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Maximal Inequalities for Bessel Processes

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It is proved that the uniform law of large numbers (over a random parameter set) for the α-dimensional ($\alpha \geq 1$) Bessel process $Z = (Z_t)_{t \geq 0}$ started at 0 is valid:

$$E\left(\max_{0 \leq t \leq T} \left| \frac{Z_t^2}{\alpha} - t \right| \right) \leq \frac{12}{\sqrt{\alpha}} E(T)$$

for all stopping times T for Z. The rate obtained (on the right-hand side) is shown to be the best possible. The following inequality is gained as a consequence:

$$\sqrt{E\left(\max_{0 \leq t \leq T} Z_t^2\right)} \leq G(\alpha)\sqrt{E(T)}$$

for all stopping times T for Z, where the constant $G(\alpha)$ satisfies

$$\frac{G(\alpha)}{\sqrt{\alpha}} = 1 + \mathrm{O}\left(\frac{1}{\sqrt{\alpha}}\right)$$

as $\alpha \to \infty$. This answers a question raised in [4]. The method of proof relies upon representing the Bessel process as a time changed geometric Brownian motion. The main emphasis of the paper is on the method of proof and on the simplicity of solution.

Keywords: Bessel process; Uniform law of large numbers; Stopping time; (Geometric) Brownian motion (with drift); Time change; Ito's formula; Burkholder–Gundy's inequality; Optimal stopping; Doob's maximal inequality; Infinitesimal operator; Trap; Reflecting boundary; Entrance boundary

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1. FORMULATION OF THE PROBLEM

The problem which motivated the present paper appeared in [4] and is described as follows. A continuous non-negative Markov process $Z=(Z_t)_{t\geq 0}$ is called a *Bessel process* of dimension $\alpha \in \mathbf{R}$, if its infinitesimal operator is given by

$$L_Z = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\alpha-1}{2x}\frac{\partial}{\partial x} \tag{1.1}$$

and the boundary point 0 is a *trap* if $\alpha \leq 0$, a *reflecting boundary* if $0<\alpha<2$, and an *entrance boundary* if $\alpha \geq 2$. (For more information about Bessel processes we shall refer the reader to [4,5,7–9,11,13,14].) The Bessel processes of dimension $\alpha \geq 1$ are submartingales. The Bessel processes of dimension $\alpha \leq 0$ are supermartingales. However, the Bessel processes of dimension $0<\alpha<1$ are not semimartingales. The Bessel process Z of dimension $\alpha = n \in \mathbf{N}$ may be realized as the radial part of the n-dimensional Brownian motion $B^{(n)} = (B_1(t), \ldots, B_n(t))_{t\geq 0}$:

$$Z_t = \sqrt{\sum_{k=1}^{n} B_k^2(t)} \quad (t \geq 0), \tag{1.2}$$

where $(B_1(t))_{t\geq 0}, \ldots, (B_n(t))_{t\geq 0}$ are mutually independent (standard) Brownian motions.

The results on optimal stopping for Bessel process $Z=(Z_t)_{t\geq 0}$ of dimension $\alpha>0$ due to Dubins *et al.* in [4] (Theorem 5, p. 254) yield the following inequality (here and in the sequel E denotes the expectation corresponding to the Bessel process started at 0):

$$E\Big(\max_{0\leq t\leq T} Z_t\Big) \leq \gamma(\alpha)\sqrt{E(T)} \tag{1.3}$$

for all stopping times T for Z, where $\gamma(\alpha) = \sqrt{4s_1(\alpha)}$ with $s_1(\alpha)$ being a (unique) root of the equation $g_*(s)=0$ for the functions $s \mapsto g_*(s)$ which is a (unique) non-negative solution of the differential equation

$$\frac{2}{\alpha-2} g'(s) g(s) \left(1 - \left(\frac{g(s)}{s}\right)^{\alpha-2}\right) = 1 \quad (s \geq 0) \tag{1.4}$$

such that $g_*(s) \leq s$ and $g_*(s)/s \to 1$ as $s \to \infty$. (For $\alpha = 2$ Eq. (1.4) reads as follows:

$$2g'(s)g(s)\log\left(\frac{s}{g(s)}\right) = 1 \quad (s \geq 0) \tag{1.4$'$}$$

which is obtained by passing to the limit in (1.4) as $\alpha \to 2$.)

It has been shown in [4] (Theorem 7, p. 259) that we have

$$\frac{\gamma(\alpha)}{\sqrt{\alpha}} \to 1 \quad (\alpha \to \infty). \tag{1.5}$$

The problem which was left open is described by the following words (see p. 259 in [4]): "It is of great interest be able to find the function $\gamma = \gamma(\alpha)$ or, at least, to study its properties." The present paper is devoted to clarifications and refinements of the underlying structure for this problem, and to the presentation of its solution in the form of a rate of convergence in (1.5) (after a reformulation of the inequality (1.3) to a stronger and proper form). In this process we discover a fact of independent interest: The uniform law of large numbers (over a *random* parameter set) for Bessel processes. To the best of our knowledge this sort of uniform law of large numbers has not been studied previously. We think that this fact is by itself of theoretical and practical interest, and we intend to write more about it elsewhere.

Instead of going into a description of our method and results obtained, we find it of greater priority at the moment to display two general facts about the problem just stated.

