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On Ozeki's Inequality

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We shall obtain the best bound in Ozeki's inequality which estimates the difference of Cauchy's inequality. We also give an operator version of Ozeki's inequality which extends an inequality on the variance of an operator.

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1 INTRODUCTION

This paper is in continuation to our preceding note [3] on Ozeki's inequality. We first recall it [7, p.121, 8]: if $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ are n-tuples of real numbers satisfying

$$0 \le m_1 \le a_i \le M_1 \quad \text{and} \quad 0 \le m_2 \le b_i \le M_2 \tag{1.1}$$

for $i=1,\dots,n$, then

$$\sum a_i^2 \sum b_i^2 - \left(\sum a_i b_i\right)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2. \tag{1.2}$$

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This is complementary to Cauchy's inequality. Recently, we initiated to extend an operator version of this inequality, which is motivated by the work on the covariance and variance of operators due to Fujii *et al.* [1].

An operator version of (1.2) might be given via diagonal matrices as follows: If A and B are commuting, and satisfy the conditions $0 \le m_1 \le A \le M_1$ and $0 \le m_2 \le B \le M_2$, then for a unit vector x

$$(A^2x, x)(B^2x, x) - (ABx, x)^2 \le \tfrac{1}{4}(M_1M_2 - m_1m_2)^2. \tag{1.3}$$

However, (1.3) is incorrect in general, which we could show by using 3×3 matrices (cf. [3]). In [3], we pointed out that the initial inequality (1.2) itself is false, by the simple counterexample:

$$a = (1, 1, 0), \qquad b = (0, 1, 1), \tag{1.4}$$

and we proposed the following

THEOREM 1.1 (cf. [3, Theorem 2]) *If a and b satisfy the condition* (1.1), *then*

$$\sum {a_k}^2 \sum {b_k}^2 - \left(\sum a_k b_k\right)^2 \le \frac{n^2}{2}(M_1M_2 - m_1m_2)^2 \tag{1.5}$$

instead of (1.2).

We also showed in [3] an operator version of (1.5) which was proved by using inequalities (given in [1]) on the covariance and the variance of operators.

In this paper, following the original idea due to Ozeki [8] we give an improvement of (1.5) which is best possible. Roughly, the constant $n^2/2$ in (1.5) is sharpened by $n^2/3$. We note here that the example (1.4) is suggestive of our discussion below, for which the left hand side of (1.2) is $3 = 3^2/3$.

As an application, we show the following inequality instead of (1.3): If A and B are commuting positive operators on a Hilbert space H and satisfy the conditions $0 \le m_1 \le A \le M_1$ and $0 \le m_2 \le B \le M_2$, then for a unit vector $x \in H$,

$$(A^2x, x)(B^2x, x) - (ABx, x)^2 \le \tfrac{1}{3}(M_1M_2 - m_1m_2)^2. \tag{1.6}$$

We also give a refinement of this inequality, and in particular, we show that if $(1+m_1/M_1)(1+m_2/M_2) \geq 2$, then

$$(A^2x,x)(B^2x,x) - (ABx,x)^2 \leq \tfrac{1}{4}(M_1M_2 - m_1m_2)^2.$$

This is an extension of the variance inequality given in [1]:

$$(A^2x,x) - (Ax,x)^2 \leq \tfrac{1}{4}(M_1 - m_1)^2. \tag{1.7}$$

For two positive operators without commutativity condition, using the geometric mean $A^2\sharp_{1/2}B^2$ of A^2 and B^2 in Kubo–Ando theory [4], we shall show that if $\gamma = \max\{m_1/M_1, m_2/M_2\}$, then

$$(A^2x,x)(B^2x,x) - (A^2\sharp_{1/2}B^2x,x)^2 \leq \frac{1}{4\gamma^2}(M_1M_2 - m_1m_2)^2.$$

2 OZEKI'S INEQUALITY

The aim of this section is to prove the following theorem in which we correct the constant $n^2/4$ of Ozeki's inequality (1.2) and at the same time we improve the constant $n^2/2$ of the inequality (1.5). Though the original proof of (1.2) includes some minor overlooks, it will be right methodologically. Basically we owe our proof to the original idea of Ozeki [8].

THEOREM 2.1 *Let $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_n)$ be n-tuples of real numbers satisfying*

$$0 \leq m_1 \leq a_i \leq M_1 \quad \text{and} \quad 0 \leq m_2 \leq b_i \leq M_2 \qquad (i=1,\ldots,n).$$

Then,

$$\sum a_i^2 \sum b_i^2 - \left(\sum a_ib_i\right)^2 \leq \frac{n^2}{3}(M_1M_2 - m_1m_2)^2. \tag{2.1}$$

To prove this theorem we write the left hand side of (2.1) as $T=T(a,b)$, where a and b are taken over the n-dimensional rectangles $[m_1,M_1]^n$ and $[m_2,M_2]^n$, respectively. We then state some basic facts on $T(a,b)$.

