t Q_{\rho(t)} y + P_{\rho(t)} y \\ &\quad = -T_t Q_t G_t^{-1} B_t Q_{\rho(t)} y + P_{\rho(t)} y = Q_{\rho(t)} y + P_{\rho(t)} y = y, \end{aligned} \tag{2.42}$$

where we have already used a result of Lemma 2.3 that $\widetilde{Q} = -T_t Q_t G_t^{-1} B_t$ is a projector onto $\ker A_{\rho(t)}$. So $Ł_t = \Omega_t$.

(ii) Let $y \in \Omega_t \cap \ker A_{\rho(t)}$. Then $y \in \Omega_t$ and $P_{\rho(t)} y = 0$. Since $\Omega_t = Ł_t$, we have $y \in Ł_t$. This means that $Q_{\rho(t)} y = T_t Q_t G_t^{-1} B_t P_{\rho(t)} y + T_t Q_t G_t^{-1} f(t,y) = T_t Q_t G_t^{-1} f(t, Q_{\rho(t)} y)$. From the assumption $f(t,0) = 0$, it follows that $\|Q_{\rho(t)} y\| \leqslant L_t \|T_t Q_t G_t^{-1}\| \|Q_{\rho(t)} y\| = \gamma_t \|Q_{\rho(t)} y\|$. The fact $\gamma_t < 1$ implies that $Q_{\rho(t)} y = 0$. Thus $y = P_{\rho(t)} y + Q_{\rho(t)} y = 0$. The lemma is proved. □

Remark 2.9. (1) By virtue of Lemma 2.8, we find out that the solution space $Ł_t$ is independent from the choice of projector Q_t and operator T_t.

(2) Since $G^{-1}_{\rho(t_0)}A_{\rho(t_0)} = P_{\rho(t_0)}$ and $A_{\rho(t_0)}P_{\rho(t_0)} = A_{\rho(t_0)}$, the initial condition (2.33) is equivalent to the condition $A_{\rho(t_0)}x(t_0) = A_{\rho(t_0)}x_0$. This implies that the initial condition is not also dependent on choice of projectors.

(3) Noting that if $x(t)$ is a solution of (2.3) with the initial condition (2.33), then $x(t) \in Ł_t$ for all $t \geqslant t_0$. Conversely, let $x_0 \in Ł_t = \Omega_t$ and let $x(s;t,x_0)$, $s \geqslant t$, be the solution of (2.3) satisfying the initial condition $P_{\rho(t)}(x(t;t,x_0) - x_0) = 0$. We see that $x(t;t,x_0) = P_{\rho(t)}x + g_t(P_{\rho(t)}x) = P_{\rho(t)}x_0 + g_t(P_{\rho(t)}x_0) = x_0$. This means that there exists a solution of (2.3) passing $x_0 \in Ł_t$.

2.3. Quasilinear Implicit Dynamic Equations

Now we consider a quasilinear implicit dynamic equation of the form

$$A_t x^{\Delta} = f(t,x), \tag{2.43}$$

with $A_{\cdot} \in C_{\mathrm{rd}}(\mathbb{T}^k, \mathbb{R}^{m\times m})$ and $f : \mathbb{T} \times \mathbb{R}^m \to \mathbb{R}^m$ assumed to be continuously differentiable in the variable x and continuous in (t,x).

Suppose that rank $A_t = r$ for all $t \in \mathbb{T}$. We keep all assumptions on the projector Q_t and operator T_t stated in Section 2.2.

Equation (2.43) is said to be of index-1 if the matrix

$$\widetilde{G}_t := A_t - f'_x(t,x)T_tQ_t \tag{2.44}$$

is invertible for every $t \in \mathbb{T}$ and $x \in \mathbb{R}^m$.

Denote

$$S(t,x) = \left\{z \in \mathbb{R}^m,\ f'_x(t,x)z \in \mathrm{im}A_t\right\}; \quad \ker A_t = N_t. \tag{2.45}$$

Further introduce the set

$$\Omega_t = \left\{x \in \mathbb{R}^m,\ f(t,x) \in \mathrm{im}A_t\right\}, \tag{2.46}$$

containing all solutions of (2.43). The subspace $S(t,x)$ manifests its geometrical meaning

$$S(t,x) = T_x\Omega_t \quad \text{for } x \in \Omega_t, \tag{2.47}$$

where T_x is the tangent space of Ω_t at the point x.

Suppose that (2.43) is of index-1. Then, by Lemma 2.2, this condition is equivalent to one of the following conditions:

(1) $S(t,x) \oplus N_{\rho(t)} = \mathbb{R}^m$,

(2) $S(t,x) \cap N_{\rho(t)} = \{0\}$.

(3) Let $B_t \in \mathbb{R}^{m\times m}$ be a matrix such that the matrix $G_t = A_t - B_t T_t Q_t$ is invertible (we can choose $B_t = f'_x(t,0)$, e.g.). From the relation

$$\begin{aligned}\widetilde{G}_t &= A_t - B_t T_t Q_t + B_t T_t Q_t - f'_x(t,x) T_t Q_t \\ &= G_t + \left(B_t - f'_x(t,x)\right) T_t Q_t \\ &= \left[I + \left(B_t - f'_x(t,x)\right) T_t Q_t G_t^{-1}\right] G_t,\end{aligned} \tag{2.48}$$

it follows that

$$I + \left(B_t - f'_x(t,x)\right) T_t Q_t G_t^{-1} \tag{2.49}$$

is invertible.

Lemma 2.10. *Suppose that the bounded linear operator triplet: $\mathbb{M} : X \to Y$, $\mathbb{P} : Y \to Z$, $\mathbb{N} : Z \to X$ is given, where X, Y, Z are Banach spaces. Then the operator $I - \mathbb{MPN}$ is invertible if and only if $I - \mathbb{PNM}$ is invertible.*

Proof. See [17, Lemma 1]. □

By virtue of (2.49) and Lemma 2.10, we get that

$$I + T_t Q_t G_t^{-1}\left(B_t - f'_x(t,x)\right) \text{ is invertible.} \tag{2.50}$$

Now we come to split (2.43). Multiplying both sides of (2.43) by $P_t G_t^{-1}$ and $Q_t G_t^{-1}$, respectively, and putting $u = P_{\rho(t)} x$, $v = Q_{\rho(t)} x$, we obtain

$$\begin{gathered} u^{\Delta} = \left(P_{\rho(t)}\right)^{\Delta}(u+v) + P_t G_t^{-1} f(t, u+v), \\ 0 = T_t Q_t G_t^{-1} f(t, u+v). \end{gathered} \tag{2.51}$$

Consider the function

$$k(t,u,v) := T_t Q_t G_t^{-1} f(t, u+v). \tag{2.52}$$

We see that

$$\frac{\partial k}{\partial v}(t,u,v) h = T_t Q_t G_t^{-1} f'_x(t, u+v) h, \tag{2.53}$$

where $h \in Q_{\rho(t)} \mathbb{R}^m$.

Let $h \in Q_{\rho(t)}\mathbb{R}^m$ be a vector satisfying $T_tQ_tG_t^{-1}f'_x(t,u+v)h = 0$. Paying attention to $T_tQ_tG_t^{-1}B_th = -h$, we have

$$-T_tQ_tG_t^{-1}f'_x(t,u+v)h = \left[I + T_tQ_tG_t^{-1}\left(B_t - f'_x(t,u+v)\right)\right]h. \tag{2.54}$$

Therefore, by (2.50) we get $h = 0$. This means that $(\partial k/\partial v)(t,u,v)|_{Q_{\rho(t)}\mathbb{R}^m}$ is an isomorphism of $Q_{\rho(t)}\mathbb{R}^m$. By the implicit function theorem, equation $k(t,u,v) = 0$ has a unique solution $v = g_t(u)$. Moreover, the function $v = g_t(u)$ is continuous in (t,u) and continuously differentiable in u. Its derivative is

$$\frac{\partial g_t(u)}{\partial u} = \left[-T_tQ_tG_t^{-1}f'_x(t,u+g_t(u))|_{Q_{\rho(t)}\mathbb{R}^m}\right]^{-1}T_tQ_tG_t^{-1}f'_x(t,u+g_t(u))|_{P_{\rho(t)}\mathbb{R}^m}. \tag{2.55}$$

Then, by substituting $v = g_t(u)$ into the first equation of (2.51) we come to

$$u^\Delta = \left(P_{\rho(t)}\right)^\Delta\left(u + g_t(u)\right) + P_tG_t^{-1}f\left(t, u + g_t(u)\right). \tag{2.56}$$

It is obvious that the ordinary dynamic equation (2.56) with the initial condition

$$u(t_0) = P_{\rho(t_0)}x_0 \tag{2.57}$$

is locally uniquely solvable and the solution $x(t;t_0,x_0)$ of (2.43) with the initial condition (2.33) can be expressed by $x(t;t_0,x_0) = u(t;t_0,x_0) + g_t(u(t;t_0,x_0))$.