First note from (6.7) + (6.8) in [4] that:

$$\sup_T E\left(\max_{0 \leq t \leq T} Z_t - cT\right) = \frac{s_1(\alpha)}{c} \tag{1.6}$$

for all $c > 0$, where the supremum is taken over all stopping times T for Z, and is attained at $T^* = T^*(\alpha, c)$, which is according to (1.5) in [4] defined by

$$T^* = \inf\left\{t > 0 \mid S_t \geq s_c(\alpha),\, Z_t \leq g_*(S_t)\right\} \tag{1.7}$$

where $S_t = \max_{0 \le s \le t} Z_s$ and $s_c(\alpha)$ is a (unique) root of the equation $g_*(s) = 0$ for the function $s \mapsto g_*(s)$ which (in the same manner) solves Eq. (1.4) when the number 1 on the right-hand side is replaced by $1/c$. From (1.6) we find

$$E\left(\max_{0 \le t \le T} Z_t\right) \le \inf_{c>0}\left(cE(T) + \frac{s_1(\alpha)}{c}\right) = \sqrt{4s_1(\alpha)E(T)} \tag{1.8}$$

for all stopping times T for Z, where the infimum is attained at $c = (s_1(\alpha)/ET)^{1/2}$.

In particular, if we have

$$E(T^*) = ET^*(\alpha, c) = s_1(\alpha)/c^2 \tag{1.9}$$

then we would also have the equality in (1.8), which in turn would show that the constant $\gamma(\alpha)$ (defined through (1.4)) is the best possible in (1.3). Although neither (1.9) has been explicitly derived in [4], nor $\gamma(\alpha)$ has shown the best possible in (1.3) (except for $\alpha = 1$ when $\gamma(1) = \sqrt{2}$; see [6] for a simple proof), as indicated by A. Shiryaev (personal communication), the identity (1.9) should follow from the proof and methods in [4]. In fact, a closer look at (1.8), combined with the extreme property of T^* in (1.6), shows that (1.9) indeed holds. Thus, the constant $\gamma(\alpha)$ defined above is the best possible constant in the inequality (1.3) (being valid for all stopping times T for Z). In this context it is interesting to observe the underlying phenomenon when taking the infimum over all $c > 0$ in (1.8), that it is sufficient to treat $E(T) = E(T^*) = ET^*(\alpha, c)$ as a constant which does not depend on c. (For a similar phenomenon see [6].)

The next we want to address is the proof of (1.5) in [4], and in this context a fundamental result due to Davis. The proof of (1.5) in [4] was heavily based upon the result of Davis in [3]:

$$E\left(\max_{0 \le t \le T}\left|\frac{Z_t}{\sqrt{\alpha}} - \sqrt{t}\right|\right) \le \frac{C_1 E\sqrt{T}}{\sqrt{\alpha}} \tag{1.10}$$

being valid for all stopping times T for Z whenever $\alpha \in \mathbf{N}$, where $C_1 > 0$ is a (universal) constant. Note by a sup-norm property that

$$\left| \frac{1}{\sqrt{\alpha}} \Big(\max_{0 \le t \le T} Z_t \Big) - \sqrt{T} \right| \le \max_{0 \le t \le T} \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} \right|. \tag{1.11}$$

Dividing by $E\sqrt{T}$ through (1.11) and using (1.10) we get:

$$\left| \frac{1}{\sqrt{\alpha} E\sqrt{T}} E\Big(\max_{0 \le t \le T} Z_t \Big) - 1 \right| \le E \left| \frac{1}{\sqrt{\alpha} E\sqrt{T}} \Big(\max_{0 \le t \le T} Z_t \Big) - \frac{\sqrt{T}}{E\sqrt{T}} \right| \le \frac{C_1}{\sqrt{\alpha}}. \tag{1.12}$$

Hence we see that the (best) constant $D(\alpha)$ defined by

$$D(\alpha) = \sup_T \left(\frac{1}{E\sqrt{T}} E\Big(\max_{0 \le t \le T} Z_t \Big) \right) \tag{1.13}$$

(where the supremum is taken over all stopping times T for Z) satisfies

$$\left| \frac{D(\alpha)}{\sqrt{\alpha}} - 1 \right| \le \frac{C_1}{\sqrt{\alpha}}. \tag{1.14}$$

In fact, the constant $D(\alpha)$ is (by definition) the best constant in the inequality

$$E\Big(\max_{0 \le t \le T} Z_t \Big) \le D(\alpha) E\sqrt{T} \tag{1.15}$$

being valid for all stopping times T for Z. Since $E\sqrt{T} \le \sqrt{E(T)}$ by Jensen's inequality, we see that $\gamma(\alpha) \le D(\alpha)$. On the other hand, it is well known that for the hitting time $T_x = \inf\{t > 0 \mid Z_t = x\}$ we have $E(T_x) = x^2/\alpha$ whenever $x > 0$. Inserting this into (1.3) we get $1 \le \gamma(\alpha)/\sqrt{\alpha}$. Taking all these facts together we conclude:

$$1 \le \frac{\gamma(\alpha)}{\sqrt{\alpha}} \le \frac{D(\alpha)}{\sqrt{\alpha}}, \tag{1.16}$$

$$0 \le \frac{\gamma(\alpha)}{\sqrt{\alpha}} - 1 \le \frac{D(\alpha)}{\sqrt{\alpha}} - 1 \le \frac{C_1}{\sqrt{\alpha}} \tag{1.17}$$

for all $\alpha \in \mathbf{N}$ (where C_1 is the (universal) constant from Davis' result (1.10)).