LEMMA 2.2 ([7, p. 84]) *Let* $\Delta=\{(i,j);\ i,j\in\mathbb{Z},\ 1\le i<j\le n\}$. *Then*

$$T(a,b)=\sum_{(i,j)\in\Delta}(a_ib_j-a_jb_i)^2.$$

LEMMA 2.3 (cf. [5]) *$T(a,b)$ is a separately convex function with respect to a and b, that is,*

$$T(pa+p'a',b)\le pT(a,b)+p'T(a',b),\quad p\in[0,1],\ p'=1-p,$$

and

$$T(a,qb+q'b')\le qT(a,b)+q'T(a,b'),\quad q\in[0,1],\ q'=1-q.$$

Consequently, $T(a,b)$ attains its maximum at a point $(a,\ b)\in[m_1,M_1]^n\times[m_2,M_2]^n$ such that a and b are extremal points, namely, vertices of $[m_1,M_1]^n$ and $[m_2,M_2]^n$, respectively.

LEMMA 2.4 ([2, p. 261]) *For $c\in\mathbb{R}^n_+$, let $\underline{c}=(\underline{c_1},\ldots,\underline{c_n})$ (resp. $\overline{c}=(\overline{c_1},\ldots,\overline{c_n})$) be the rearrangement of c in decreasing (resp. increasing) order. Then $\sum\underline{a_k}\overline{b_k}\le\sum a_kb_k$ and $\sum\overline{a_k}\underline{b_k}\le\sum a_kb_k$, so that $T(\underline{a},\overline{b})\ge T(a,b)$ and $T(\overline{a},\underline{b})\ge T(a,b)$ for $a,b\in\mathbb{R}^n_+$.*

Proof of the theorem First note that an extremal point of an n-dimensional rectangle $[m,M]^n$ is a point, all of whose components are either m or M. Hence from Lemmas 2.3 and 2.4 we may estimate $T(a,b)$ for a,b such that

$$a=(\underbrace{M_1,\ldots,M_1}_{s},\ \underbrace{m_1,\ldots,m_1}_{n-s}),\quad b=(\underbrace{m_2,\ldots,m_2}_{t},\ \underbrace{M_2,\ldots,M_2}_{n-t}),\tag{2.2}$$

where s and t are nonnegative integers not greater than n. (It suffices to discuss for a in decreasing order and b in increasing order.) It would be more convenient to write $T=T(s,t)$. Without loss of generality we may assume that $M_1=M_2=1$, and conveniently we put $m_1=\alpha$ and $m_2=\beta$. Now we consider the following two cases.

Case I $\quad 0 \le t \le s \le n$. We can write a and b as follows:

$$a = (\underbrace{1,\dots,1}_{t},\underbrace{1,\dots,1}_{s-t},\underbrace{\alpha,\dots,\alpha}_{n-s}) \text{ and } b = (\underbrace{\beta,\dots,\beta}_{t},\underbrace{1,\dots,1}_{s-t},\underbrace{1,\dots,1}_{n-s}).$$

Let $P = \{1,\dots,t\}$, $Q = \{t+1,\dots,s\}$ and $R = \{s+1,\dots,n\}$. Then the index set Δ is divided into the six subsets

$$(P \times P) \cap \Delta, \quad (P \times Q) \cap \Delta, \quad (P \times R) \cap \Delta,$$
$$(Q \times Q) \cap \Delta, \quad (Q \times R) \cap \Delta \quad \text{and} \quad (R \times R) \cap \Delta.$$

Recall that $T(a,b) = \sum_{(i,j)\in\Delta}(a_ib_j - a_jb_i)^2$ by Lemma 2.2. If $(i,j) \in (P \times P) \cap \Delta$, $(Q \times Q) \cap \Delta$ or $(R \times R) \cap \Delta$, then from the above assumption we have

$$a_ib_j - a_jb_i = 0,$$

so that T is the sum with respect to (i,j) in the remaining sets.

