J. of Inequal. & Appl., 1998, Vol. 2, pp. 57–69
Reprints available directly from the publisher
Photocopying permitted by license only

On the Existence of Positive Solutions for Certain Difference Equations and Inequalities

CH.G. PHILOS

Department of Mathematics, University of Ioannina, P.O. Box 1186, 451 10 Ioannina, Greece, e-mail: cphilos@cc.uoi.gr

(Received 5 December 1996)

Some new results on the existence of positive solutions for certain difference equations and inequalities are established. The difference equations and inequalities considered are discrete analogues of some integrodifferential equations and inequalities.

Keywords: Difference equation; Difference inequality; Solution; Positive solution

1991 Mathematics Subject Classification: Primary 39A10

1 INTRODUCTION

Within the past two decades, the study of difference equations has acquired a new significance. This comes about, in large part, from the fact that the application of the theory of difference equations is rapidly increasing to various fields such as numerical analysis, control theory, finite mathematics, statistics, economics, biology and computer science. Difference equations arise frequently in the study of biological models, in the formulation and analysis of discrete-time systems, the discretization methods for differential equations, the study of deterministic chaos, etc. For the basic theory of difference equations and its applications the reader is referred to the books by Agarwal [1], Kelley and Peterson [5], and Lakshmikantham and Trigiante [11]. See also Chapter 7 of the book by Györi and Ladas [2] for some results on the oscillation theory of difference equations.

This paper is dealing with the problem of the existence and the nonexistence of positive solutions of certain difference equations and inequalities, which can be considered as discrete versions of some integrodifferential equations and inequalities. The papers by Jaroš and Stavroulakis [4], Kiventidis [6], Kordonis and Philos [7], and Ladas, Philos and Sficas [9] are the only works devoted to such difference equations and inequalities. For some results on the existence of positive solutions of integrodifferential equations (and inequalities), we choose to refer to Györi and Ladas [3], Kiventidis [6], Ladas, Philos and Sficas [10], Philos [12,13], and Philos and Sficas [14].

The results of the present paper are motivated by the oscillation and nonoscillation criteria for linear delay or advanced difference equations, which have been very recently obtained by Kordonis and Philos [8].

Throughout the paper, by $\mathbb{N}$ we will denote the set of all nonnegative integers and the set of all integers will be denoted by $\mathbb{Z}$. Moreover, if $n_0 \in \mathbb{N}$, $\mathbb{N}_{n_0}$ stands for the set $\{n \in \mathbb{N}\colon n \geq n_0\}$; similarly, if $n_0 \in \mathbb{Z}$, $\mathbb{Z}_{n_0}$ stands for the set $\{n \in \mathbb{Z}\colon n \geq n_0\}$. Furthermore, the forward difference operator Δ will be considered to be defined as usual, i.e.

$$\Delta S_n = S_{n+1} - S_n$$

for any sequence (S_n) of real numbers; moreover, if (S_n) is a sequence of real numbers, we define

$$\Delta^0 S_n = S_n \quad \text{and} \quad \Delta^i S_n = \Delta(\Delta^{i-1} S_n) \ \ (i = 1, 2, \ldots).$$

Consider the difference equations

$$\Delta A_n + p_n \sum_{j=0}^{n} K_{n-j} A_j = 0 \qquad (\mathrm{E}_1)$$

and

$$\Delta A_n + q_n \sum_{j=-\infty}^{n} K_{n-j} A_j = 0 \qquad (\mathrm{E}_2)$$

as well as the difference inequalities

$$\Delta B_n + p_n \sum_{j=0}^{n} K_{n-j} B_j \leq 0 \tag{I_1}$$

and

$$\Delta B_n + q_n \sum_{j=-\infty}^{n} K_{n-j} B_j \leq 0, \tag{I_2}$$

where *the coefficients* $(p_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{Z}}$ *and the kernel* $(K_n)_{n\in\mathbb{N}}$ *are sequences of nonnegative real numbers.*