Now suppose further that $f(t,x)$ satisfies the Lipschitz condition in x and we can find a matrix B_t such that

$$\left[T_tQ_tG_t^{-1}f'_x(t,x)|_{Q_{\rho(t)}\mathbb{R}^m}\right]^{-1}T_tQ_tG_t^{-1}f'_x(t,x)|_{P_{\rho(t)}\mathbb{R}^m} \tag{2.58}$$

is bounded for all $t \in \mathbb{T}$ and $x \in \mathbb{R}^m$. Then, the right-hand side of (2.56) also satisfies the Lipschitz condition. Thus, from the global existence theorem (see [12]), (2.56) with the initial condition (2.57) has a unique solution defined on $[t_0, \sup\mathbb{T})$.

Therefore, we have the following theorem.

Theorem 2.11. *Given an index-1 quasilinear implicit dynamic equation* (2.43), *then there holds the following.*

(1) Equation (2.43) *is locally solvable, that is, for any $t_0 \in \mathbb{T}^k$, $x_0 \in \mathbb{R}^m$, there exists a unique solution $x(t;t_0,x_0)$ of* (2.43), *defined on $[t_0,b)$ with some $b \in \mathbb{T}$, $b > t_0$, satisfying the initial condition* (2.33).

(2) Moreover, if $f(t,x)$ satisfies the Lipschitz condition in x and we can find a matrix B_t such that

$$\left[T_tQ_tG_t^{-1}f'_x(t,x)|_{Q_{\rho(t)}\mathbb{R}^m}\right]^{-1}T_tQ_tG_t^{-1}f'_x(t,x)|_{P_{\rho(t)}\mathbb{R}^m} \tag{2.59}$$

is bounded, then this solution is defined on $[t_0, \sup \mathbb{T})$ *and we have the expression*

$$x(t; t_0, x_0) = u(t; t_0, x_0) + g_t(u(t; t_0, x_0)), \quad t \geqslant t_0, \tag{2.60}$$

where $u(t; t_0, x_0)$ *is the solution of* (2.56) *with* $u(t_0) = P_{\rho(t_0)} x_0$.

Remark 2.12. (1) We note that the expression $T_t Q_t G_t^{-1} B_t$ depends only on choosing the matrix B_t.

(2) The assumption that $[T_t Q_t G_t^{-1} f_x'(t, x)|_{Q_{\rho(t)}\mathbb{R}^m}]^{-1} T_t Q_t G_t^{-1} f_x'(t, x)|_{P_{\rho(t)}\mathbb{R}^m}$ is bounded for a matrix function B_t seems to be too strong. In Section 3, we show a condition for the global solvability via Lyapunov functions.

(3) If $x_0 \in \Omega_t$, there exists $z \in \mathbb{R}^m$ satisfying $A_t z = f(t, x_0)$. Hence, $T_t Q_t G_t^{-1} f(t, x_0) = 0$. Therefore, by the same argument as in Section 2.2, we can prove that for every $x_0 \in \Omega_t$, there is a unique solution passing through x_0.

3. Stability Theorems of Implicit Dynamic Equations

For the reason of our purpose, in this section we suppose that $\mathbb{T}$ is an upper unbounded time scale, that is, $\sup \mathbb{T} = \infty$. For a fixed $\tau \in \mathbb{T}$, denote $\mathbb{T}_\tau = \{t \in \mathbb{T},\ t \geqslant \tau\}$.

Consider an implicit dynamic equation of the form

$$A_t x^\Delta = f(t, x), \quad t \in \mathbb{T}_\tau, \tag{3.1}$$

where $A_\cdot \in C_{\mathrm{rd}}(\mathbb{T}_\tau^k, \mathbb{R}^{m \times m})$ and $f(\cdot, \cdot) \in C_{\mathrm{rd}}(\mathbb{T}_\tau \times \mathbb{R}^m, \mathbb{R}^m)$.

First, we suppose that for each $t_0 \in \mathbb{T}_\tau^k$, (3.1) with the initial condition

$$A_{\rho(t_0)}(x(t_0) - x_0) = 0 \tag{3.2}$$

has a unique solution defined on $\mathbb{T}_{t_0}$. The condition ensuring the existence of a unique solution can be refered to Section 2. We denote the solution with the initial condition (3.2) by $x(t) = x(t; t_0, x_0)$. Remember that we look for the solution of (3.1) in the space $C_N^1(\mathbb{T}_\tau^k, \mathbb{R}^m)$. Let $f(t, 0) = 0$ for all $t \in \mathbb{T}_\tau$, which follows that (3.1) has the trivial solution $x \equiv 0$.

We mention again that $\Omega_t = \{x \in \mathbb{R}^m,\ f(t, x) \in \operatorname{im} A_t\}$. Noting that if $x(t) = x(t; t_0, x_0)$ is the solution of (3.1) and (3.2) then $x(t) \in \Omega_t$ for all $t \in \mathbb{T}_{t_0}$.

Definition 3.1. The trivial solution $x \equiv 0$ of (3.1) is said to be

(1) A-stable (resp., P-stable) if, for each $\epsilon > 0$ and $t_0 \in \mathbb{T}_\tau^k$, there exists a positive $\delta = \delta(t_0, \epsilon)$ such that $\|A_{\rho(t_0)} x_0\| < \delta$ (resp., $\|P_{\rho(t_0)} x_0\| < \delta$) implies $\|x(t; t_0, x_0)\| < \epsilon$ for all $t \geqslant t_0$,

(2) A-uniformly (resp., P-uniformly) stable if it is A-stable (resp., P-stable) and the number δ mentioned in the part (1). of this definition is independent of t_0,

(3) A-asymptotically (resp., P-asymptotically) stable if it is stable and for each $t_0 \in \mathbb{T}_\tau^k$, there exist positive $\delta = \delta(t_0)$ such that the inequality $\|A_{\rho(t_0)} x_0\| < \delta$ (resp., $\|P_{\rho(t_0)} x_0\| < \delta$) implies $\lim_{t \to \infty} \|x(t; t_0, x_0)\| = 0$. If δ is independent of t_0, then the corresponding stability is A-uniformly asymptotically (P-uniformly asymptotically) stable,

(4) A-uniformly globally asymptotically (resp., P-uniformly globally asymptotically) stable if for any $\delta_0 > 0$ there exist functions $\delta(\cdot)$, $T(\cdot)$ such that $\|A_{\rho(t_0)}x_0\| < \delta(\epsilon)$ (resp., $\|P_{\rho(t_0)}x_0\| < \delta(\epsilon)$) implies $\|x(t;t_0,x_0)\| < \epsilon$ for all $t \geqslant t_0$ and if $\|A_{\rho(t_0)}x_0\| < \delta_0$ (resp., $\|P_{\rho(t_0)}x_0\| < \delta_0$) then $\|x(t;t_0,x_0)\| < \epsilon$ for all $t \geqslant t_0 + T(\epsilon)$,

(5) P-exponentially stable if there is positive constant α with $-\alpha \in \mathcal{R}^+$ such that for every $t_0 \in \mathbb{T}_\tau^k$ there exists an $N = N(t_0) \geqslant 1$, the solution of (3.1) with the initial condition $P_{\rho(t_0)}(x(t_0) - x_0) = 0$ satisfies $\|x(t;t_0,x_0)\| \leqslant N\|P_{\rho(t_0)}x_0\|e_{-\alpha}(t,t_0)$, $t \geqslant t_0$, $t \in \mathbb{T}_\tau$. If the constant N can be chosen independent of t_0, then this solution is called P-uniformly exponentially stable.