Since $\Delta_1 = (P \times Q) \cap \Delta$ has $t(s-t)$ elements and $a_ib_j - a_jb_i = 1 - \beta$ for $(i,j) \in \Delta_1$, we see that

$$\sum_{\Delta_1} = \sum_{(i,j)\in\Delta_1} (a_ib_j - a_jb_i)^2 = t(s-t)(1-\beta)^2.$$

Similarly if we put $\sum_{\Delta_2}$ and $\sum_{\Delta_3}$ the sums of T with respect to $(i,j) \in \Delta_2 = (P \times R) \cap \Delta$ and $\Delta_3 = (Q \times R) \cap \Delta$, respectively, then we have

$$\sum_{\Delta_2} = t(n-s)(1-\alpha\beta)^2 \quad \text{and} \quad \sum_{\Delta_3} = (s-t)(n-s)(1-\alpha)^2.$$

Hence

$$\begin{aligned} T = T(s,t) &= \sum_{\Delta_1} + \sum_{\Delta_2} + \sum_{\Delta_3} \\ &= t(s-t)(1-\beta)^2 + t(n-s)(1-\alpha\beta)^2 + (s-t)(n-s)(1-\alpha)^2. \end{aligned} \tag{2.3}$$

Since $\alpha, \beta \in [0,1]$, we have

$$T \le \{t(s-t) + t(n-s) + (s-t)(n-s)\}(1-\alpha\beta).$$

Note that for real numbers x, y and z, $xy + yz + zx \le \frac{1}{3}(x + y + z)^2$ (and the equality holds if $x = y = z$). Hence we have

$$\begin{aligned} S(s,t) &:= t(s-t) + t(n-s) + (s-t)(n-s) \\ &\le \frac{1}{3}\{t + (s-t) + (n-s)\}^2 = \frac{n^2}{3}, \end{aligned} \tag{2.4}$$

so that $T \le n^2/3\ (1-\alpha\beta)^2$. We remark that the equality in (2.4) holds when $t = s - t = n - s$, that is, $s = 2n/3$ and $t = n/3$. Hence if $n = 3\nu$ for some integer $\nu \ge 1$, then $S(s, t)$ really attains $n^2/3$. We here recall that the example (1.4) is nothing but the case when $\nu = 1$ (and $\alpha = \beta = 0$).

Case II $0 \le s \le t \le n$. As in Case I, we obtain

$$\begin{aligned} T = {} & s(t-s)(\beta - \alpha\beta)^2 + s(n-t)(1-\alpha\beta)^2 \\ & + (t-s)(n-t)(\alpha - \alpha\beta)^2. \end{aligned} \tag{2.5}$$

Hence by a similar argument as in Case I we have $T \le (n^2/3)\ (1-\alpha\beta)^2$.

We have shown that the constant $n^2/3$ is a best bound in Ozeki's inequality (2.1) and is the best with respect to n, when $n = 3\nu$. Considering the best constant for a general integer n, we have:

THEOREM 2.5 *Under the same assumption as in Theorem* 2.1,

$$\sum a_i^2 \sum b_i^2 - \left(\sum a_i b_i\right)^2 \le \begin{cases} (n^2/3)(M_1M_2 - m_1m_2)^2, \\ \quad \text{if } n = 3\nu, \\ ((n^2-1)/3)(M_1M_2 - m_1m_2)^2, \\ \quad \text{if } n = 3\nu \pm 1. \end{cases}$$

Moreover, in each case the constant is best possible with respect to n.

Proof It suffices to discuss the cases when $n = 3\nu \pm 1$. Let us use the same notations as in the proof of Theorem 2.1. We may assume $t \le s$. Recall that

$$S = S(s,t) = t(s-t) + t(n-s) + (s-t)(n-s),$$

and that the constant $n^2/3$ in Theorem 2.1 was obtained from the inequality $S \le n^2/3$. Hence we have to show here that S attains $(n^2-1)/3$

as its maximum. Consider the case when $n = 3\nu + 1$. Since

$$\begin{aligned} S &= -s^2 + (n+t)s - t^2 \\ &= -\left(s - \frac{2n}{3}\right)^2 - \left(s - \frac{2n}{3}\right)\left(t - \frac{n}{3}\right) + \left(t - \frac{n}{3}\right)^2 + \frac{n^2}{3}, \end{aligned}$$

if we assume $S = constant$, then $c := (n^2/3) - S$ is constant, and the equation

$$\left(s - \frac{2n}{3}\right)^2 + \left(s - \frac{2n}{3}\right)\left(t - \frac{n}{3}\right) - \left(t - \frac{n}{3}\right)^2 = c$$

expresses an ellipse with center at $(2n/3, n/3)(2\nu + \frac{2}{3}, \nu + \frac{1}{3})$ in the st-plane (if s and t move continuously). Let

$$\Delta' = \{(s,t);\ s \le t,\ s = 0, \ldots, 3\nu + 1,\ t = 0, \ldots, 3\nu + 1\}.$$

Then in order to obtain the maximum of S, we have to find the (smallest) ellipse which has no point of Δ' in the interior and which passes through at least one point of Δ'. By a simple computation we can see that the desired ellipse is the one corresponding to $c = \frac{1}{3}$, on which lie the three points $(2\nu, \nu)$, $(2\nu + 1, \nu)$ and $(2\nu + 1, \nu + 1)$ of Δ'. Hence the maximum of S is $(n^2/3) - c = (n^2 - 1)/3$. For the case when $n = 3\nu - 1$ we can obtain the same value $(n^2 - 1)/3$ as the maximum of S similarly.