The preceding lines on the proof of (1.5) as given in [4] indicate the following two facts which in essence motivated our work in the sequel. First, we feel that due to the monotonicity (in α) of the drift term of the infinitesimal operator in (1.1), a certain stability theory for solutions of stochastic differential equations should make it clear that the sample paths of the Bessel processes are, roughly speaking, monotone in $\alpha \geq 1$, and therefore (1.10), and thus (1.17) as well, should be valid for all (real) $\alpha \geq 1$. We will, however, choose another way towards (1.17) which is in our opinion simpler and more instructive, and which will give us a more precise information on the constant C_1 in (1.17). Second, the validity of the inequality (1.15) indicates an essential reformulation of the problem about (1.3) stated above; the left-hand side in (1.3) should be increased (by Jensen's inequality) to read as follows:

$$\sqrt{E\left(\max_{0 \leq t \leq T} Z_t^2\right)} \leq G(\alpha)\sqrt{E(T)} \tag{1.18}$$

being valid for all stopping times T for Z where $G(\alpha)$ is a numerical constant. In Section 2 we will see that (1.18) (and thus (1.3) in the initial problem as well) could be thought of as a maximal Doob-type inequality for geometric Brownian motion (or Bessel process itself). In Theorem 3.1 (and Remark 1 following it) we will show that (1.18) holds, and the (best) constant $G(\alpha)$ satisfies the same rate (with a specified and sharp constant $C_1 > 0$) as $\gamma(\alpha)$ in (1.17) for $\alpha \uparrow \infty$. This is obtained as a consequence of the uniform law of large numbers for Bessel process Z of dimension $\alpha \geq 1$:

$$E\left(\max_{0 \leq t \leq T}\left|\frac{Z_t^2}{\alpha} - t\right|\right) \leq \frac{12}{\sqrt{\alpha}}E(T) \tag{1.19}$$

being valid for all stopping times T for Z and all $\alpha \geq 1$. It should be noted if $\alpha = n \in \mathbf{N}$, then by (1.2) we see that (1.19) gets the following form:

$$E\left(\max_{0 \leq t \leq T}\left|\frac{1}{n}\sum_{k=1}^{n} B_k^2(t) - E\big(B_1^2(t)\big)\right|\right) \leq \frac{12}{\sqrt{n}}E(T) \tag{1.20}$$

which is precisely the uniform law of large numbers (over the random time interval $[0, T]$) for the (randomly) parametrized family of sequences of independent and identically distributed random variables $\left(\{B_k^2(t)\}_{k\geq 1} \mid t \in [0, T]\right)$, where T is any stopping time for Z.

In Section 4 we generalize and extend the results obtained either in this paper or in the paper of Davis [3] (see also [2]). Finally, we shall conclude this section by pointing out that our main emphasis in this paper is on the method of proof and on the simplicity of solution. In this context note that the two main ingredients of the proof are: Bessel process as a time changed geometric Brownian motion (Section 2) and the square representation of Bessel process (Section 3).

2. BESSEL PROCESS AS A TIME CHANGED GEOMETRIC BROWNIAN MOTION

Our main aim in this section is to show that (1.18) (and thus (1.3) in the initial problem as well) could be thought of as a maximal Doob-type inequality for geometric Brownian motion (or Bessel process itself). We shall moreover see that the best constant (which equals 2) in Doob's maximal inequality (see [15]) could be in this particular case decreased to get as close to 1 as one desires (by enlarging the drift). We think that this fact is by itself of theoretical and practical interest.

We shall begin by considering geometric Brownian motion. For this, let $B = (B_t)_{t\geq 0}$ be standard Brownian motion, and let $\mu \in \mathbf{R}$ be given and fixed. Then $X_t = B_t + \mu t$ is a Brownian motion with drift, and $Y_t = \mathrm{e}^{X_t}$ defines a geometric Brownian motion. The infinitesimal operators of $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are respectively given by

$$L_X = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x}, \tag{2.1}$$

$$L_Y = \frac{1}{2}x^2\frac{\partial^2}{\partial x^2} + \left(\mu + \frac{1}{2}\right)x\frac{\partial}{\partial x}. \tag{2.2}$$

From this we see that a division of L_Y by x^2 gives the infinitesimal operator of a Bessel process (recall (1.1)). This fact motivates us to apply "change of time scale" transformation (see [14] p.175) with $\rho(y)=y^2$. Thus, define

$$\varphi(t) = \int_0^t \rho(Y_s)\,\mathrm{d}s = \int_0^t Y_s^2\,\mathrm{d}s \tag{2.3}$$

for $t \geq 0$. Then φ is continuous and strictly increasing with $\sup_{t\geq 0}\varphi(t) = +\infty$ if $\mu \geq 0$. Thus φ has the inverse $\tau_t = \varphi^{-1}(t)$, and it is easily verified that we have

$$\tau_t = \int_0^t \frac{1}{Y(\tau_s)^2}\,\mathrm{d}s \tag{2.4}$$

for $t \geq 0$. Moreover, each τ_t is a stopping time for B. Putting

$$Z_t = Y_{\tau_t} = \exp\big(X_{\tau_t}\big) = \exp\big(B_{\tau_t} + \mu\tau_t\big) \tag{2.5}$$

for $t \geq 0$, we find that the infinitesimal operator of the process $Z = (Z_t)_{t\geq 0}$ is given by

$$L_Z = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\mu + \frac{1}{2}\right)\frac{1}{x}\frac{\partial}{\partial x}. \tag{2.6}$$

Thus, the process Z is a Bessel process of dimension $\alpha = 2(1+\mu) \geq 2$ started at e^x under P_x. Observe if T is a stopping time for Z, then τ_T is a stopping time for B. Note also that from (2.4) and (2.5) we get

$$\tau_T = \int_0^T \frac{1}{Z_s^2}\,\mathrm{d}s \tag{2.7}$$

for any stopping time T for Z.