Remark 3.2. From $G_t^{-1}A_t = P_t$ and $A_t = A_tP_t$, the notions of A-stable and P-stable as well as A-asymptotically stable and P-asymptotically stable are equivalent. Therefore, in the following theorems we will omit the prefixes A and P when talking about stability and asymptotical stability. However, the concept of A-uniform stability implies P-uniform stability if the matrices A_t are uniformly bounded and P-uniform stability implies A-uniform stability if the matrices G_t are uniformly bounded.

Denote

$$\digamma := \{\phi \in C([0,a),\mathbb{R}_+), \phi(0) = 0,\ \phi \text{ is strictly increasing; } a > 0\}, \tag{3.3}$$

and $\mathfrak{D}(\phi)$ is the domain of definition of ϕ.

Proposition 3.3. *The trivial solution $x \equiv 0$ of* (3.1) *is A-uniformly (resp., P-uniformly) stable if and only if there exists a function $\varphi \in \digamma$ such that for each $t_0 \in \mathbb{T}_\tau^k$ and any solution $x(t;t_0,x_0)$ of* (3.1) *the inequality*

$$\|x(t;t_0,x_0)\| \leqslant \varphi(\|A_{\rho(t_0)}x_0\|), \quad (\textit{resp.,}\ \|x(t;t_0,x_0)\| \leqslant \varphi(\|P_{\rho(t_0)}x_0\|)) \quad \forall t \geqslant t_0, \tag{3.4}$$

holds, provided $\|A_{\rho(t_0)}x_0\| \in \mathfrak{D}(\varphi)$ (resp., $\|P_{\rho(t_0)}x_0\| \in \mathfrak{D}(\varphi)$).

Proof. We only need to prove the proposition for the A-uniformly stable case.

Sufficiency. Suppose there exists a function $\varphi \in \digamma$ satisfying (3.4) for each $\epsilon > 0$; we take $\delta = \delta(\epsilon) > 0$ such that $\varphi(\delta) < \epsilon$, that is, $\varphi^{-1}(\epsilon) > \delta$. If $x(t;t_0,x_0)$ is an arbitrary solution of (3.1) and $\|A_{\rho(t_0)}x_0\| < \delta$, then $\|x(t;t_0,x_0)\| \leqslant \varphi(\|A_{\rho(t_0)}x_0\|) < \varphi(\delta) < \epsilon$, for all $t \geqslant t_0$.

Necessity. Suppose that the trivial solution $x \equiv 0$ of (3.1) is A-uniformly stable, that is, for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $t_0 \in \mathbb{T}_\tau^k$ the inequality $\|A_{\rho(t_0)}x_0\| < \delta$ implies $\|x(t;t_0,x_0)\| < \epsilon$, for all $t \geqslant t_0$. For the sake of simplicity in computation, we choose $\delta(\epsilon) < \epsilon$. Denote

$$\gamma(\epsilon) = \sup\{\delta(\epsilon) : \delta(\epsilon) \text{ has such a property}\}. \tag{3.5}$$

It is clear that $\gamma(\epsilon)$ is an increasing positive function in ϵ. Further, $\gamma(\epsilon) \leqslant \epsilon$ and by definition, there holds

$$\|A_{\rho(t_0)}x_0\| < \gamma(\epsilon) \text{ then } \|x(t;t_0,x_0)\| < \epsilon \quad \forall t \geqslant t_0. \tag{3.6}$$

By putting

$$\beta(\epsilon) := \frac{1}{\epsilon}\int_0^{\epsilon} \gamma(t)dt, \tag{3.7}$$

it is seen that

$$\beta \in \mathcal{F}, \quad 0 < \beta(\epsilon) < \gamma(\epsilon) \leqslant \epsilon. \tag{3.8}$$

Let $\varphi : [0, \sup \beta) \longrightarrow \mathbb{R}_+$ be the inverse function of β. It is clear that φ also belongs to $\mathcal{F}$.

For $t \geqslant t_0$, we denote $\epsilon_t = \|x(t; t_0, x_0)\|$. If $\epsilon_t = 0$, then $\|x(t; t_0, x_0)\| = \epsilon_t = 0 \leqslant \varphi(\|A_{\rho(t_0)}x_0\|)$ $\forall t \geqslant t_0$ by $\varphi \in \mathcal{F}$ (remember that $x(t; t_0, x_0) = 0$ does not imply that $x(\cdot; t_0, x_0) \equiv 0$). Consider the case where $\epsilon_t > 0$. If $\|A_{\rho(t_0)}x_0\| < \beta(\epsilon_t)$, then by the relations (3.6) and (3.8) we have $\|x(s; t_0, x_0)\| < \epsilon_t$, $\forall s \geqslant t_0$. In particular, $\|x(t; t_0, x_0)\| < \epsilon_t$ which is a contradiction. Thus $\|A_{\rho(t_0)}x_0\| \geqslant \beta(\epsilon_t)$, this implies $\|x(t; t_0, x_0)\| = \epsilon_t \leqslant \varphi(\|A_{\rho(t_0)}x_0\|)$, $\forall t \geqslant t_0$, provided $\sup \beta > \|A_{\rho(t_0)}x_0\|$.

The proposition is proved. □

Similarly, we have the following proposition.

Proposition 3.4. *The trivial solution $x \equiv 0$ of (3.1) is A-stable (resp., P-stable) if and only if for each $t_0 \in \mathbb{T}_\tau^k$ and any solution $x(t; t_0, x_0)$ of (3.1) there exists a function $\varphi_{t_0} \in \mathcal{F}$ such that there holds the following:*

$$\|x(t; t_0, x_0)\| \leqslant \varphi_{t_0}\left(\left\|A_{\rho(t_0)}x_0\right\|\right) \quad \left(\textit{resp.}, \ \|x(t; t_0, x_0)\| \leqslant \varphi_{t_0}\left(\left\|P_{\rho(t_0)}x_0\right\|\right)\right) \quad \forall t \geqslant t_0, \tag{3.9}$$

provided $\|A_{\rho(t_0)}x_0\| \in \mathfrak{D}(\varphi_{t_0})$ (resp., $\|P_{\rho(t_0)}x_0\| \in \mathfrak{D}(\varphi_{t_0})$).

In order to use the Lyapunov function technique related to (3.1), we suppose that $A_{\rho(t)} \in C^1_{\mathrm{rd}}(\mathbb{T}_\tau^k, \mathbb{R}^{m\times m})$. By using (2.3), we can define the derivative of the function $V : \mathbb{T}_\tau \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$ along every solution curve as follows:

$$\begin{aligned} V^{\Delta}_{(3.10)}(t, A_{\rho(t)}x) &= V_t^{\Delta}(t, A_{\rho(t)}x) \\ &\quad + \int_0^1 \left\langle V_x'\left(\sigma(t), A_{\rho(t)}x + h\mu(t)\left(A_{\rho(t)}x\right)^{\Delta}\right), \left(A_{\rho(t)}x\right)^{\Delta} \right\rangle dh. \end{aligned} \tag{3.10}$$

Remark 3.5. Note that when the function V is independent of t and even if the vector field associated with the implicit dynamic equation (3.1) is autonomous, the derivative $V^{\Delta}_{(3.10)}$ may depend on t.

Theorem 3.6. *Assume that there exist a constant $c > 0$, $-c \in \mathcal{R}^+$ and a function $V : \mathbb{T}_\tau \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$ being rd-continuous and a function $\psi \in \mathcal{F}$, ψ defined on $[0, \infty)$ satisfying*

(1) *$\psi(\|x\|) \leqslant V(t, A_{\rho(t)}x)$ for all $x \in \Omega_t$ and $t \in \mathbb{T}_\tau$,*

(2) *$V^{\Delta}_{(3.10)}(t, A_{\rho(t)}x) \leqslant (c/(1 - c\mu(t)))V(t, A_{\rho(t)}x)$, for any $x \in \Omega_t$ and $t \in \mathbb{T}_\tau^k$.*

Assume further that (3.1) *is locally solvable. Then,* (3.1) *is globally solvable, that is, every solution with the initial condition* (3.2) *is defined on* $\mathbb{T}_{t_0}$.