3 REFINEMENT OF OZEKI'S INEQUALITY

In the preceding section for the extremal n-tuples a and b having the forms in (2.2), we showed, for the Case I: $0 \le t \le s \le n$,

$$T = t(s-t)(1-\beta)^2 + t(n-s)(1-\alpha\beta)^2 + (s-t)(n-s)(1-\alpha)^2,$$

and for Case II: $0 \le s \le t \le n$,

$$\begin{aligned} T = s(t-s)(\beta - \alpha\beta)^2 + s(n-t)(1-\alpha\beta)^2 \\ + (t-s)(n-t)(\alpha - \alpha\beta)^2. \end{aligned}$$

Ozeki's inequality (2.1) was obtained as an estimation of these identities. In this section we try to estimate their maximum precisely. Consider the first case, we put

$$x = t, \qquad y = s - t, \qquad z = n - s,$$
$$A = (1-\beta)^2, \qquad B = (1 - \alpha\beta)^2, \qquad C = (1 - \alpha)^2.$$

Then the problem is to compute the maximum of

$$T = Axy + Bxz + Cyz \tag{3.1}$$

under the conditions

$$x \geq 0, \ \ y \geq 0, \ \ z \geq 0 \quad \text{and} \quad x + y + z = n. \tag{3.2}$$

Though x, y and z are discrete variables, we for a moment assume that they are continuous. Without loss of generality we assume $ABC \neq 0$, or equivalently $\alpha \neq 1$ and $\beta \neq 1$. To get the maximum of T, let

$$U = T - \lambda(x + y + z - n),$$

where λ is the Lagrange multiplier. Then solving the equations

$$\begin{cases} U_x = Ay + Bz - \lambda = 0, \\ U_y = Ax + Cz - \lambda = 0, \\ U_z = Bx + Cy - \lambda = 0, \end{cases}$$

we have

$$x = \frac{(A + B - C)\lambda}{2AB}, \quad y = \frac{(C + A - B)\lambda}{2CA}, \quad z = \frac{(B + C - A)\lambda}{2BC}. \tag{3.3}$$

From the identity $x + y + z = n$, we then have

$$\lambda = \frac{2ABCn}{D}, \tag{3.4}$$

where $D = 2AB + 2BC + 2CA - A^2 - B^2 - C^2$. If we express D using α and β, then

$$D = \{4 - (1 + \alpha)(1 + \beta)\}(1 + \alpha)(1 + \beta)(1 - \alpha)^2(1 - \beta)^2 \tag{3.5}$$

(and so that $D > 0$). In fact, putting $p = 1 - \beta$, $q = 1 - \alpha\beta$ and $r = 1 - \alpha$, or $A = p^2$, $B = q^2$ and $C = r^2$, we have

$$\begin{aligned} D &= 2p^2q^2 + 2q^2r^2 + 2r^2p^2 - p^4 - q^4 - r^4 \\ &= (p + q + r)(-p + q + r)(p - q + r)(p + q - r) \\ &= (3 - \alpha - \beta - \alpha\beta)(1 - \alpha + \beta - \alpha\beta)(1 - \alpha - \beta + \alpha\beta) \\ &\quad \times (1 + \alpha - \beta - \alpha\beta) \\ &= \{4 - (1 + \alpha)(1 + \beta)\}(1 + \alpha)(1 + \beta)(1 - \alpha)^2(1 - \beta)^2. \end{aligned}$$

Now note that from (3.4) we can rewrite (3.2) as follows:

$$x = \frac{(A + B - C)Cn}{D}, \quad y = \frac{(C + A - B)Bn}{D}, \quad z = \frac{(B + C - A)An}{D}. \tag{3.6}$$

Since $B = (1 - \alpha\beta)^2 \geq (1 - \alpha)^2 = C$ and similarly $B \geq A$, we see that $x \geq 0$ and $z \geq 0$. To guarantee $y \geq 0$ we have to assume

$$C + A - B = (1 - \alpha)^2 + (1 - \beta)^2 - (1 - \alpha\beta)^2 \geq 0, \tag{3.7}$$

or equivalently

$$(1 + \alpha)(1 + \beta) \leq 2.$$

Hence, if the condition (3.7) is satisfied then the maximum $T_{\max}$ of T is attained for x, y and z of (3.6), and then from (3.1) we obtain

$$T_{\max} = \frac{ABCn^2}{D} = \frac{(1 - \alpha\beta)^2 n^2}{\{4 - (1 + \alpha)(1 + \beta)\}(1 + \alpha)(1 + \beta)}. \tag{3.8}$$

If α and β are rational, then for a sufficiently large integer n the values of x, y and z of (3.6) become integers, and then T really attains its maximum $T_{\max}$ of (3.8).