Consider also the more general difference equations

$$(-1)^{m+1}\Delta^m A_n + p_n \sum_{j=0}^{n} K_{n-j} A_j = 0 \tag{$E_1[m]$}$$

and

$$(-1)^{m+1}\Delta^m A_n + q_n \sum_{j=-\infty}^{n} K_{n-j} A_j = 0 \tag{$E_2[m]$}$$

as well as the more general difference inequalities

$$(-1)^{m+1}\Delta^m B_n + p_n \sum_{j=0}^{n} K_{n-j} B_j \leq 0 \tag{$I_1[m]$}$$

and

$$(-1)^{m+1}\Delta^m B_n + q_n \sum_{j=-\infty}^{n} K_{n-j} B_j \leq 0, \tag{$I_2[m]$}$$

where *m is a positive integer.*

For $m = 1$, ($E_1[m]$) and ($E_2[m]$) lead to (E_1) and (E_2) respectively, and ($I_1[m]$) and ($I_2[m]$) take the forms (I_1) and (I_2) respectively.

If $n_0 \in \mathbb{N}$, by a *solution on* $\mathbb{N}_{n_0}$ of ($E_1[m]$) [resp. of ($I_1[m]$)] we mean a sequence of real numbers $(A_n)_{n\in\mathbb{N}}$ [resp. $(B_n)_{n\in\mathbb{N}}$], which satisfies

($E_1[m]$) [resp. ($I_1[m]$)] for every $n \in \mathbb{N}_{n_0}$. In particular, a *solution on* $\mathbb{N}$ of ($I_1[m]$) is a sequence of real numbers $(B_n)_{n\in\mathbb{N}}$ satisfying ($I_1[m]$) for all $n \in \mathbb{N}$.

If n_0 is an integer, then a *solution on* $\mathbb{Z}_{n_0}$ of ($E_2[m]$) [resp. of ($I_2[m]$)] is a sequence of real numbers $(A_n)_{n\in\mathbb{Z}}$ [resp. $(B_n)_{n\in\mathbb{Z}}$], which satisfies ($E_2[m]$) [resp. ($I_2[m]$)] for every $n \in \mathbb{Z}_{n_0}$. Also, a sequence of real numbers $(B_n)_{n\in\mathbb{Z}}$, which satisfies ($I_2[m]$) for all $n \in \mathbb{Z}$, is called a *solution on* $\mathbb{Z}$ of ($I_2[m]$).

In the sequel, we will use the convention that

$$\prod_{\sigma}^{\sigma-1} = 1.$$

Our main results will be given in Sections 2, 3 and 4.

In Section 2, we prove that, if ($I_1[m]$) has a positive and bounded solution on $\mathbb{N}$ and n_0 is a positive integer, then under some conditions there exists also a solution on $\mathbb{N}_{n_0}$ of ($E_1[m]$) which is positive on $\mathbb{N}$. We also show that, if $n_0 \in \mathbb{Z}$ and some conditions hold, the existence of a positive solution on $\mathbb{Z}$ of ($I_2[m]$) which is bounded on $\mathbb{N}$ implies the existence of a solution on $\mathbb{Z}_{n_0}$ of ($E_2[m]$) which is positive on $\mathbb{Z}$.

In Section 3, sufficient conditions are obtained for (E_1) to have a solution on $\mathbb{N}_{n_0}$, where $n_0 \in \mathbb{N}$ with $n_0 > 0$, which is positive on $\mathbb{N}$ and tends to zero as $n \to \infty$. Similarly, sufficient conditions are given for the existence of a solution on $\mathbb{Z}_{n_0}$, where $n_0 \in \mathbb{Z}$, of (E_2) which is positive on $\mathbb{Z}$ and tends to zero as $n \to \infty$.

Section 4 deals with the nonexistence of positive solutions of (I_1) and (I_2) [and, in particular, of (E_1) and (E_2)]. More precisely, necessary conditions are given for (E_1), or more generally for (I_1), to have solutions on $\mathbb{N}_{n_0}$, where $n_0 \in \mathbb{N}$, which are positive on $\mathbb{N}$. Analogously, necessary conditions are derived for (E_2), or more generally for (I_2), to have solutions on $\mathbb{Z}_{n_0}$, where $n_0 \in \mathbb{Z}$, which are positive on $\mathbb{Z}$.

2 EXISTENCE OF POSITIVE SOLUTIONS

The following elementary lemma is needed for the proofs of Theorems 2.1 and 2.2 below. This lemma has been very recently stated and proved by Kordonis and Philos [7].