Proof. Denote

$$W(t,x) = V(t,x)e_{-c}(t,t_0). \tag{3.11}$$

By the condition (2), we have

$$\begin{aligned} W^{\Delta}_{(3.10)}(t, A_{\rho(t)}x) &= V^{\Delta}_{(3.10)}(t, A_{\rho(t)}x)e_{-c}(\sigma(t),t_0) - cV(t, A_{\rho(t)}x)e_{-c}(t,t_0) \\ &\leqslant \frac{c}{1-c\mu(t)}V(t, A_{\rho(t)}x)(1-c\mu(t))e_{-c}(t,t_0) - cV(t, A_{\rho(t)}x)e_{-c}(t,t_0) = 0. \end{aligned} \tag{3.12}$$

Therefore, for all $t \geqslant t_0$

$$W(t, A_{\rho(t)}x(t)) - W(t_0, A_{\rho(t_0)}x(t_0)) = \int_{t_0}^{t} W^{\Delta}_{(3.10)}(\tau, A_{\rho(\tau)}x(\tau))\Delta\tau \leqslant 0. \tag{3.13}$$

From the condition (1), it follows that

$$e_{-c}(t,t_0)\varphi(\|x(t)\|) \leqslant W(t, A_{\rho(t)}x(t)) \leqslant W(t_0, A_{\rho(t_0)}x(t_0)) = V(t_0, A_{\rho(t_0)}x(t_0)) \tag{3.14}$$

or

$$\|x(t)\| \leqslant \varphi^{-1}(V(t_0, A_{\rho(t_0)}x(t_0))e_{\ominus(-c)}(t,t_0)) = \varphi^{-1}(V(t_0, A_{\rho(t_0)}x(t_0))e_{(c/(1-c\mu(t)))}(t,t_0)). \tag{3.15}$$

The last inequality says that the solution $x(t)$ can be lengthened on $\mathbb{T}_{t_0}$, that is, (3.1) is globally solvable. □

Theorem 3.7. *Assume that there exist a function* $V : \mathbb{T}_\tau \times \mathbb{R}^m \to \mathbb{R}_+$ *being rd-continuous and a function* $\varphi \in F$, φ *defined on* $[0,\infty)$ *satisfying the conditions*

(1) $V(t,0) \equiv 0$ *for all* $t \in \mathbb{T}_\tau$,

(2) $\varphi(\|x\|) \leqslant V(t, A_{\rho(t)}x)$ *for all* $x \in \Omega_t$ *and* $t \in \mathbb{T}_\tau$,

(3) $V^{\Delta}_{(3.10)}(t, A_{\rho(t)}x) \leqslant 0$ *for any* $x \in \Omega_t$ *and* $t \in \mathbb{T}^k_\tau$.

Assume further that (3.1) *is locally solvable. Then the trivial solution of* (3.1) *is stable.*

Proof. By virtue of Theorem 3.6 and the conditions (2) and (3), it follows that (3.1) is globally solvable. Suppose on the contrary that the trivial solution $x \equiv 0$ of (3.1) is not stable. Then, there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists a solution $x(t)$ of (3.1) satisfying $\|A_{\rho(t_0)}x(t_0)\| < \delta$ and $\|x(t_1;t_0,x(t_0))\| \geqslant \epsilon_0$ for some $t_1 \geqslant t_0$. Put $\epsilon_1 = \varphi(\epsilon_0)$.

By the assumption that $V(t_0,0)=0$ and $V(t,x)$ is rd-continuous, we can find $\delta_0>0$ such that if $\|y\|<\delta_0$ then $V(t_0,y)<\epsilon_1$. With given $\delta_0>0$, let $x(t)$ be a solution of (3.1) such that $\|A_{\rho(t_0)}x(t_0)\|<\delta_0$ and $\|x(t_1;t_0,x(t_0))\|\geqslant\epsilon_0$ for some $t_1\geqslant t_0$.

Since $x(t)\in\Omega_t$ and by the condition (3),

$$\int_{t_0}^{t_1} V^{\Delta}_{(3.10)}\left(t,A_{\rho(t)}x(t)\right)\Delta t = V\left(t_1,A_{\rho(t_1)}x(t_1)\right)-V\left(t_0,A_{\rho(t_0)}x(t_0)\right)\leqslant 0. \tag{3.16}$$

Therefore, $V(t_1,A_{\rho(t_1)}x(t_1))\leqslant V(t_0,A_{\rho(t_0)}x(t_0))<\epsilon_1$. Further, $x(t_1)\in\Omega_{t_1}$ and by the condition (2) we have $V(t_1,A_{\rho(t_1)}x(t_1))\geqslant\varphi(\|x(t_1)\|)\geqslant\varphi(\epsilon_0)=\epsilon_1$. This is a contradiction. The theorem is proved. □

Theorem 3.8. *Assume that there exist a function $V:\mathbb{T}_\tau\times\mathbb{R}^m\to\mathbb{R}_+$ being rd-continuous and functions $\varphi,\phi\in\mathcal{F}$, φ defined on* $[0,\infty)$, $\delta\in C_{\mathrm{rd}}([t_0,\infty),[0,\infty))$ *such that*

$$\int_{t_0}^{t}\delta(s)\Delta s\longrightarrow\infty\quad\text{as } t\longrightarrow\infty, \tag{3.17}$$

satisfying the conditions

(1) $\lim_{x\to 0}V(t,x)=0$ *uniformly in* $t\in\mathbb{T}_\tau$,

(2) $\varphi(\|x\|)\leqslant V(t,A_{\rho(t)}x)$ *for all $x\in\Omega_t$ and $t\in\mathbb{T}_\tau$*,

(3) $V^{\Delta}_{(3.10)}(t,A_{\rho(t)}x)\leqslant-\delta(t)\phi(\|A_{\rho(t)}x\|)$ *for any $x\in\Omega_t$ and $t\in\mathbb{T}^k_\tau$.*

Further, (3.1) *is locally solvable. Then the trivial solution of* (3.1) *is asymptotically stable.*

Proof. Also from Theorem 3.6 and the conditions (2) and (3), it implies that (3.1) is globally solvable.

And since $V^{\Delta}_{(3.10)}(t,A_{\rho(t)}x)\leqslant-\delta(t)\phi(\|A_{\rho(t)}x\|\leqslant 0$, the trivial solution of (3.1) is stable by Theorem 3.7. Consider a bounded solution $x(t)$ of (3.1). First, we show that $\liminf_{t\to\infty}V(t,A_{\rho(t)}x(t))=0$. Assume on the contrary that $\inf_{t\in\mathbb{T}_{t_0}}V(t,A_{\rho(t)}x(t))>0$. From the condition (1), it follows that $\inf_{t\in\mathbb{T}_{t_0}}\|A_{\rho(t)}x(t)\|:=r>0$. By the condition (3), we have

$$\begin{aligned}
V\left(t,A_{\rho(t)}x(t)\right) &= V\left(t_0,A_{\rho(t_0)}x(t_0)\right)+\int_{t_0}^{t}V^{\Delta}_{(3.10)}\left(s,A_{\rho(s)}x(s)\right)\Delta s\\
&\leqslant V\left(t_0,A_{\rho(t_0)}x(t_0)\right)-\int_{t_0}^{t}\delta(s)\phi\left(\|A_{\rho(s)}x(s)\|\right)\Delta s\leqslant V(t_0,x(t_0))\\
&\quad-\phi(r)\int_{t_0}^{t}\delta(s)\Delta s\longrightarrow-\infty,
\end{aligned} \tag{3.18}$$

as $t\to\infty$, which gets a contradiction.