If (3.7) does not hold, that is, if

$$(1+\alpha)(1+\beta) \geq 2, \tag{3.9}$$

then

$$\begin{aligned} T &= Axy + Bxz + Cyz \\ &= Ax(n-x-z) + Bxz + C(n-x-z)z \\ &= (B-C-A)xz + Ax(n-x) + C(n-z)z, \end{aligned}$$

and since

$$xz \leq \left(\frac{x+z}{2}\right)^2 \leq \frac{n^2}{4}, \quad x(n-x) \leq \left(\frac{x+n-x}{2}\right)^2 = \frac{n^2}{4}$$

and

$$(n-z)z \leq \left(\frac{n-z+z}{2}\right)^2 = \frac{n^2}{4},$$

we have

$$\begin{aligned} T &\leq (B-C-A)\left(\frac{n^2}{4}\right) + A\left(\frac{n^2}{4}\right) + C\left(\frac{n^2}{4}\right) \\ &= \frac{Bn^2}{4} = \frac{n^2}{4}(1-\alpha\beta)^2. \end{aligned}$$

If n is even, then putting $x = z = n/2$, $y = 0$, we see that T really attains $(n^2/4)(1-\alpha\beta)^2$.

For the second case: $0 \leq s \leq t \leq n$, if we put

$$\begin{gathered} x = s, \qquad y = t - s, \qquad z = n - t, \\ A = (\beta - \alpha\beta)^2, \quad B = (1-\alpha\beta)^2, \quad C = (\alpha - \alpha\beta)^2, \end{gathered}$$

then we have the same problem as in the first case. Since in this case

$$\begin{aligned} B - C - A &= (1-\alpha\beta)^2 - (\alpha - \alpha\beta)^2 - (\beta - \alpha\beta)^2 \\ &= (1-\alpha)^2(1-\beta)^2 \geq 0, \end{aligned}$$

we have the condition (3.9), so that as above we have

$$T \leq T_{\max} = \frac{Bn^2}{4} = \frac{n^2}{4}(1-\alpha\beta)^2,$$

and if n is even, $T_{\max}$ is attained for $x = y = n/2$, $y = 0$.

Now summarizing the above argument, we have a refinement of Theorem 2.1:

THEOREM 3.1 *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples of real numbers satisfying*

$$0 \leq m_1 \leq a_i \leq M_1, \ 0 \leq m_2 \leq b_i \leq M_2 \quad (i = 1, \ldots, n) \text{ and } M_1 M_2 \neq 0.$$

Put $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then

(i) *if* $(1+\alpha)(1+\beta) \leq 2$ *then*

$$\sum a_i^2 \sum b_i^2 - \left(\sum a_i b_i\right)^2 \leq \frac{n^2(M_1M_2 - m_1m_2)^2}{\{4-(1+\alpha)(1+\beta)\}(1+\alpha)(1+\beta)}; \tag{3.10}$$

(ii) *if* $(1+\alpha)(1+\beta) \geq 2$ *then*

$$\sum a_i^2 \sum b_i^2 - \left(\sum a_i b_i\right)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2. \tag{3.11}$$

The constants $n^2/\{4-(1+\alpha)(1+\beta)\}(1+\alpha)(1+\beta)$ in (3.10) and $n^2/4$ in (3.11) are best possible: For (i), if both α and β are rational, then if we choose a sufficiently large integer n, then for suitable extremal n-tuples a and b the equality sign of (3.10) holds. For (ii), if n is an even integer then for the extremal n-tuples a and b such that the first halves of a_i and b_i equal to M_1 and m_2, respectively, and the latter halves of them equal to m_1 and M_2, respectively, then the equality sign of (3.11) holds.

COROLLARY 3.2 *If* $0 \leq m \leq a_i \leq M$ $(i = 1, \ldots, n)$, *then*

$$\frac{1}{n}\sum a_i^2 - \left(\frac{\sum a_i}{n}\right)^2 \leq \frac{(M-m)^2}{4}. \tag{3.12}$$

Proof Putting $b_i = 1$ in (3.11) and dividing both the sides by n^2, we obtain (3.12).

The above inequality (3.12) is nothing but the variance inequality of the numerical case.

4 EXTENSIONS AND OPERATOR VERSIONS OF OZEKI'S INEQUALITY

In this section, extending Ozeki's inequality (2.1) and the refinements (3.10) and (3.11), we show their operator versions for commuting positive operators. We also try to obtain an operator version of Ozeki's inequality without commutativity condition.