LEMMA *Let $(S_n)_{n\in\mathbb{Z}_\nu}$ where $\nu \in \mathbb{Z}$, be a positive and bounded sequence of real numbers such that*

$$(-1)^m \Delta^m S_n \geq 0 \quad \textit{for all } n \in \mathbb{Z}_\nu.$$

Then

$$(-1)^i \Delta^i S_n \geq 0 \quad \textit{for every } n \in \mathbb{Z}_\nu \quad (i = 0, 1, \ldots, m-1, m).$$

Moreover, if $(T_n)_{n\in\mathbb{Z}_\nu}$ is a sequence of real numbers such that

$$(-1)^m \Delta^m S_n \geq T_n \quad \textit{for } n \in \mathbb{Z}_\nu,$$

then

$$S_n \geq \sum_{j_1=n}^{\infty} \sum_{j_2=j_1}^{\infty} \cdots \sum_{j_m=j_{m-1}}^{\infty} T_{j_m} \quad \textit{for all } n \in \mathbb{Z}_\nu.$$

Theorems 2.1 and 2.2 below are basic tools in order to establish sufficient conditions for the existence of positive solutions of the difference equations ($E_1[m]$) and ($E_2[m]$) [and, in particular, of (E_1) and (E_2)] respectively. These theorems with $m = 1$ will be used in Section 3.

Theorem 2.1 has been very recently established by Kordonis and Philos [7] (Theorem 4.1) for the special case where $p_n = 1$ for $n \in \mathbb{N}$. This theorem has been previously given by Ladas, Philos and Sficas [9] (Theorem 2) when $m = 1$ and $p_n = 1$ for $n \in \mathbb{N}$. Theorem 2.2 was very recently stated and proved by Kordonis and Philos [7] (Theorem 4.2) for the particular case where $q_n = 1$ for $n \in \mathbb{Z}$.

The method of proof of Theorem 2.1 is similar to that of Theorem 4.1 in [7] and, when $m = 1$, to that of Theorem 2 in [9]. Also, the technique applied in proving Theorem 2.2 is the same with that used in proving Theorem 4.2 in [7]. So, the proofs of Theorems 2.1 and 2.2 will be omitted.

THEOREM 2.1 *Let $(B_n)_{n\in\mathbb{N}}$ be a positive and bounded solution on $\mathbb{N}$ of the inequality* ($I_1[m]$). *Moreover, let $n_0 \in \mathbb{N}$ with $n_0 > 0$ and suppose that*

$$p_n > 0 \quad \textit{for all } n \in \mathbb{N} \textit{ with } n \geq n_0 - 1 \qquad (C_1(n_0))$$

and that: $(\mathrm{H}(n_0))$ *There exist integers* $n_0' \in \{0, 1, \ldots, n_0 - 1\}$ *and* $n_0'' \in \{1, 2, \ldots, n_0\}$ *such that*

$$K_{n_0'} > 0 \quad \textit{and} \quad K_{n_0''} > 0.$$

Then there exists a solution $(A_n)_{n \in \mathbb{N}}$ *on* $\mathbb{N}_{n_0}$ *of* $(\mathrm{E}_1[m])$, *which is positive on* $\mathbb{N}$ *and such that*

$$A_n \leq B_n \quad \textit{for every } n \in \mathbb{N},$$

$$\lim_{n \to \infty} \Delta^i A_n = 0 \quad (i = 0, 1, \ldots, m - 1)$$

and

$$(-1)^{m+1} \Delta^m A_n + p_n \sum_{j=0}^{n} K_{n-j} A_j \leq 0 \quad \textit{for } n = 0, 1, \ldots, n_0 - 1.$$

THEOREM 2.2 *Suppose that*:
(H) *There exists an integer* $n_\star > 0$ *such that*

$$K_{n_\star} > 0.$$

Let $(B_n)_{n \in \mathbb{Z}}$ *be a positive solution on* $\mathbb{Z}$ *of* $(\mathrm{I}_2[m])$, *which is bounded on* $\mathbb{N}$. *Moreover, let* $n_0 \in \mathbb{Z}$ *and assume that*

$$q_n > 0 \quad \textit{for all } n \in \mathbb{Z} \textit{ with } n \geq n_0 - 1. \qquad (\mathrm{C}_2(n_0))$$