Thus, $\inf_{t\in\mathbb{T}_{t_0}}V(t,A_{\rho(t)}x(t))=0$. Further, from the condition (3) for any $s\leqslant t$ we get

$$V\left(t,A_{\rho(t)}x(t)\right)-V\left(s,A_{\rho(s)}x(s)\right)=\int_{s}^{t}V^{\Delta}_{(3.10)}\left(\tau,A_{\rho(\tau)}x(\tau)\right)\Delta\tau\leqslant 0. \tag{3.19}$$

This means that $V(t, A_{\rho(t)}x(t))$ is a decreasing function. Consequently,

$$\lim_{t\to\infty} V\big(t, A_{\rho(t)}x(t)\big) = \inf_{t\in\mathbb{T}_{t_0}} V\big(t, A_{\rho(t)}x(t)\big) = 0, \tag{3.20}$$

which follows that $\lim_{t\to\infty}\|x(t)\| = 0$ by the condition (2). $\square$

Theorem 3.9. *Suppose that there exist a function $a \in \mathcal{F}$, a defined on $[0,\infty)$, and a function $V \in C_{\mathrm{rd}}(\mathbb{T}_\tau \times \mathbb{R}^m, \mathbb{R}_+)$ such that*

(1) $\lim_{x\to 0} V(t,x) = 0$ *uniformly in $t \in \mathbb{T}_\tau$ and $a(\|x\|) \leqslant V(t, A_{\rho(t)}x)$ for all $x \in \Omega_t$ and $t \in \mathbb{T}_\tau$,*

(2) $V^{\Delta}_{(3.10)}(t, A_{\rho(t)}x) \leqslant 0$, *for any $x \in \Omega_t$ and $t \in \mathbb{T}^k_\tau$.*

Assume further that (3.1) *is locally solvable. Then, the trivial solution of* (3.1) *is A-uniformly stable.*

Proof. The proof is similar to the one of Theorem 3.7 with a remark that since $\lim_{x\to 0} V(t,x) = 0$ uniformly in $t \in \mathbb{T}_\tau$, we can find $\delta_0 > 0$ such that if $\|y\| < \delta_0$ then $\sup_{t\in\mathbb{T}_\tau} V(t,y) < \epsilon_1$.

The proof is complete. $\square$

Remark 3.10. The conclusion of Theorem 3.9 is still true if the condition (1) is replaced by *"there exist two functions $a, b \in \mathcal{F}$, a defined on $[0,\infty)$ and a function $V \in C_{\mathrm{rd}}(\mathbb{T}_\tau \times \mathbb{R}^m, \mathbb{R}_+)$ such that $a(\|x\|) \leqslant V(t, A_{\rho(t)}x) \leqslant b(\|A_{\rho(t)}x\|)$ for all $x \in \Omega_t$ and $t \in \mathbb{T}_\tau$".*

We present a theorem of uniform global asymptotical stability.

Theorem 3.11. *If there exist functions $a, b, c \in \mathcal{F}$, a defined on $[0,\infty)$, and a function $V \in C_{\mathrm{rd}}(\mathbb{T}_\tau \times \mathbb{R}^m, \mathbb{R}_+)$ satisfying*

(1) $a(\|x\|) \leqslant V(t, A_{\rho(t)}x) \leqslant b(\|A_{\rho(t)}x\|)$ *for all $x \in \Omega_t$ and $t \in \mathbb{T}_\tau$,*

(2) $V^{\Delta}_{(3.10)}(t, A_{\rho(t)}x) \leqslant -c(\|A_{\rho(t)}x)\|)$ *for any $x \in \Omega_t$ and $t \in \mathbb{T}^k_\tau$.*

Assume further that (3.1) *is locally solvable. Then, the trivial solution of* (3.1) *is A-uniformly globally asymptotically stable.*

Proof. Let $\delta_0 > 0$ be given. Define $\delta(\epsilon) = \min\{b^{-1}(a(\epsilon)), \delta_0\}$ and

$$T(\epsilon) = \max_{t\in\mathbb{T}} \mu(t) + \frac{2b(\delta_0)}{c(\delta(\epsilon))}. \tag{3.21}$$

($T(\epsilon)$ is not necessary in $\mathbb{T}$).

Let $x(t)$ be a solution of (3.1) with $\|A_{\rho(t_0)}x(t_0)\| < \delta(\epsilon)$. From the condition (2), we see that

$$V\big(t, A_{\rho(t)}x(t)\big) - V\big(t_0, A_{\rho(t_0)}x(t_0)\big) = \int_{t_0}^{t} V^{\Delta}_{(3.10)}\big(s, A_{\rho(s)}x(s)\big)\Delta s \leqslant 0. \tag{3.22}$$

Therefore,

$$a(\|x(t)\|) \leqslant V\big(t, A_{\rho(t)}x(t)\big) \leqslant V\big(t_0, A_{\rho(t_0)}x(t_0)\big) \leqslant b\big(\big\|A_{\rho(t_0)}x(t_0)\big\|\big) < b(\delta(\epsilon)) \leqslant a(\epsilon). \quad (3.23)$$

Hence, $\|x(t)\| < \epsilon$ for all $t \geqslant t_0$.

Because the trivial solution of (3.1) is A-uniformly stable, we only need to show that there exists a $t^* \in [t_0, t_0 + T(\epsilon)]$ such that $\|A_{\rho(t^*)}x(t^*)\| < \delta(\epsilon)$. Assume that such a t^* does not exist, that is $\|A_{\rho(t)}x(t)\| \geqslant \delta(\epsilon)$ for all $t \in [t_0, t_0 + T(\epsilon)]$. From the condition (2), we get

$$\begin{aligned} &V\big(t_0 + T(\epsilon), A_{\rho(t_0+T(\epsilon))}x(t_0 + T(\epsilon))\big) + \int_{t_0}^{t_0+T(\epsilon)} c\big(\big\|A_{\rho(s)}x(s)\big\|\big)\Delta s \\ &\quad\leqslant V\big(t_0, A_{\rho(t_0)}x(t_0)\big) \\ &\quad\leqslant b\big(\big\|A_{\rho(t_0)}x(t_0)\big\|\big)b \leqslant (\delta_0). \end{aligned} \quad (3.24)$$

Since $V \geqslant 0$,

$$c(\delta(\epsilon))T(\epsilon) \leqslant b(\delta) \Longrightarrow T(\epsilon)\frac{b(\delta_0)}{c(\delta(\epsilon))}, \quad (3.25)$$

which contradicts the definition of $T(\epsilon)$ in (3.21). The proof is complete. □

When $A_{\rho(t)}$ is not differentiable, one supposes that there exists a Δ-differentiable projector Q_t onto $\ker A_t$ and $(Q_{\rho(t)})^{\Delta}$ is rd-continuous on $\mathbb{T}_\tau^k$; moreover, $Q_{\rho(t)} = Q_t$ for all $t \in (\mathbb{T}_\tau)_{\mathrm{rd}}^{\mathrm{ls}}$. Let $P_t = I - Q_t$.

We choose matrix functions $T_t, B_t \in C_{\mathrm{rd}}(\mathbb{T}_\tau^k, \mathbb{R}^{m\times m})$ such that $T_t|_{\ker A_t}$ is an isomorphism between $\ker A_t$ and $\ker A_{\rho(t)}$ and the matrix $G_t = A_t - B_tT_tQ_t$ is invertible. Define

$$V_{(3.26)}^{\Delta}\big(t, P_{\rho(t)}x\big) = V_t^{\Delta}\big(t, P_{\rho(t)}x\big) + \int_0^1 \Big\langle V_x'\Big(\sigma(t), P_{\rho(t)}x + h\mu(t)\big(P_{\rho(t)}x\big)^{\Delta}\Big), \big(P_{\rho(t)}x\big)^{\Delta}\Big\rangle dh, \quad (3.26)$$

where $(P_{\rho(t)}x)^{\Delta} = (P_{\rho(t)})^{\Delta}x + P_tG_t^{-1}f(t,x)$ (see (2.51)).

From now on we remain following the above assumptions on the operators Q_t, T_t, B_t whenever $V_{(3.26)}^{\Delta}(t, P_{\rho(t)}x)$ is mentioned.

By the same argument as Theorem 3.6, we have the following theorem.

Theorem 3.12. *Assume that there exist a constant $c > 0$, $-c \in \mathcal{R}^+$ and a function $V : \mathbb{T}_\tau\times\mathbb{R}^m \to \mathbb{R}_+$ being rd-continuous and a function $\varphi \in \mathcal{F}$, φ defined on $[0,\infty)$ satisfying*

(1) $\varphi(\|x\|) \leqslant V(t, P_{\rho(t)}x)$ *for all* $x \in \Omega_t$ *and* $t \in \mathbb{T}_\tau$,

(2) $V_{(3.26)}^{\Delta}(t, P_{\rho(t)}x) \leqslant (c/(1 - c\mu(t)))V(t, P_{\rho(t)}x)$, *for any* $x \in \Omega_t$ *and* $t \in \mathbb{T}_\tau^k$.