Next we want to show that the inequality (1.18) (and thus (1.3) as well) may be viewed as a Doob maximal inequality for geometric Brownian motion Y (or Bessel process Z itself). First, we show if T is a stopping time for Z such that the stopped martingale

$$\left(\mathrm{e}^{2B(t\wedge\tau_T)-2t\wedge\tau_T}\right)_{t\geq 0} \tag{2.8}$$

is uniformly integrable, then we have

$$E_x(T) = \frac{1}{2(1+\mu)}\Big(E_x(Z_T^2) - e^{2x}\Big), \tag{2.9}$$

where P_x is the probability measure under which Brownian motion B starts at $x \in \mathbf{R}$, and E_x denotes the expectation with respect to P_x.

Indeed, by Doob's optional sampling theorem we find

$$\begin{aligned} E_x(T) &= E_x\big(\varphi(\tau_T)\big) = E_x\Big(\int_0^{\tau_T} Y_s^2\, ds\Big) \\ &= \int_0^\infty E_x\Big(e^{2B_s + 2\mu s} \cdot I(s < \tau_T)\Big)\, ds \\ &= \int_0^\infty e^{2\mu s + 2s} E_x\Big(e^{2B_s - 2s} \cdot I(s < \tau_T)\Big)\, ds \\ &= \int_0^\infty e^{2\mu s + 2s} E_x\Big(e^{2B(\tau_T) - 2\tau_T} \cdot I(s < \tau_T)\Big)\, ds \\ &= E_x\Big(e^{2B(\tau_T) - 2\tau_T} \int_0^{\tau_T} e^{2(1+\mu)s}\, ds\Big) \\ &= \frac{1}{2(1+\mu)}\Big(E_x\Big(e^{2B(\tau_T) + 2\mu\tau_T}\Big) - E_x\Big(e^{2B(\tau_T) - 2\tau_T}\Big)\Big) \\ &= \frac{1}{2(1+\mu)}\Big(E_x(Y_{\tau_T}^2) - e^{2x}\Big). \end{aligned} \tag{2.10}$$

This establishes (2.9) and the proof of this fact is complete.

Next, letting $x \to -\infty$ in (2.9) and using the Feller property of the Bessel process, we see that the inequality (1.3) gets the following form:

$$E\Big(\max_{0 \le t \le T} Z_t\Big) \le \frac{\gamma(\alpha)}{\sqrt{\alpha}} \sqrt{E(Z_T^2)}. \tag{2.11}$$

This inequality can be seen as a passage to the limit (for $x \to -\infty$) in

$$E_x\Big(\max_{0 \le t \le \tau_T} Y_t\Big) \le \frac{\gamma(\alpha)}{\sqrt{\alpha}} \sqrt{E_x(Y_{\tau_T}^2)} \tag{2.12}$$

being valid for any stopping time T for Z. Since Y is a non-negative submartingale (for $\mu \ge -1/2$, i.e. $\alpha \ge 1$), then by Doob's maximal inequality (and Jensen's inequality) we get

$$E_x\Big(\max_{0\le t\le \tau_T} Y_t\Big) \le \sqrt{E_x\Big(\max_{0\le t\le \tau_T} Y_t^2\Big)} \le 2\sqrt{E_x(Y_{\tau_T}^2)} \tag{2.13}$$

whenever the stopped martingale (2.8) is uniformly integrable. The constant 2 is generally known to be the best possible in the last (Doob's) inequality (see [15]). Hence by a limiting argument we arrive to (2.11) (with the constant 2 instead of $\gamma(\alpha)/\sqrt{\alpha}$), and thus (1.3) with $\gamma(\alpha)/\sqrt{\alpha} \to 1$ (as $\alpha \to \infty$) shows that in the Doob's inequality (2.12) (for geometric Brownian motion) the "best" constant 2 may be replaced by a number which is as close to 1 as desired (by enlarging the drift μ). In fact, we will see (Theorem 3.1) that the same phenomenon holds for the "right" Doob's inequality (the second one) in (2.13) (for geometric Brownian motion), by proving that the left-hand term in (2.12) can be increased (with the same asymptotic behaviour of the constant) to the second term in (2.13) (which is precisely the inequality (1.18)).

Note that from (1.3), (1.18) and (2.13) we get

$$1 \le \frac{\gamma(\alpha)}{\sqrt{\alpha}} \le \frac{G(\alpha)}{\sqrt{\alpha}} \le 2 \tag{2.14}$$

for all $\alpha \ge 2$. (For completeness in Theorem 3.1 (to cover the case $1 < \alpha < 2$ as well) observe that the last inequality extends to all $\alpha \ge 1$. This is easily seen by the fact that from (1.1) for $f(x) = x^2$ we get $L_Z(f) \equiv \alpha$. Thus $(Z_t^2 - \alpha t)_{t\ge 0}$ is a (local) martingale. Therefore by Doob's optional sampling theorem and Fatou's lemma we have

$$E(Z_T^2) \le \liminf_{t\to\infty} E(Z_{t\wedge T}^2) = \liminf_{t\to\infty} \alpha E(t \wedge T) = \alpha E(T) \tag{2.15}$$

which suffices (by Doob's inequality). This shows that (2.14) holds for all $\alpha \ge 1$.)

In fact, the estimate (2.14) could be refined to read as follows:

$$1 \le \frac{\gamma(\alpha)}{\sqrt{\alpha}} \le \frac{G(\alpha)}{\sqrt{\alpha}} \le \left(\frac{\alpha - 2}{\alpha - 4}\right)^{(\alpha-2)/4} \tag{2.16}$$

for all $\alpha > 4$, which gives a better estimate when $\alpha > 6$. (Note, however, that the right-hand side in (2.16) tends to $\sqrt{\mathrm{e}}$ as $\alpha \to \infty$.)