Then there exists a solution $(A_n)_{n \in \mathbb{Z}}$ *on* $\mathbb{Z}_{n_0}$ *of* $(\mathrm{E}_2[m])$, *which is positive on* $\mathbb{Z}$ *and such that*

$$A_n \leq B_n \quad \textit{for every } n \in \mathbb{Z},$$

$$\lim_{n \to \infty} \Delta^i A_n = 0 \quad (i = 0, 1, \ldots, m - 1)$$

and

$$(-1)^{m+1} \Delta^m A_n + q_n \sum_{j=-\infty}^{n} K_{n-j} A_j \leq 0 \quad \textit{for } n \in \mathbb{Z} \textit{ with } n < n_0.$$

Remark 2.1 It is easy to see that a positive solution on $\mathbb{N}$ of (I_1) is always decreasing and so it is bounded. Also, every positive solution on $\mathbb{Z}$ of (I_2) is necessarily decreasing and hence it is always bounded on $\mathbb{N}$.

3 SUFFICIENT CONDITIONS FOR THE EXISTENCE OF POSITIVE SOLUTIONS

Now, we concentrate our interest to the difference equations (E_1) and (E_2) and we will apply Theorems 2.1 and 2.2 respectively (with $m=1$) to obtain sufficient conditions for the existence of positive solutions.

THEOREM 3.1 *Assume that there exists a sequence* $(\gamma_n)_{n\in\mathbb{N}}$ *of real numbers in* $(0,1)$ *such that*

$$p_n \sum_{j=0}^{n} \frac{1}{\gamma_j} K_j < 1 \quad \textit{for every } n \in \mathbb{N}$$

and

$$\prod_{r=n-j}^{n-1} \left(1 - p_r \sum_{i=0}^{r} \frac{1}{\gamma_i} K_i\right) \geq \gamma_j \quad \textit{for all } n \in \mathbb{N} \textit{ and } j = 0, 1, \ldots, n.$$

Moreover, let $n_0 \in \mathbb{N}$ *with* $n_0 > 0$ *and suppose that* ($C_1(n_0)$) *and* ($H(n_0)$) *hold.*

Then there exists a solution on $\mathbb{N}_{n_0}$ *of* (E_1), *which is positive* (*on* $\mathbb{N}$) *and tends to zero as* $n \to \infty$.

Proof Set

$$B_n = \prod_{r=0}^{n-1} \left(1 - p_r \sum_{j=0}^{r} \frac{1}{\gamma_j} K_j\right) \quad \text{for } n \in \mathbb{N}.$$

We observe that $B_n > 0$ for all $n \in \mathbb{N}$. So, by Theorem 2.1 (cf. Remark 2.1), it suffices to show that $(B_n)_{n\in\mathbb{N}}$ is a solution on $\mathbb{N}$ of (I_1). To this end, we have for every $n \in \mathbb{N}$

$$\Delta B_n + p_n \sum_{j=0}^{n} K_{n-j} B_j = \Delta B_n + p_n \sum_{j=0}^{n} K_j B_{n-j}$$
$$= \left[\left(1 - p_n \sum_{j=0}^{n} \frac{1}{\gamma_j} K_j \right) - 1 \right] B_n$$
$$+ p_n \left[\sum_{j=0}^{n} K_j \left\{ \prod_{r=n-j}^{n-1} \left(1 - p_r \sum_{i=0}^{r} \frac{1}{\gamma_i} K_i \right) \right\}^{-1} \right] B_n$$
$$= p_n \left[- \sum_{j=0}^{n} \frac{1}{\gamma_j} K_j + \sum_{j=0}^{n} K_j \left\{ \prod_{r=n-j}^{n-1} \left(1 - p_r \sum_{i=0}^{r} \frac{1}{\gamma_i} K_i \right) \right\}^{-1} \right] B_n$$
$$= p_n \left\{ \sum_{j=0}^{n} \left[- \frac{1}{\gamma_j} + \left\{ \prod_{r=n-j}^{n-1} \left(1 - p_r \sum_{i=0}^{r} \frac{1}{\gamma_i} K_i \right) \right\}^{-1} \right] K_j \right\} B_n \leq 0.$$