Assume further that (3.1) *is locally solvable. Then,* (3.1) *is globally solvable.*

Theorem 3.13. *Assume that* (3.1) *is locally solvable. Then, the trivial solution* $x \equiv 0$ *of* (3.1) *is stable if there exist a function* $V : \mathbb{T}_\tau \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$ *being rd-continuous and a function* $\psi \in \mathcal{F}$, ψ *defined on* $[0, \infty)$ *such that*

(1) $V(t, 0) \equiv 0$ *for all* $t \in \mathbb{T}_\tau$,

(2) $V(t, P_{\rho(t)}y) \geqslant \psi(\|y\|)$ *for all* $y \in \Omega_t$ *and* $t \in \mathbb{T}_\tau$,

(3) $V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x) \leqslant 0$ *for all* $x \in \Omega_t$ *and* $t \in \mathbb{T}^k_\tau$.

Proof. Assume that there is a function V satisfying the assertions (1), (2), and (3) but the trivial solution $x \equiv 0$ of (3.1) is not stable. Then, there exist a positive $\epsilon_0 > 0$ and a $t_0 \in \mathbb{T}^k_\tau$ such that $\forall \delta > 0$; there exists a solution $x(t) = x(t; t_0, x_0)$ of (3.1) satisfying $\|P_{\rho(t_0)}x_0\| < \delta$ and $x(t_1; t_0, x_0) \geqslant \epsilon_0$, for some $t_1 \geqslant t_0$. Let $\epsilon_1 = \psi(\epsilon_0)$. Since $V(t_0, 0) = 0$, it is possible to find a $\delta = \delta(\epsilon_0, t_0) > 0$ satisfying $V(t_0, P_{\rho(t_0)}z) < \epsilon_1$ when $\|P_{\rho(t_0)}z\| < \delta$, $z \in \mathbb{R}^m$. Consider the solution $x(t)$ satisfying $\|P_{\rho(t_0)}x_0\| < \delta$ and $x(t_1; t_0, x_0) \geqslant \epsilon_0$ for a $t_1 \geqslant t_0$.

From the assumption (3), it follows that

$$\int_{t_0}^{t_1} V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x(t))\Delta t = V(t_1, P_{\rho(t_1)}x(t_1)) - V(t_0, P_{\rho(t_0)}x_0) \leqslant 0. \tag{3.27}$$

This implies

$$V(t_0, P_{\rho(t_0)}x_0) \geqslant V(t_1, P_{\rho(t_1)}x(t_1)) \geqslant \psi\|(x(t_1))\| \geqslant \psi(\epsilon_0) = \epsilon_1. \tag{3.28}$$

We get a contradiction because $\epsilon_1 > V(t_0, P_{\rho(t_0)}x_0)$ when $\|P_{\rho(t_0)}x_0\| < \delta$.

The proof of the theorem is complete. □

Theorem 3.14. *Assume that* (3.1) *is locally solvable. If there exist two functions* $a, b \in \mathcal{F}$, *a defined on* $[0, \infty)$ *and a function* $V : \mathbb{T}_\tau \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$ *being rd-continuous such that*

(1) $a(\|x\|) \leqslant V(t, P_{\rho(t)}x) \leqslant b(\|P_{\rho(t)}x\|)$ *for all* $x \in \Omega_t$ *and* $t \in \mathbb{T}_\tau$,

(2) $V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x) \leqslant 0$ *for all* $x \in \Omega_t$ *and* $t \in \mathbb{T}^k_\tau$,

then the trivial solution of (3.1) *is P-uniformly stable.*

Proof. The proof is similar to the one of Theorem 3.9. □

Theorem 3.15. *If there exist functions* $a, b, c \in \mathcal{F}$, *a defined on* $[0, \infty)$ *and a function* $V \in C_{\mathrm{rd}}(\mathbb{T}_\tau \times \mathbb{R}^m, \mathbb{R}_+)$ *satisfying*

(1) $a(\|x\|) \leqslant V(t, P_{\rho(t)}x) \leqslant b(\|P_{\rho(t)}x\|)$ *for all* $x \in \Omega_t$ *and* $t \in \mathbb{T}_\tau$,

(2) $V^{\Delta}_{(3.10)}(t, P_{\rho(t)}x) \leqslant -c(\|P_{\rho(t)}x)\|)$ *for any* $x \in \Omega_t$ *and* $t \in \mathbb{T}^k_\tau$.

Assume further that (3.1) *is locally solvable. Then, the trivial solution of* (3.1) *is P-uniformly globally asymptotically stable.*

Proof. Similarly to the proof of Theorem 3.11. □

It is difficult to establish the inverse theorem for Theorems from 3.7 to 3.15, that is, if the trivial solution of (3.1) is stable, there exists a function V satisfying the assertions in the above theorems. However, if the structure of the time scale $\mathbb{T}$ is rather simple we have the following theorem.

Theorem 3.16. *Suppose that $\mathbb{T}_\tau$ contains no right-dense points and the trivial solution $x \equiv 0$ of* (3.1) *is P-uniformly stable. Then, there exists a function $V : \mathbb{T}_\tau \times U \to \mathbb{R}_+$ being rd-continuous satisfying the conditions (1), (2), and (3) of Theorem 3.13, where U is an open neighborhood of 0 in $\mathbb{R}^m$.*

Proof. Suppose the trivial solution of (3.1) is P-uniformly stable. Due to Proposition 3.3, there exist functions $\varphi \in \mathcal{F}$ such that for any solution $x(t; t_0, x_0)$ of (3.1), we have

$$\|x(t; t_0, x_0)\| \leqslant \varphi\left(\left\|P_{\rho(t_0)}x_0\right\|\right) \quad \forall t \geqslant t_0, \tag{3.29}$$

provided $\|P_{\rho(t_0)}x_0\| \in \mathfrak{D}(\varphi)$.

Let $\mathfrak{D}(\varphi) = [0, a)$ and $U = \{x : \|x\| < a\}$. For any $z \in \mathbb{R}^m$ satisfying $\|P_{\rho(t_0)}z\| < a$ and $t \in \mathbb{T}_\tau$, we put

$$V(t, z) := \sup_{s \geqslant t}\|x(s; t, z)\|, \tag{3.30}$$

where $x(s; t, z)$ is the unique solution of (3.1) satisfying the initial condition $P_{\rho(t)}x(t) = P_{\rho(t)}z$. It is seen that V is defined for all z satisfying $\|P_{\rho(t_0)}z\| \in \mathfrak{D}(\varphi)$, $V(t, 0) \equiv 0$, and $V(t, x) \in C_{\mathrm{rd}}(\mathbb{T}_\tau \times \mathbb{R}^m, \mathbb{R}_+)$.

Let $y \in \Omega_t$. By the definition, $V(t, P_{\rho(t)}y) = \sup_{s \geqslant t}\|x(s; t, P_{\rho(t)}y)\| \geqslant \|x(t; t, P_{\rho(t)}y)\|$. From (2.60), $x(s; t, P_{\rho(t)}y) = u(s; t, P_{\rho(t)}y) + g(s, u(s; t, P_{\rho(t)}y))$ for all $s \in \mathbb{T}_t$. In particular, $x(t; t, P_{\rho(t)}y) = P_{\rho(t)}y + g(t, P_{\rho(t)}y) = y$. Thus, $V(t, P_{\rho(t)}y) \geqslant \|y\|$ $\forall y \in \Omega_t$, $t \in \mathbb{T}_\tau$. Hence, we have the assertion (2) of the theorem.