For this, recall that (1.3) reduces to (2.11), and thus to (2.12) and (2.13) as well (at least for not too large stopping times (which suffices) such that the stopped martingale (2.8) is uniformly integrable). The same holds for (1.18) and (2.13) (the second inequality). Let us therefore look at (2.13). By Doob's maximal inequality being applied to the submartingale $\exp(2B_t/\mu + 2t)$ for $t \geq 0$ with $p = \mu > 1$, or equivalently $\alpha > 4$, we get

$$\begin{aligned} E_x\Big(\max_{0\leq t\leq \tau_T} Y_t\Big) &\leq \sqrt{E_x\Big(\max_{0\leq t\leq \tau_T} Y_t^2\Big)} = \sqrt{E_x\Big(\max_{0\leq t\leq \tau_T} \mathrm{e}^{(2B_t/\mu+2t)\mu}\Big)} \\ &\leq \sqrt{\Big(\frac{\mu}{\mu-1}\Big)^{\mu} E_x\Big(e^{\left(2B(\tau_T)/\mu+2\tau_T\right)\mu}\Big)} \\ &= \sqrt{\Big(\frac{\mu}{\mu-1}\Big)^{\mu}}\sqrt{E_x(Y_{\tau_T}^2)} \end{aligned} \tag{2.17}$$

from where (2.16) follows (as above) by using $\mu = \alpha/2 - 1$. (Recall that $(\mu/(\mu-1))^{\mu} \to \mathrm{e}$ as $\mu \to \infty$.) This completes the proof of the claim.

3. THE SQUARE REPRESENTATION OF BESSEL PROCESS

In this section we present the main results of the paper. We will begin by deriving the square representation of Bessel process which is shown of fundamental importance in our proof below.

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at $x \in \mathbf{R}$ under P_x, let $X_t = B_t + \mu t$ be Brownian motion with drift $\mu \in \mathbf{R}$, and let $Y_t = \mathrm{e}^{X_t}$ be geometric Brownian motion. Then the infinitesimal operators of $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are respectively given by (2.1) and (2.2). Function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ defined by (2.3) is continuous and strictly increasing with $\sup_{t \geq 0} \varphi(t) = +\infty$ if $\mu \geq 0$. Thus, for $\mu \geq 0$, or equivalently $\alpha \geq 2$, the "change of time scale" is applicable (as indicated in Section 2), and for the inverse $\tau_t = \varphi^{-1}(t)$ (which is given by (2.4)) we know that the process $Z_t = Y_{\tau_t}$ is a Bessel process of dimension $\alpha = 2(1 + \mu) \geq 2$ started at e^x under P_x (with infinitesimal operator given by (2.6)). By Ito's formula we get

$$e^{X_t} = e^x + \int_0^t e^{X_s}\, d(B_s + x) + \frac{2\mu + 1}{2} \int_0^t e^{X_s}\, ds \tag{3.1}$$

for all $t \geq 0$. Inserting τ_t instead of t in (3.1), we should note that

$$W_t = \int_0^{\tau_t} e^{X_s}\, d(B_s + x) \quad (t \geq 0) \tag{3.2}$$

defines an $(\mathcal{F}_{\tau_t})_{t\geq 0}$ – Wiener process (since the quadratic variation equals $\varphi(\tau_t) = t$). Moreover, by putting $s = \tau_u$ and using $\tau'(u) = 1/\varphi'(s)$ we find

$$\int_0^{\tau_t} e^{X_s}\, ds = \int_0^t \frac{1}{e^{X_{\tau_s}}}\, ds \tag{3.3}$$

for all $t \geq 0$. Thus, from (3.1)–(3.3) we get

$$Z_t = e^x + W_t + \frac{2\mu + 1}{2} \int_0^t \frac{1}{Z_s}\, ds \tag{3.4}$$

for all $t \geq 0$. From this by Ito's formula we derive the square representation of Bessel process:

$$Z_t^2 = e^{2x} + 2 \int_0^t Z_s\, dW_s + \alpha t \tag{3.5}$$

for all $t \geq 0$. (For completeness in Theorem 3.1 (to cover the case $1 < \alpha < 2$ as well) we will need that this representation extends to the case $\alpha \geq 1$ (the case $\alpha = 1$ is evident). The proof of this fact is not as simple (as in the case $\alpha \geq 2$) and could be given by showing that the (local) martingale $(Z_t^2 - \alpha t)_{t\geq 0}$ (recall the lines following (2.14)) has the quadratic variation equal to $(4 \int_0^t Z_s^2\, ds)_{t\geq 0}$. This can be deduced from definition of quadratic variation if one uses (the well-known fact) that the quadratic variation $[Z]_t$ of Bessel process Z equals t for $t \geq 0$. We will not present this in more detail here, since we are (in accordance with the initial problem about (1.3)) mainly interested in the asymptotic behaviour when $\alpha \to \infty$ (that is α large), and our emphasis on simplicity applies exactly for $\alpha \geq 2$ when the Bessel process Z does not hit zero, and therefore the "change of time scale" for geometric Brownian motion (Section 2) works globally (otherwise, freely speaking, it works until Bessel process Z first time hits zero). Thus our main emphasis in Theorem 3.1 below is on the case when $\alpha \geq 2$. (The case $\alpha < 1$ appears of no interest in this context and is not considered.)