THEOREM 3.2 *Suppose that* (H) *holds. Assume also that there exists a sequence* $(\delta_n)_{n \in \mathbb{N}}$ *of real numbers in* $(0, 1)$ *such that*

$$q_n \sum_{j=0}^{\infty} \frac{1}{\delta_j} K_j < 1 \quad \textit{for every } n \in \mathbb{Z}$$

and

$$\prod_{r=n-j}^{n-1} \left(1 - q_r \sum_{i=0}^{\infty} \frac{1}{\delta_i} K_i \right) \geq \delta_j \quad \textit{for all } n \in \mathbb{Z} \textit{ and } j \in \mathbb{N}.$$

Moreover, let $n_0 \in \mathbb{Z}$ *and suppose that* $(C_2(n_0))$ *is satisfied.*

Then there exists a solution on $\mathbb{Z}_{n_0}$ *of* (E_2), *which is positive* (*on* $\mathbb{Z}$) *and tends to zero as* $n \to \infty$.

Proof Set

$$B_n = \prod_{r=-\infty}^{n-1} \left(1 - q_r \sum_{j=0}^{\infty} \frac{1}{\delta_j} K_j \right) \quad \text{for } n \in \mathbb{Z}.$$

Clearly, $B_n > 0$ for all $n \in \mathbb{Z}$ and, by Theorem 2.2 (cf. Remark 2.1), it is enough to verify that $(B_n)_{n \in \mathbb{Z}}$ is a solution on $\mathbb{Z}$ of (I_2). Indeed, for every $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\Delta B_n + q_n \sum_{j=-\infty}^{n} K_{n-j} B_j &= \Delta B_n + q_n \sum_{j=0}^{\infty} K_j B_{n-j} \\
&= \left[\left(1 - q_n \sum_{j=0}^{\infty} \frac{1}{\delta_j} K_j\right) - 1\right] B_n \\
&\quad + q_n \left[\sum_{j=0}^{\infty} K_j \left\{\prod_{r=n-j}^{n-1} \left(1 - q_r \sum_{i=0}^{\infty} \frac{1}{\delta_i} K_i\right)\right\}^{-1}\right] B_n \\
&= q_n \left[-\sum_{j=0}^{\infty} \frac{1}{\delta_j} K_j + \sum_{j=0}^{\infty} K_j \left\{\prod_{r=n-j}^{n-1} \left(1 - q_r \sum_{i=0}^{\infty} \frac{1}{\delta_i} K_i\right)\right\}^{-1}\right] B_n \\
&= q_n \left\{\sum_{j=0}^{\infty} \left[-\frac{1}{\delta_j} + \left\{\prod_{r=n-j}^{n-1} \left(1 - q_r \sum_{i=0}^{\infty} \frac{1}{\delta_i} K_i\right)\right\}^{-1}\right] K_j \right\} B_n \leq 0.
\end{aligned}$$

4 NECESSARY CONDITIONS FOR THE EXISTENCE OF POSITIVE SOLUTIONS

We study, in this last section, the problem of the nonexistence of positive solutions of (I_1) and (I_2) [and, in particular, of (E_1) and (E_2)].

THEOREM 4.1 *Let $n_0 \in \mathbb{N}$. Assume that there exists a nonnegative integer N so that*

$$U_N(n^\star) \geq 1 - p_{n^\star+1} \sum_{j=0}^{n^\star - n_0} K_{j+1} \quad \textit{for some } n^\star \in \mathbb{N}_{n_0}$$

and, provided that $N > 0$,

$$U_i(n) < 1 \quad \textit{for all } n \in \mathbb{N}_{n_0} \quad (i = 0, 1, \ldots, N-1),$$

where

$$U_0(n) = p_n \sum_{j=0}^{n-n_0} K_j \quad \textit{for } n \in \mathbb{N}_{n_0}$$

and, when $N>0$, *for* $i=0,1,\ldots,N-1$

$$U_{i+1}(n)=p_n\sum_{j=0}^{n-n_0}K_j\left\{\prod_{r=n-j}^{n-1}[1-U_i(r)]\right\}^{-1}\quad \textit{for } n\in\mathbb{N}_{n_0}.$$

Then there is no solution on $\mathbb{N}_{n_0}$ *of* (I_1) [*and, in particular, of* (E_1)], *which is positive on* $\mathbb{N}$.