Due to the unique solvability of (3.1), we have $x(s; t, P_{\rho(t)}y) = x(s; \sigma(t), x(\sigma(t), t, P_{\rho(t)}y))$ with $s \geqslant \sigma(t)$. Therefore, $V(t, P_{\rho(t)}y) = \sup_{s \geqslant t}\|x(s; t, P_{\rho(t)}y)\|$ and

$$\begin{aligned} V\left(\sigma(t), P_{\rho(\sigma(t))}x\left(\sigma(t), t, P_{\rho(t)}y\right)\right) &= \sup_{s \geqslant \sigma(t)}\left\|x\left(s; \sigma(t), x\left(\sigma(t), t, P_{\rho(t)}y\right)\right)\right\| \\ &= \sup_{s \geqslant \sigma(t)}\left\|x\left(s; t, P_{\rho(t)}y\right)\right\| \leqslant V\left(t, P_{\rho(t)}y\right). \end{aligned} \tag{3.31}$$

This implies

$$V^{\Delta}_{(3.26)}\left(t, P_{\rho(t)}y(t)\right) = \frac{V\left(\sigma(t), P_{\rho(\sigma(t))}x\left(\sigma(t), t, P_{\rho(t)}y\right)\right) - V\left(t, P_{\rho(t)}y\right)}{\mu(t)} \leqslant 0. \tag{3.32}$$

The proof is complete. ◻

Now we give an example on using Lyapunov functions to test the stability of equations. The following result finds out that the stability of a linear equation will be ensured if nonlinear perturbations are sufficiently small Lipschitz.

Consider a nonlinear equation of the form (2.3)

$$Ax^{\Delta} = Bx + f(t, x), \tag{3.33}$$

where A and B are constant matrices with ind $(A, B) = 1$, $f(t, 0) = 0\ \forall t \in \mathbb{T}$, and $f(t, x)$ satisfing the Lipschitz condition

$$\|f(t, x) - f(t, y)\| < L\|x - y\|, \tag{3.34}$$

where L is sufficiently small. Let Q be defined by (2.9) with $T_t = I$ and $G = A - BQ, P = I - Q$. By Theorem 2.7, we see that there exists a unique solution satisfying the condition $P(x(t_0) - x_0) = 0$ for any $x_0 \in \mathbb{R}^m$.

Besides, also consider the homogeneous equation associated to (3.33)

$$Ax^{\Delta} = Bx, \tag{3.35}$$

and suppose this equation has index-1. As in Section 2, multiplying (3.33) by PG^{-1} we get

$$(Px)^{\Delta} = Mx + PG^{-1}f(t, x), \tag{3.36}$$

where $M = PG^{-1}B = PG^{-1}BP$.

Note that the general solution of (3.35) is

$$x(t; t_0, x_0) = e_M(t, t_0)Px(t_0) = \exp(tM)\left(\prod_{s\in I_{t,t_0}} (I + \mu(s)M) \exp(-\mu(s)M)\right)Px(t_0), \quad t \geqslant t_0, \tag{3.37}$$

in there I_{t,t_0} is denoted the set of right-scattered points of the interval $[t_0, t)$.

Denote $\sigma(A, B) = \{\lambda : \det(\lambda A - B) = 0\}$. It is easy to show that the trivial solution $x \equiv 0$ of (3.35) is P-uniformly exponentially stable if and only if $\sigma(A, B) \subset S$, where S is the domain of uniform exponential stability of $\mathbb{T}$. On the exponential stable domain of a time scale, we can refer to [10, 18, 19]. By the definition of exponential stability, it implies that the graininess function of the time scale $\mathbb{T}$ is upper bounded. Let $\mu^* = \sup_{t\in\mathbb{T}}\mu(t)$.

We denote the set

$$U = \begin{cases} \left\{\lambda : \left|\lambda + \dfrac{1}{\mu^*}\right| \leqslant \dfrac{1}{\mu^*}\right\} & \text{if } \mu^* \neq 0 \\ \{\lambda : \Re\lambda < 0\} & \text{if } \mu^* = 0, \end{cases} \tag{3.38}$$

and suppose $\sigma(A, B) \subset U$. Since $U \subset S$, this condition implies that (3.35) is P-uniformly exponentially stable.

If $\mu^* \neq 0$, define

$$H = \mu^* \sum_{k=0}^{\infty} \left(I + \mu^* M^{\top}\right)^n P^{\top} F P (I + \mu^* M)^n + Q^{\top} F Q, \tag{3.39}$$

where the matrix F is supposed to be symmetric positive definite. It is clear that H is symmetric positive definite.

Since $\sigma(A, B) \subset U$, the above series is convergent. Further, for any $k \geqslant 0$ we have

$$\begin{aligned}
&\left(I+\mu^* M^{\top}\right)^{k+1} P^{\top} F P(I+\mu^* M)^{k+1}-\left(I+\mu^* M^{\top}\right)^{k} P^{\top} F P(I+\mu^* M)^{k} \\
&\quad=\left(I+\mu^* M^{\top}\right)^{k+1} P^{\top} F P\left((I+\mu^* M)^{k+1}-(I+\mu^* M)^{k}\right) \\
&\qquad+\left(\left(I+\mu^* M^{\top}\right)^{k+1}-\left(I+\mu^* M^{\top}\right)^{k}\right) P^{\top} F P(I+\mu^* M)^{k} \\
&\quad=\left(I+\mu^* M^{\top}\right) \mu^*\left(I+\mu^* M^{\top}\right)^{k} P^{\top} F P \\
&\qquad\times(I+\mu^* M)^{k} M+\mu^* M^{\top}\left(I+\mu^* M^{\top}\right)^{k} P^{\top} F P(I+\mu^* M)^{k} .
\end{aligned} \tag{3.40}$$

Thus,

$$\begin{aligned}
&\left(I+\mu^* M^{\top}\right)^{n+1} P^{\top} F P(I+\mu^* M)^{n+1}-P^{\top} F P \\
&\quad=\left(I+\mu^* M^{\top}\right) \sum_{k=0}^{n} \mu^*\left(I+\mu^* M^{\top}\right)^{k} P^{\top} F P(I+\mu^* M)^{k} M \\
&\qquad+\mu^* M^{\top} \sum_{k=0}^{n}\left(I+\mu^* M^{\top}\right)^{k} P^{\top} F P(I+\mu^* M)^{k} .
\end{aligned} \tag{3.41}$$

Letting $n \to \infty$ and paying attention to $\lim_{n\to\infty}(I+\mu^* M^{\top})^{n} P^{\top} F P(I+\mu^* M)^{n}=0$, we obtain

$$-P^{\top} F P=\left(I+\mu^* M^{\top}\right) H M+M^{\top} H=H M+M^{\top} H+\mu^* M^{\top} H M. \tag{3.42}$$

In the case where $\mu^*=0$ and F is symmetric positive definite, by putting

$$H=\int_{0}^{\infty} \exp\left(t M^{\top}\right) P^{\top} F P \exp(t M) d t+Q^{\top} F Q, \tag{3.43}$$

we can examine easily that the matrix H also satisfies (3.42), H is symmetric and positive definite.

Theorem 3.17. *Suppose that $\sigma(A, B) \subset U$ and the homogeneous equation* (3.35) *is of index-1 and the constant L is sufficiently small. Then, the trivial solution $x \equiv 0$ of* (3.33) *is P-uniformly globally asymptotically stable.*