We begin with a weighted version of Ozeki's inequality.

THEOREM 4.1 *Let $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ be n-tuples of real numbers satisfying the condition* (1.1). *Suppose that $\{w_1,\dots,w_n\}$ is an n-tuple of nonnegative numbers and $\sum w_k = w$. Then*

$$\sum w_k a_k^2 \sum w_k b_k^2 - \left(\sum w_k a_k b_k\right)^2 \le \frac{w^2}{3}(M_1M_2 - m_1m_2)^2. \tag{4.1}$$

Proof We may assume that all w_k (and w) are rational, so that, multiplying by an integer both the sides of (4.1), we can turn all w_k (and w) into integers. Then from Theorem 2.1 we can deduce the inequality easily.

For two nonnegative integrable functions defined on a measure space we have

THEOREM 4.2 *Let μ be a positive measure on X, $\mu(X)=1$, and let f and g be functions in $L^2(X)$ satisfying $0\le m_1\le f\le M_1$ and $0\le m_2\le g\le M_2$, respectively. Then we have*

$$\int f^2\,\mathrm{d}\mu \int g^2\,\mathrm{d}\mu - \left(\int fg\,\mathrm{d}\mu\right)^2 \le \frac{1}{3}(M_1M_2 - m_1m_2)^2. \tag{4.2}$$

Proof To obtain the inequality, let $\{E_1,\dots,E_n\}$ be a decomposition of X, and let $x_k\in E_k$ $(k=1,\dots,n)$. Then from Theorem 4.1 we have

$$\sum f(x_k)^2\mu(E_k)\sum g(x_k)^2\mu(E_k) - \left(\sum f(x_k)g(x_k)\mu(E_k)\right)^2 \le \tfrac{1}{3}(M_1M_2 - m_1m_2)^2.$$

Hence as the limit of decomposition we have the desired inequality.

Let A and B be two self-adjoint operators on a Hilbert space H. Then if they are commuting, there exist commuting spectral families $E^A(\cdot)$ and $E^B(\cdot)$ corresponding to A and B such that for a polynomial $p(A, B)$ in A, B

$$p(A,B) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(s,t)\,\mathrm{d}E^A(s)\,\mathrm{d}E^B(t)$$

([7, p. 287]). Hence for $p(A,B) = A^m B^n$ we have

$$(A^m B^n x, x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(s,t)\,\mathrm{d}(E^A(s)E^B(t)x, x), \quad x \in H.$$

Now as an operator version of Theorem 2.1 we have

THEOREM 4.3 *Let A and B be commuting self-adjoint operators satisfying $0 \le m_1 \le A \le M_1$ and $0 \le m_2 \le B \le M_2$ respectively. Then for a unit vector $x \in H$,*

$$(A^2x,x)(B^2x,x) - (ABx,x)^2 \le \tfrac{1}{3}(M_1M_2 - m_1m_2)^2. \tag{4.3}$$

Proof Let $\mathrm{d}\mu = \mathrm{d}\mu(s,t) = \mathrm{d}(E^A(s)E^B(t)x, x) = \mathrm{d}\|E^A(s)E^B(t)x\|^2$. Then μ is a positive measure on the rectangle $[m_1, M_1] \times [m_2, M_2]$ and its total mass $= 1$. Hence from Theorem 4.2 we have

$$\begin{aligned}(A^2x,x)(B^2x,x) - (ABx,x)^2 &= \int s^2\,\mathrm{d}\mu \int t^2\,\mathrm{d}\mu - \left(\int s\,\mathrm{d}\mu \int t\,\mathrm{d}\mu\right)^2 \\ &\le \tfrac{1}{3}(M_1M_2 - m_1m_2)^2.\end{aligned}$$

Starting from Theorem 2.1 (Eq. (2.1)), we have successively deduced the inequalities (4.1), (4.2) and (4.3). In parallel, starting from Theorem 3.1 we can successively deduce corresponding inequalities. In particular, we can show

THEOREM 4.4 *Let A and B be commuting self-adjoint operators on H satisfying the same conditions as in Theorem 4.3, and assume $M_1M_2 \neq 0$.*

Put $\alpha = m_1/M_1$ *and* $\beta = m_2/M_2$. *Then for any unit vector* $x \in H$,

(i) *if* $(1+\alpha)(1+\beta) \le 2$, *then*

$$(A^2x,x)(B^2x,x) - (ABx,x)^2 \le \frac{(M_1M_2 - m_1m_2)^2}{\{4-(1+\alpha)(1+\beta)\}(1+\alpha)(1+\beta)};$$

(ii) if $(1+\alpha)(1+\beta) \ge 2$, *then*

$$(A^2x,x)(B^2x,x) - (ABx,x)^2 \le \tfrac{1}{4}(M_1M_2 - m_1m_2)^2.$$

If $B = 1$, the identity operator on H, then putting $M_2 = m_2 = 1$, from (ii) we have

$$(A^2x,x) - (Ax,x)^2 \le \tfrac{1}{4}(M_1 - m_1)^2.$$

This is nothing but the variance inequality of A[1, Theorem 2] in noncommutative probability.