Proof Assume, for the sake of contradiction, that the difference inequality (I_1) has a solution $(B_n)_{n\in\mathbb{N}}$ on $\mathbb{N}_{n_0}$, which is positive on $\mathbb{N}$. Then from (I_1) it follows that $\Delta B_n\le 0$ for all $n\in\mathbb{N}_{n_0}$ and hence the sequence $(B_n)_{n\in\mathbb{N}_{n_0}}$ is decreasing.

We first show that

$$B_{n+1}\le[1-U_N(n)]B_n\quad \text{for every } n\in\mathbb{N}_{n_0}. \tag{$\star$}$$

For this purpose, by taking into account the fact that the sequence $(B_n)_{n\in\mathbb{N}_{n_0}}$ is decreasing, from (I_1) we obtain for any $n\in\mathbb{N}_{n_0}$

$$\begin{aligned}0&\ge \Delta B_n+p_n\sum_{j=0}^{n}K_{n-j}B_j=B_{n+1}-B_n+p_n\sum_{j=0}^{n}K_jB_{n-j}\\ &\ge B_{n+1}-B_n+p_n\sum_{j=0}^{n-n_0}K_jB_{n-j}\ge B_{n+1}-B_n+p_n\left(\sum_{j=0}^{n-n_0}K_j\right)B_n\\ &=B_{n+1}-\left(1-p_n\sum_{j=0}^{n-n_0}K_j\right)B_n\equiv B_{n+1}-[1-U_0(n)]B_n.\end{aligned}$$

So, we have

$$B_{n+1}\le[1-U_0(n)]B_n\quad \text{for every } n\in\mathbb{N}_{n_0}.$$

Thus, $(\star)$ is proved when $N=0$. Let us assume that $N>0$. Then from the last inequality it follows that

$$B_{n-j}\ge\left\{\prod_{r=n-j}^{n-1}[1-U_0(r)]\right\}^{-1}B_n\quad \text{for } n\in\mathbb{N}_{n_0} \text{ and } j=0,1,\ldots,n-n_0.$$

Hence, from (I_1) we get for $n \in \mathbb{N}_{n_0}$

$$\begin{aligned}
0 &\geq \Delta B_n + p_n \sum_{j=0}^{n} K_{n-j} B_j = B_{n+1} - B_n + p_n \sum_{j=0}^{n} K_j B_{n-j} \\
&\geq B_{n+1} - B_n + p_n \sum_{j=0}^{n-n_0} K_j B_{n-j} \\
&\geq B_{n+1} - B_n + p_n \left[\sum_{j=0}^{n-n_0} K_j \left\{ \prod_{r=n-j}^{n-1} [1 - U_0(r)] \right\}^{-1} \right] B_n \\
&= B_{n+1} - \left[1 - p_n \sum_{j=0}^{n-n_0} K_j \left\{ \prod_{r=n-j}^{n-1} [1 - U_0(r)] \right\}^{-1} \right] B_n \\
&\equiv B_{n+1} - [1 - U_1(n)] B_n.
\end{aligned}$$

Therefore,

$$B_{n+1} \leq [1 - U_1(n)] B_n \quad \text{for every } n \in \mathbb{N}_{n_0}.$$

So, $(\star)$ is true if $N = 1$. Repeating the above procedure in the case where $N > 1$, we can finally establish $(\star)$.

Next, we will prove that

$$B_{n+1} > \left(p_{n+1} \sum_{j=0}^{n-n_0} K_{j+1} \right) B_n \quad \text{for all } n \in \mathbb{N}_{n_0}. \qquad (\star\star)$$

Indeed, since the sequence $(B_n)_{n \in \mathbb{N}_{n_0}}$ is decreasing, from (I_1) we obtain for every $n \in \mathbb{N}_{n_0}$

$$\begin{aligned}
0 &\geq \Delta B_{n+1} + p_{n+1} \sum_{j=0}^{n+1} K_{n+1-j} B_j = B_{n+2} - B_{n+1} + p_{n+1} \sum_{j=0}^{n+1} K_j B_{n+1-j} \\
&> -B_{n+1} + p_{n+1} \sum_{j=1}^{n-n_0+1} K_j B_{n+1-j} \geq -B_{n+1} + p_{n+1} \left(\sum_{j=1}^{n-n_0+1} K_j \right) B_n \\
&= -B_{n+1} + \left(p_{n+1} \sum_{j=0}^{n-n_0} K_{j+1} \right) B_n,
\end{aligned}$$

which gives $(\star\star)$.