It is the representation (3.5) we shall heavily use in the sequel. (We would like to point out that our main motivation for such a representation comes from (2.9)–(2.11) where we realized that the square of Bessel process plays an essential role for the understanding of the inequality (1.3).)

THEOREM 3.1 *Let $Z = (Z_t)_{t \geq 0}$ be a Bessel process of dimension $\alpha \geq 1$ started at* 0 *under P, and let T be any stopping time for Z. Then the uniform law of large numbers is satisfied*:

$$E\left(\max_{0 \leq t \leq T} \left| \frac{Z_t^2}{\alpha} - t \right| \right) \leq \frac{12}{\sqrt{\alpha}} E(T). \tag{3.6}$$

In particular, the inequality is valid:

$$\sqrt{E\left(\max_{0 \leq t \leq T} Z_t^2 \right)} \leq G(\alpha) \sqrt{E(T)}, \tag{3.7}$$

where the (*universal*) *constant $G(\alpha)$ satisfies*:

$$\frac{G(\alpha)}{\sqrt{\alpha}} = 1 + \mathrm{O}\left(\frac{1}{\sqrt{\alpha}} \right) \tag{3.8}$$

as $\alpha \to \infty$. (*For more specific evaluations of the constant $G(\alpha)$ see Remark* **1**.)

Proof We shall begin our proof by showing that (3.7) with (3.8) holds. Then, by using (3.7), we shall derive (3.6).

(3.7) + (3.8): From (3.5) by Burkholder–Gundy's inequality and Hölder's inequality we obtain

$$\begin{aligned}
E_x\Big(\max_{0\le t\le T} Z_t^2\Big) &\le \mathrm{e}^{2x} + 2E_x\Big(\max_{0\le t\le T}\int_0^t Z_s\,\mathrm{d}W_s\Big) + \alpha E_x(T)\\
&\le \mathrm{e}^{2x} + 2K_1E_x\Big(\Big(\int_0^T Z_s^2\,\mathrm{d}s\Big)^{1/2}\Big) + \alpha E_x(T)\\
&\le \mathrm{e}^{2x} + 2K_1E_x\Big(\max_{0\le t\le T} Z_t\cdot\sqrt{T}\Big) + \alpha E_x(T)\\
&\le \mathrm{e}^{2x} + 2K_1\Big(E_x\Big(\max_{0\le t\le T} Z_t^2\Big)\Big)^{1/2}\\
&\qquad \times\sqrt{E(T)} + \alpha E_x(T) \qquad (3.9)
\end{aligned}$$

for all $x\in\mathbf{R}$. Letting $x\to-\infty$ (and using the Feller property of the Bessel process) we get

$$E\Big(\max_{0\le t\le T} Z_t^2\Big) \le K\Big(E\Big(\max_{0\le t\le T} Z_t^2\Big)\Big)^{1/2}\sqrt{E(T)} + \alpha E(T), \qquad (3.10)$$

where $K = 2K_1$ with $K_1 > 0$ from (3.9). Denoting $a = E\big(\max_{0\le t\le T} Z_t^2\big)$ and $b = E(T)$, we see that (3.10) may be written as follows:

$$\frac{a}{b} \le K\sqrt{\frac{a}{b}} + \alpha. \qquad (3.11)$$

From (3.11) we easily find (by solving the underlying quadratic equation):

$$\sqrt{a} \le \left(\frac{K}{2} + \sqrt{\alpha + \frac{K^2}{4}}\right)\sqrt{b}. \qquad (3.12)$$

Hence by definition of a and b, we get (3.7) with

$$G(\alpha) = \left(\frac{K}{2} + \sqrt{\alpha + \frac{K^2}{4}}\right). \qquad (3.13)$$

Thus (3.8) is easily seen as well. This completes the first part of the proof.

(3.6): From (3.5) we have

$$\left|Z_t^2 - \alpha t\right| \leq \mathrm{e}^{2x} + 2\left|\int_0^t Z_s \,\mathrm{d}W_s\right| \tag{3.14}$$

for all $t \geq 0$. Hence, by Burkholder–Gundy's inequality and Hölder's inequality (as in (3.9)), and (3.7) just proved, we get

$$\begin{aligned} E_x\left(\max_{0\leq t\leq T}\left|Z_t^2 - \alpha t\right|\right) &\leq \mathrm{e}^{2x} + 2E_x\left(\max_{0\leq t\leq T}\left|\int_0^t Z_s\,\mathrm{d}W_s\right|\right) \\ &\leq \mathrm{e}^{2x} + 2K_1\left(E_x\left(\max_{0\leq t\leq T} Z_t^2\right)\right)^{1/2}\sqrt{E_x(T)} \\ &\leq \mathrm{e}^{2x} + 2K_1 G(\alpha) E_x(T). \end{aligned} \tag{3.15}$$

Letting $x \to -\infty$ and dividing by α in (3.15), we obtain

$$E\left(\max_{0\leq t\leq T}\left|\frac{Z_t^2}{\alpha} - t\right|\right) \leq K\frac{G(\alpha)}{\alpha}E(T), \tag{3.16}$$

where $K = 2K_1$ with $K_1 > 0$ from (3.15) (or (3.9)). Moreover, it is well known (see [10]) that one may take $K_1 = 3$. (This is the constant from Burkholder–Gundy's inequality (see [1]) used in (3.9) and (3.15)). Thus one may take $K = 6$, and from (2.14) we see that $G(\alpha)/\alpha \leq 2/\sqrt{\alpha}$. This shows the validity of (3.6) with the constant equal to 12, and the proof is complete.

Remarks We state several facts which are aimed to refine and additionally clarify the statement and proof of Theorem 3.1.