If we do not assume that the operators A and B commute, then the inequality (4.3) is false in general. In fact, let

$$A = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}, \quad x = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

Then $(A^2x,x)(B^2x,x) - (ABx,x)^2 = 256$. On the other hand since the characteristic values of A and B coincide and they are $3 \pm \sqrt{5}$, we can put $m_1 = m_2 = 3 - \sqrt{5}$ and $M_1 = M_2 = 3 + \sqrt{5}$, so that

$$\tfrac{1}{3}(M_1M_2 - m_1m_2)^2 = 240 < 256.$$

To consider the operator version of Ozeki's inequality without commutativity assumption, we here introduce the s-geometric mean $A\sharp_s B$ of positive (invertible) operators A and B defined (by Kubo–Ando [4]) as

$$A\sharp_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2} \quad (0 \le s \le 1).$$

THEOREM 4.5 *Let A and B be positive operators on H satisfying* $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$, *respectively. Then for any unit vector $x \in H$,*

$$(A^2x, x)(B^2x, x) - (A^2\sharp_{1/2}B^2x, x)^2 \le \frac{1}{4\gamma^2}(M_1M_2 - m_1m_2)^2, \qquad (4.4)$$

where $\gamma = \max\{m_1/M_1, m_2/M_2\}$.

Proof By the variance inequality [1], if C is a positive operator on H such that $0 < m \le C \le M$, then for any unit vector $x \in \mathrm{H}$,

$$(C^2x, x) - (Cx, x)^2 \le \tfrac{1}{4}(M - m)^2. \qquad (4.5)$$

If we replace x by $x/\|x\|$ in (4.5), then we have

$$(C^2x, x)(x, x) - (Cx, x)^2 \le \tfrac{1}{4}(M - m)^2(x, x)^2. \qquad (4.6)$$

Furthermore, if we replace C by $(B^{-1}A^2B^{-1})^{1/2}$ and x by Bx in (4.6), then we have

$$\begin{aligned}(A^2x, x)(B^2x, x) &- (B(B^{-1}A^2B^{-1})^{1/2}Bx, x)^2 \\ &\le \tfrac{1}{4}(M' - m')^2(Bx, Bx)^2 \le \tfrac{1}{4}(M' - m')^2M_2^4. \qquad (4.7)\end{aligned}$$

Here the corresponding M' and m' are determined as follows; since

$$\frac{m_1^2}{M_2^2} \le m_1^2B^{-2} \le B^{-1}A^2B^{-1} \le M_1^2B^{-2}\frac{M_1^2}{m_2^2},$$

we have $m_1/M_2 \le (B^{-1}A^2B^{-1})^{1/2} \le M_1/m_2$, so that we may put $M' = M_1/m_2$ and $m' = m_1/M_2$ in (4.7). Hence, nothing $B(B^{-1}A^2B^{-1})^{1/2}\ B = B^2\sharp_{1/2}A^2$, we have

$$(A^2x, x)(B^2x, x) - (B^2\sharp_{1/2}A^2x, x)^2 \le \frac{M_2^2}{4m_2^2}(M_1M_2 - m_1m_2)^2. \qquad (4.8)$$

If we exchange A and B in (4.8), then we have

$$(A^2x,x)(B^2x,x) - (A^2\sharp_{1/2}B^2x,x)^2 \leq \frac{M_1^2}{4m_1^2}(M_1M_2 - m_1m_2)^2. \quad (4.9)$$

Now since $B^2\sharp_{1/2}A^2 = A^2\sharp_{1/2}B^2$ (cf. [4]), we can obtain the desired inequality (4.4) from (4.8) and (4.9).

Concluding this paper, we note the covariance–variance inequality due to Fujii *et al.* [1].

THEOREM A [1, Theorem 3] *If A and B are self-adjoint operators on H such that $m_1 \leq A \leq M_1$* and *$m_2 \leq B \leq M_2$ respectively, then*

$$|(ABx,x) - (Ax,x)(Bx,x)| \leq \tfrac{1}{4}(M_1 - m_1)(M_2 - m_2). \quad (4.10)$$

In [1], applying this inequality, they induced several classical inequalities such as the Kantrovich inequality and the Heinz–Kato–Furuta inequality skillfully, and also proved the following result concerning the difference in the Hölder–McCarthy inequality (cf. [6]).