Proof. Let H be a symmetric and positive definite (constant) matrix satisfying (3.42). Consider the Lyapunov function $V(x) := x^{\top}Hx$. The derivative of V along the solution of (3.33) is

$$\begin{aligned}
V^{\Delta}_{(3.26)}(Px) &= \left((Px)^{\Delta}\right)^{\top} H(Px)^{\sigma} + (Px)^{\top}H(Px)^{\Delta} \\
&= \left(Mx + PG^{-1}f(t,x)\right)^{T} H\left(Px + \mu(t)\left(Mx + PG^{-1}f(t,x)\right)\right) \\
&\quad + (Px)^{\top}H\left(Mx + PG^{-1}f(t,x)\right) \\
&= \left(Mx + PG^{-1}f(t,x)\right)^{\top} HPx + \mu(t)\left(Mx + PG^{-1}f(t,x)\right)^{T} H\left(Mx + PG^{-1}f(t,x)\right) \\
&\quad + (Px)^{\top}H\left(Mx + PG^{-1}f(t,x)\right) \\
&\leqslant \left(Mx + PG^{-1}f(t,x)\right)^{\top} HPx + \mu^{*}\left(Mx + PG^{-1}f(t,x)\right)^{\top} H\left(Mx + PG^{-1}f(t,x)\right) \\
&\quad + (Px)^{\top}H\left(Mx + PG^{-1}f(t,x)\right) \\
&= (Px)^{\top}\left(M^{\top}H + HM + \mu^{*}M^{\top}HM\right)Px + \left(PG^{-1}f(t,x)\right)^{\top} \\
&\quad \times H\left(Px + \mu^{*}Mx + \mu^{*}PG^{-1}f(t,x)\right) \\
&\quad + (Px)^{\top}H\left(I + \mu^{*}M^{\top}\right)PG^{-1}f(t,x) \\
&= -(Px)^{\top}P^{\top}FPPx + \left(PG^{-1}f(t,x)\right)^{\top} H\left(Px + \mu^{*}Mx + \mu^{*}PG^{-1}f(t,x)\right) \\
&\quad + (Px)^{\top}H\left(I + \mu^{*}M^{\top}\right)PG^{-1}f(t,x) \\
&= -(Px)^{\top}FPx + \left(PG^{-1}f(t,x)\right)^{\top} H\left(Px + \mu^{*}MPx + \mu^{*}PG^{-1}f(t,x)\right) \\
&\quad + (Px)^{\top}H\left(I + \mu^{*}M^{\top}\right)PG^{-1}f(t,x).
\end{aligned} \tag{3.44}$$

From the Lipschitz condition and (2.25), it is seen that $\|Qx\| \leqslant K\|Px\|$ where $K = (\|QG^{-1}B\| + L\|QG^{-1}\|)/(1 - L\|QG^{-1}\|)$. Therefore,

$$\left\|f(t,x)\right\| \leqslant L(1+K)\|Px\|. \tag{3.45}$$

Combining this inequality and the above appreciation, we see that when L is sufficiently small there exists $\beta > 0$ such that

$$V^{\Delta}_{(3.26)}(Px) \leqslant -\beta\|Px\|^{2}. \tag{3.46}$$

By Theorem 3.15, (3.33) is P-uniformly globally asymptotically stable. □

Example 3.18. Let $\mathbb{T} = \cup_{k=0}^{\infty}[2k, 2k+1]$ and consider

$$A_t x^{\Delta} = B_t x + f(t, x), \tag{3.47}$$

with

$$A_t = (t+1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_t = \begin{pmatrix} -t-2 & 0 \\ 0 & -t-1 \end{pmatrix}, \qquad f(t,x) = \frac{\sin x_1}{t+1}(0,1)^{\top}. \tag{3.48}$$

We have $\ker A_t = \operatorname{span}\{(0,1)^{\top}\}$, rank $A_t = 1$ for all $t \in \mathbb{T}$. It is easy to verify that $Q_t = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the canonical projector onto $\ker A_t$, $P_t = I - Q_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let us choose $T_t = I$. We see that

$$G_t = A_t - B_t T_t Q_t = (t+1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.49}$$

Since $t \geqslant 0$, $\det G_t = (t+1)^2 \neq 0$, (3.47) has index-1.

It is obvious that $\|f(t,w_1) - f(t,w_2)\| \leqslant (1/(t+1))\|w_1 - w_2\|$, $\forall w_1, w_2 \in \mathbb{R}^2$. Further, $\gamma_t = L_t\|T_t Q_t G_t^{-1}\| = 1/(t+1)^2 < 1$, for all $t \in \mathbb{T}$. Thus, according to Theorem 2.7 for each $t_0 \in \mathbb{T}$, (3.47) with the initial condition $P_{\rho(t_0)}x(t_0) = P_{\rho(t_0)}x_0$ has the unique solution.

It is easy to compute, $G_t^{-1} = (1/(t+1))\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $T_t Q_t G_t^{-1} B_t P_{\rho(t)} x = (0,0)^{\top}$, $T_t Q_t G_t^{-1} f(t,x) = (\sin x_1/(t+1)^2)(0,1)^{\top}$, where $x = (x_1, x_2)^{\top}$, $P_t G_t^{-1} B_t = (-1/(t+1))\left(\begin{smallmatrix} t+2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $P_t G_t^{-1} f(t,x) = (0,0)^{\top}$.

Therefore, $u(t) = P_{\rho(t)}x(t)$ satisfies $u^{\Delta} = -(1/(t+1))\left(\begin{smallmatrix} t+2 & 0 \\ 0 & 0 \end{smallmatrix}\right)u$. Moreover, we have

$$Ł_t = \left\{ x = (x_1, x_2)^{\top} \in \mathbb{R}^2,\ x_2 = \frac{\sin x_1}{(t+1)^2} \right\}. \tag{3.50}$$

Let the Lyapunov function be $V(t,x) := 2\|x\|$, $t \in \mathbb{T}$, $x \in \mathbb{R}^2$.

Put $x = (x_1, x_2)^{\top} \in Ł_t$, we have $V(t, P_{\rho(t)}x) = 2\|P_{\rho(t)}x\| = 2|x_1|$ and

$$\|x\| = \left(x_1^2 + x_2^2\right)^{1/2} = \left(x_1^2 + \frac{\sin^2 x_1}{(t+1)^4}\right)^{1/2} \leqslant \left(x_1^2 + \sin^2 x_1\right)^{1/2} \leqslant 2|x_1|. \tag{3.51}$$

Hence,

$$\|x\| \leqslant V\left(t, P_{\rho(t)}x\right) = 2\left\|P_{\rho(t)}x\right\|, \qquad \forall x \in Ł_t,\ t \in \mathbb{T}. \tag{3.52}$$

We have for any solution $x(t)$ of (3.47) and $t \in \mathbb{T}$ (noting that $t \geqslant 0$),

(+) if t is right-scattered then $V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x(t)) \overset{u=P_{\rho(t)}x}{=} 2(\|u(t+1)\| - \|u(t)\|) = 2(\|u(t) + u^{\Delta}(t)\| - \|u(t)\|) = 2(u_1^2/(t+1)^2 + u_2^2)^{1/2} - (u_1^2 + u_2^2)^{1/2} \leqslant 0$,

(+) if t is right-dense then $V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x(t)) = \langle V'_x(t, P_{\rho(t)}x), F(t, P_{\rho(t)}x)\rangle = -2(t+2)u_1^2/(t+1)\|u\| \leqslant 0$, where $u = (u_1, u_2)^{\top}$, $F(t,u) = (-1/(t+1))\left(\begin{smallmatrix} t+2 & 0 \\ 0 & 0 \end{smallmatrix}\right)u$.

In both two cases, we have $V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x(t)) \leqslant 0$, so the trivial solution of (3.47) is P-uniformly stable by Theorem 3.14.

Note that if we let $V(t,x) := \|x\|^2,\ t \in \mathbb{T},\ x \in \mathbb{R}^2$ then the result is still true. Indeed, by the simple calculations we obtain

(a) $a(\|x\|) \leqslant V(t, P_{\rho(t)}x) \leqslant b(\|P_{\rho(t)}x\|)$, $\forall x \in Ł_t$, $t \in \mathbb{T}$, where $a, b \in \mathcal{F}$ defined by $a(y) = (1/2)y^2, b(y) = y^2, y \in \mathbb{R}_+$,

(b) $V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x(t)) \overset{u=P_{\rho(t)}x}{=} \langle 2u, F(t,u)\rangle + \mu(t)\|F(t,u)\|^2$. Thus,

$$V^{\Delta}_{(3.26)}(t, P_{\rho(t)}x(t)) \overset{u=(u_1,u_2)^{\top}}{=} \begin{cases} -\dfrac{2(t+2)u_1^2}{t+1} & \text{if } t \text{ is right-dense}, \\ -\dfrac{t(t+2)u_1^2}{(t+1)^2} & \text{if } t \text{ is right-scattered}, \end{cases} \leqslant 0. \tag{3.53}$$

Therefore, having the above result is obvious.

4. Conclusion

We have studied some criteria ensuring the stability for a class of quasilinear dynamic equations on time scales. So far, the inverse theorem of the theorems of the stability in Section 3 of this paper is still an open problem for an arbitrary time scale meanwhile it is true for discrete and continuous time scales.

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