1. Since one may take $K = 6$ in the last proof, from (3.13) we see that (3.7) holds with

$$G(\alpha) = G_1(\alpha) = \sqrt{\alpha + 9} + 3 \leq \sqrt{\alpha} + 6 \tag{3.17}$$

for all $\alpha \geq 1$. This (with (2.14)) gives (3.8) in the following (more specified) form:

$$1 \leq \frac{G(\alpha)}{\sqrt{\alpha}} \leq \sqrt{1 + \frac{9}{\alpha}} + \frac{3}{\sqrt{\alpha}} \leq 1 + \frac{6}{\sqrt{\alpha}} \tag{3.18}$$

for all $\alpha \geq 1$. Note, however, that dividing by $E(T)$ in (3.6), we find

$$\begin{aligned} &\left| \frac{1}{\alpha E(T)} E\Big(\max_{0 \leq t \leq T} Z_t^2 \Big) - 1 \right| \\ &\quad \leq E\left| \frac{1}{\alpha E(T)} \Big(\max_{0 \leq t \leq T} Z_t^2 \Big) - \frac{T}{E(T)} \right| \\ &\quad \leq E\left(\max_{0 \leq t \leq T} \left| \frac{Z_t^2}{\alpha E(T)} - \frac{t}{E(T)} \right| \right) \leq \frac{12}{\sqrt{\alpha}} \end{aligned} \tag{3.19}$$

for all $\alpha \geq 1$. Maximizing the left-hand side over all stopping times T for Z, we get

$$\left| \frac{G(\alpha)^2}{\alpha} - 1 \right| \leq \frac{12}{\sqrt{\alpha}} \tag{3.20}$$

for all $\alpha \geq 1$. Hence we easily conclude

$$1 \leq \frac{G(\alpha)}{\sqrt{\alpha}} \leq \sqrt{1 + \frac{12}{\sqrt{\alpha}}} \tag{3.21}$$

for all $\alpha \geq 1$. This gives a better estimate for (3.8) from the one which is obtained in the end of (3.18), but worse from the middle one in (3.18) if and only if $\alpha > 16$. Moreover, the step from (3.19) to (3.20) shows that (3.7) holds with

$$G(\alpha) = G_2(\alpha) = \sqrt{\alpha + 12\sqrt{\alpha}} \tag{3.22}$$

for all $\alpha \geq 1$. Note that $G_1(\alpha) < G_2(\alpha)$ if and only if $\alpha > 16$. Taking these facts together, we conclude that (3.7) holds with

$$G(\alpha) = \begin{cases} \sqrt{\alpha + 9} + 3, & 16 < \alpha < \infty, \\ \sqrt{\alpha + 12\sqrt{\alpha}}, & 1 \leq \alpha \leq 16. \end{cases} \tag{3.23}$$

Note that G is continuous with $G(16)/\sqrt{16} = 2$.

2. We show how (3.7) + (3.8) (in a less specified form and for $\alpha \in \mathbf{N}$) follows from the following result of Davis in [3]:

$$E\left(\max_{0 \le t \le T} \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} \right|^p \right) \le \frac{C_p E\left(T^{p/2}\right)}{\alpha^{p/2}} \tag{3.24}$$

being valid for all stopping times T for Z with $\alpha \in \mathbf{N}$, where $C_p > 0$ is some constant for $0 < p < \infty$.

From this inequality (with $p = 2$) we get

$$\left(E \left| \frac{1}{\sqrt{\alpha}} \left(\max_{0 \le t \le T} Z_t \right) - \sqrt{T} \right|^2 \right)^{1/2} \le \frac{\sqrt{C_2}\sqrt{E(T)}}{\sqrt{\alpha}}. \tag{3.25}$$

Hence by triangle inequality we obtain

$$\left| \frac{1}{\sqrt{\alpha}} \sqrt{E\left(\max_{0 \le t \le T} Z_t^2 \right)} - \sqrt{E(T)} \right| \le \frac{\sqrt{C_2}\sqrt{E(T)}}{\sqrt{\alpha}}. \tag{3.26}$$

Dividing by $\sqrt{E(T)}$ in (3.26), and then maximizing the left-hand side over all stopping times T for Z, we can conclude

$$\left| \frac{G(\alpha)}{\sqrt{\alpha}} - 1 \right| \le \frac{\sqrt{C_2}}{\sqrt{\alpha}}. \tag{3.27}$$

This establishes (3.7) + (3.8), and the proof of the claim is complete.

3. We show how (3.6) itself (in a less specified form and for $\alpha \in \mathbf{N}$) follows from Davis' result (3.24). For this, first note that

$$\begin{aligned} \left| \frac{Z_t^2}{\alpha} - t \right| &= \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} \right| \cdot \left| \frac{Z_t}{\sqrt{\alpha}} + \sqrt{t} \right| = \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} \right| \cdot \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} + 2\sqrt{t} \right| \\ &\le \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} \right|^2 + 2\sqrt{t} \left| \frac{Z_t}{\sqrt{\alpha}} - \sqrt{t} \right| \end{aligned} \tag{3.28}$$

for all $t \geq 0$. Hence, by Davis' result (3.24) and Hölder's inequality, we get

$$
\begin{aligned}
E\left(\max_{0\leq t\leq T}\left|\frac{Z_t^2}{\alpha}-t\right|\right) &\leq \frac{C_2E(T)}{\alpha}+2\sqrt{E(T)}\left(E\left(\max_{0\leq t\leq T}\left|\frac{Z_t}{\sqrt{\alpha}}-\sqrt{t}\right|^2\right)\right)^{1/2} \\
&\leq \frac{C_2E(T)}{\alpha}+2\sqrt{E(T)}\frac{\sqrt{C_2}\sqrt{E(T)}}{\sqrt{\alpha}} \\
&= \left(\frac{C_2}{\alpha}+\frac{2\sqrt{C_2}}{\sqrt{\alpha}}\right)E(T) = \mathrm{O}\left(\frac{1}{\sqrt{\alpha}}\right)E(T). \qquad (3.29)
\end{aligned}
$$

This completes the proof of the claim.