THEOREM B [1, Theorem 4] *If a positive operator A on H satisfies $m \leq A \leq M$, then for any unit vector $x \in H$,*

$$(A^{k+1}x,x) - (Ax,x)^{k+1} \leq \frac{1}{4}(M-m)^2 \sum_{i=1}^{k}(k-i+1)m^{i-1}M^{k-1} \quad (4.11)$$

for all natural numbers k.

Using the above inequalities (4.10), (4.11) and the Hölder–McCarthy inequality [6, Lemma 2.1] that

$$\begin{gathered}(A^rx,x) \leq (Ax,x)^r \quad \textit{for } 0 \leq r \leq 1 \\ ((A^rx,x) \geq (Ax,x)^r \quad \textit{for } r \geq 1), \end{gathered} \quad (4.12)$$

we have the following extension of Theorem 4.5.

THEOREM 4.6 *Let A and B be positive operators satisfying $0 < m_1 \leq A \leq M_1$* and *$0 < m_2 \leq B \leq M_2$, respectively. Suppose that $p \geq 2$, $q > 0$ and $[p]$ is the largest integer not exceeding p. Put $k = [p] - 1$ and $r = p - k - 1$.*

Then for any unit vector $x \in H$,

$$
\begin{aligned}
&(A^p x, x)(B^q x, x)^{p-1} - (B^q \sharp_{1/p} A^p x, x)^p \\
&\quad \leq F(m_1/M_2^{q/p}, M_1/m_2^{q/p}, p) M_2^{pq},
\end{aligned} \tag{4.13}
$$

where

$$
\begin{aligned}
F(m, M, p) = &\frac{1}{4}(M^r - m^r)(M^{k+1} - m^{k+1}) + \frac{M^r}{4}(M - m)^2 \\
&\times \sum_{i=1}^{k}(k - i + 1)m^{i-1}M^{k-1}.
\end{aligned}
$$

Proof We first extend the inequality (4.11) for the positive number $p = k + 1 + r$. (For a moment we assume that $0 < m \leq A \leq M$.) Since $m^t \leq A^t \leq M^t$ for $t \geq 0$, replacing A by A^r and B by A^{k+1} in (4.10), we have

$$
\begin{aligned}
&|(A^{r+k+1}x, x) - (A^r x, x)(A^{k+1}x, x)| \\
&\quad \leq \tfrac{1}{4}(M^r - m^r)(M^{k+1} - m^{k+1}).
\end{aligned} \tag{4.14}
$$

Hence by (4.11), (4.12) and (4.14) we have

$$
\begin{aligned}
&(A^p x, x) - (Ax, x)^p \\
&\quad \leq |(A^p x, x) - (A^r x, x)(A^{k+1}x, x)| + (A^r x, x)(A^{k+1}x, x) - (Ax, x)^p \\
&\quad \leq |(A^r A^{k+1}x, x) - (A^r x, x)(A^{k+1}x, x)| \\
&\qquad + (Ax, x)^r\{(A^{k+1}x, x) - (Ax, x)^{k+1}\} \\
&\quad \leq \frac{1}{4}(M^r - m^r)(M^{k+1} - m^{k+1}) + \frac{M^r}{4}(M - m)^2 \\
&\qquad \times \sum_{i=1}^{k}(k - i + 1)m^{i-1}M^{k-1},
\end{aligned}
$$

that is,

$$
(A^p x, x) - (Ax, x)^p \leq F(m, M, p). \tag{4.15}
$$

Now, to obtain (4.13), replace x by $x/\|x\|$ in (4.15), then we have

$$(A^p x, x)(x, x)^{p-1} - (Ax, x)^p \leq F(m, M, p)(x, x)^p.$$

Moreover, replace A by $(B^{-q/2}A^pB^{-q/2})^{1/p}$ and x by $B^{q/2}x$. Then we can get

$$\begin{aligned}(A^p x, x)&(B^q x, x)^{p-1} - (B^q \sharp_{1/p} A^p x, x)^p \\ &\leq F(m', M', p)(B^q x, x)^p \leq F(m', M', p)M_2^{pq}.\end{aligned}$$

To settle the corresponding values m' and M', note that

$$m_1^p/M_2^q \leq (B^{-q/2}A^pB^{-q/2}) \leq M_1^p/m_2^q,$$

or

$$m_1/M_2^{q/p} \leq (B^{-q/2}A^pB^{-q/2})^{1/p} \leq M_1/m_2^{q/p}.$$

Hence putting $m' = m_1/M_2^{q/p}$ and $M' = M_1/m_2^{q/p}$, we have the desired inequality (4.13).

It is easy to see that if we put $p = q = 2$ in (4.13) we have (4.8).

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