Finally, by combining ($\star$) and ($\star\star$), we conclude that

$$U_N(n) < 1 - p_{n+1} \sum_{j=0}^{n-n_0} K_{j+1} \quad \text{for every } n \in \mathbb{N}_{n_0}.$$

This is a contradiction and the proof of our theorem is complete.

THEOREM 4.2 *Let* $\hat{n}_0 \in \mathbb{Z}$ *and set* $n_0 = \max\{0, \hat{n}_0\}$. *Moreover, let the assumption of Theorem* 4.1 *be satisfied with* q_n *in place of* p_n *for* $n \in \mathbb{N}_{n_0}$.

Then there is no solution on $\mathbb{Z}_{\hat{n}_0}$ *of* (I_2) [*and, in particular, of* (E_2)], *which is positive on* $\mathbb{Z}$.

Proof Clearly, $n_0 \in \mathbb{N}$. Assume that (I_2) admits a solution $(B_n)_{n\in\mathbb{Z}}$ on $\mathbb{Z}_{\hat{n}_0}$, which is positive on $\mathbb{Z}$. For every $n \in \mathbb{N}_{n_0}$, we have

$$\begin{aligned} 0 &\geq \Delta B_n + q_n \sum_{j=-\infty}^{n} K_{n-j} B_j \\ &= \Delta B_n + q_n \sum_{j=-\infty}^{-1} K_{n-j} B_j + q_n \sum_{j=0}^{n} K_{n-j} B_j \\ &\geq \Delta B_n + q_n \sum_{j=0}^{n} K_{n-j} B_j. \end{aligned}$$

That is, the sequence $(B_n)_{n\in\mathbb{N}}$ is a solution on $\mathbb{N}_{n_0}$ of the difference inequality

$$\Delta B_n + q_n \sum_{j=0}^{n} K_{n-j} B_j \leq 0,$$

which is positive on $\mathbb{N}$. By Theorem 4.1, this is a contradiction and so the proof is complete.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[3] I. Györi and G. Ladas, Positive solutions of integro-differential equations with unbounded delay, *J. Integral Equations Appl.*, **4** (1992), 377–390.

[4] J. Jaroš and I.P. Stavroulakis, Necessary and sufficient conditions for oscillations of difference equations with several delays, *Utilitas Math.*, **45** (1994), 187–195.
[5] W.G. Kelley and A.C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, Inc., New York, 1991.
[6] Th. Kiventidis, Positive solutions of integrodifferential and difference equations with unbounded delay, *Glasgow Math. J.*, **35** (1993), 105–113.
[7] I.-G.E. Kordonis and Ch.G. Philos, Positive solutions of certain difference equations, submitted.
[8] I.-G.E. Kordonis and Ch.G. Philos, Oscillation and nonoscillation in linear delay or advanced difference equations, submitted.
[9] G. Ladas, Ch.G. Philos and Y.G. Sficas, Existence of positive solutions for certain difference equations, *Utilitas Math.*, **38** (1990), 119–128.
[10] G. Ladas, Ch.G. Philos and Y.G. Sficas, Oscillations of integro-differential equations, *Differential Integral Equations*, **5** (1991), 1113–1120.
[11] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, Inc., New York, 1988.
[12] Ch.G. Philos, Oscillation and nonoscillation in integrodifferential equations, *Libertas Math.*, **12** (1992), 121–138.
[13] Ch.G. Philos, Positive solutions of integrodifferential equations, *J. Appl. Math. Stochastic Anal.*, **6** (1993), 55–68.
[14] Ch.G. Philos and Y.G. Sficas, On the existence of positive solutions of integrodifferential equations, *Appl. Anal.*, **36** (1990), 189–210.