4. It is shown in Remarks **2** and **3** that the results in Theorem 3.1 (in a less specified form and for $\alpha \in \mathbf{N}$) can also be obtained from Davis' result (3.24). (We have observed this after completing our proof.) In this context it should be noted that our methods are much different, the main emphasis in our approach being put on simplicity. A more sophisticated method, we believe, should give out the best constant $G(\alpha)$ in (3.7) explicitly.

5. It should be noted (as already indicated in Section 1) that (3.6) may be viewed as a uniform law of large numbers (over a *random* parameter set) for Bessel processes. (When $\alpha \in \mathbf{N}$ then (3.6) takes the form described in (1.20), and this justifies the term.) To the best of our knowledge this sort of uniform law of large numbers has not been studied previously. We will not pursue this in more detail here, but instead will refer the reader to [12] for more information on this subject.

6. The estimate obtained in (3.6) is the best possible, in the sense if T is a stopping time for Z satisfying $T \geq 1$, then there is a constant $C > 0$ such that

$$E\left(\max_{0\leq t\leq T}\left|\frac{Z_t^2}{\alpha}-t\right|\right) \geq \frac{C}{\sqrt{\alpha}} \qquad (3.30)$$

for all $\alpha \in \mathbf{N}$. Indeed, let $\alpha = n \in \mathbf{N}$ be given and fixed. Then we have

$$Z_1^2 = \sum_{k=1}^{n} B_k^2(1) \sim \Gamma\left(\frac{n}{2},2\right). \qquad (3.31)$$

Setting $S_n = \sum_{k=1}^n B_k^2(1)$, from (3.31) we easily find

$$E\left|\frac{S_n - E(S_n)}{\sqrt{n}}\right|^2 = \frac{1}{n}\mathrm{Var}(Z_1^2) = 2. \tag{3.32}$$

This shows that the sequence $\left((S_n - E(S_n))/\sqrt{n}\,\right)_{n\geq 1}$ is uniformly integrable. Thus by the central limit theorem we may conclude

$$E\left|\frac{S_n - E(S_n)}{\sqrt{n}}\right| \to E\left|N(0,1)\right| = \sqrt{2/\pi} \tag{3.33}$$

as $n \to \infty$. Hence, given $\varepsilon > 0$ one can find $n_\varepsilon \geq 1$, such that

$$E\left|\frac{S_n - E(S_n)}{n}\right| \geq \left(\sqrt{2/\pi} - \varepsilon\right)\frac{1}{\sqrt{n}} \tag{3.34}$$

for all $n \geq n_\varepsilon$. From this, it easily follows

$$\begin{aligned} E\left(\max_{0\leq t\leq T}\left|\frac{Z_t^2}{\alpha} - t\right|\right) &\geq E\left|\frac{1}{n}\sum_{k=1}^n B_k^2(1) - 1\right| \\ &= E\left|\frac{S_n - E(S_n)}{n}\right| \geq \frac{C}{\sqrt{n}} \end{aligned} \tag{3.35}$$

for some $C > 0$ and all $n \geq 1$. This completes the proof of the claim.

4. MAXIMAL INEQUALITIES FOR BESSEL PROCESSES

We shall conclude our exposition in this paper by pointing out that the methods and facts presented above can be easily generalized and extended to derive a chain of inequalities (with rates) presented in Theorem 4.1. (The inequality (3.24) is to be recalled, and the arguments in the lines following it are to be extended to all $0 < p < \infty$. This should justify the case $\alpha \in \mathbf{N}$. The case of real $\alpha \geq 1$ is to be treated along the lines and facts used in the proof of Theorem 3.1.) We will omit the details for simplicity. The problem of finding the best values for the unspecified constants appearing below is worthy of consideration.

These inequalities could be called the *Burkholder–Davis–Gundy inequalities* for Bessel processes, and are formulated as follows.

THEOREM 4.1 *Let $Z = (Z_t)_{t \geq 0}$ be a Bessel process of dimension $\alpha \geq 1$ started at 0 under P, and let $0 < p < \infty$ be given. Then there are (universal) constants $A_p(\alpha) > 0$ and $B_p(\alpha) > 0$ such that*

$$A_p(\alpha) E\big(T^{p/2}\big) \leq E\Big(\max_{0 \leq t \leq T} Z_t^p\Big) \leq B_p(\alpha) E\big(T^{p/2}\big) \qquad (4.1)$$

for all stopping times T for Z. Moreover, the constants $A_p(\alpha)$ and $B_p(\alpha)$ in (4.1) *can be taken to satisfy the following conditions*:

$$0 \leq 1 - \frac{A_p(\alpha)}{\sqrt{\alpha}} \leq \frac{\mathbf{C}_p}{\sqrt{\alpha}} \quad (\alpha \geq 1), \qquad (4.2)$$

$$0 \leq \frac{B_p(\alpha)}{\sqrt{\alpha}} - 1 \leq \frac{\mathbf{C}_p}{\sqrt{\alpha}} \quad (\alpha \geq 1) \qquad (4.3)$$

with some (universal) constant $\mathbf{C}_p > 0$.

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