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Continuous Dependence on Modeling for Some Well-posed Perturbations of the Backward Heat Equation

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Four different well-posed regularizations of the improperly posed Cauchy problem for the backward heat equation are investigated in order to determine whether solutions of these problems depend continuously on a perturbation parameter. Using differential inequality techniques, we derive results implying continuous dependence in each case.

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1. INTRODUCTION

One method that has been used to construct solutions to the ill-posed Cauchy problem for the backward heat equation is the quasireversibility method (see [9,10] for references). The idea behind this method is to perturb the ill-posed problem into a well-posed one and use the solution of the well-posed problem to construct an approximate solution of the original problem. A number of perturbations or regularizations have been proposed. These include a biharmonic regularization [9], a pseudo-parobolic one [10,16], a hyperbolic regularization [2,10], and a

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regularization in which the initial condition is perturbed rather than the differential equation [15]. Unlike the initial value problem for the backward heat equation, each of these regularizations produces a well-posed problem containing a perturbation parameter. One question that arises is whether the solutions of these four regularizations depend continuously on this perturbation parameter. Such studies have been referred to as "continuous dependence on modeling" investigations and have been carried out for a number of both well-posed and ill-posed problems (see e.g. [3,5,6,8,11,12]).

In this paper, we shall derive inequalities from which continuous dependence on the perturbation parameter for solutions of each of the four regularizations of the Cauchy problem for the backward heat equation can be inferred. We thus consider the following four initial-boundary value problems. In each problem, D is a bounded region in $\mathbb{R}^n$ with boundary ∂D, Δ is the Laplace operator, ϵ is a small positive constant, and T is some prescribed value of time which, except in the fourth problem, may be infinity.

PROBLEM 1

$$\begin{aligned} &u_{,t} + \Delta u + \epsilon \Delta^2 u = 0 && \text{in } D \times (0,T), \\ &u = 0, \quad \Delta u = 0 && \text{on } \partial D \times [0,T], \\ &u(x,0) = f(x), && x \in D. \end{aligned} \tag{1.1}$$

This is a biharmonic perturbation first suggested by Lattés and Lions [9] as a regularization of the initial value problem for the backward heat equation. If we restrict ourselves to a finite time interval, then Ames [1] showed that provided the solutions are suitably constrained, the difference between a solution of (1.1) and the "solution" of the problem with $\epsilon = 0$ is $0(\epsilon^{\delta(t)})$ in L^2 norm, where $\delta(t)$ is an explicit function such that $0 < \delta \leq 1$ for $0 \leq t \leq T$.

The second problem we consider is

PROBLEM 2

$$\begin{aligned} &u_{,t} + \Delta u - \epsilon \Delta u_{,t} = 0 && \text{in } D \times (0,T), \\ &u = 0, && \text{on } \partial D \times [0,T], \\ &u(x,0) = f(x), && x \in D. \end{aligned} \tag{1.2}$$

Such equations have been called pseudo-parabolic by Showalter and Ting [16] and have been considered in the context of the quasiversibility method by Showalter [13,14] and Ames [1].

Our third problem involves a hyperbolic perturbation of the backward heat equation, namely

PROBLEM 3

$$
\begin{aligned}
&u_{,t} + \Delta u - \epsilon u_{,tt} = 0 && \text{in } D \times (0, T),\\
&u = 0, && \text{on } \partial D \times [0, T],\\
&u(x, 0) = f(x), \quad u_t(x, 0) = g(x), && x \in D.
\end{aligned}
\tag{1.3}
$$

Ames and Cobb [2] have recently compared solutions of (1.3) with "solutions" of the Cauchy problem for the case $\epsilon = 0$.

Finally, the fourth problem is a regularization in which the initial condition is perturbed rather than the differential equation.

PROBLEM 4

$$
\begin{aligned}
&u_{,t} + \Delta u = 0 && \text{in } D \times (0, T),\\
&u = 0, && \text{on } \partial D \times [0, T],\\
&u(x, 0) + \epsilon u(x, T) = f(x), && x \in D.
\end{aligned}
\tag{1.4}
$$

Showalter [15] calls this a "quasi-boundary-value" approximation to the initial value problem for the backward heat equation. Problem (1.4) has been shown to be well-posed for each $\epsilon > 0$ by Clark and Oppenheimer [4]. We point out that this problem is equivalent to a Tikhonov type regularization of a Fredholm integral equation of the first kind (see [7]).

Each of the following sections is devoted to obtaining continuous dependence on ϵ results for the preceding four problems. Throughout our analysis we shall employ standard indicial notation and a comma to denote partial differentiation.

2. BIHARMONIC PERTURBATION

Let us consider two solution u_1 and u_2 of (1.1) corresponding to two different nonzero values ϵ_1 and ϵ_2 but having the same initial and boundary data. Set

$$w = u_1 - u_2 \tag{2.1}$$

so that w is a solution of the problem

$$\begin{aligned} &w_{,t} + \Delta w + \epsilon_1 \Delta^2 w = (\epsilon_2 - \epsilon_1)\Delta^2 u_2 && \text{in } D \times (0,T), \\ &w = 0, \quad \Delta w = 0 && \text{on } \partial D \times [0,T], \\ &w(x,0) = 0, && x \in D. \end{aligned} \tag{2.2}$$

We assume without loss of generality that $\epsilon_1 > \epsilon_2$. Consider the following functional defined on solutions of (2.2):

$$\Phi(t) = \int_D w^2 \,\mathrm{d}x. \tag{2.3}$$

Our aim is to derive a differential inequality from which our continuous dependence results can be obtained. Differentiation (2.3) and substituting the differential equation (2.2), we see that

$$\begin{aligned} \Phi'(t) &= 2\int_D w w_{,t} \,\mathrm{d}x \\ &= 2\int_D w[-\Delta w - \epsilon_1 \Delta^2 w - (\epsilon_1 - \epsilon_2)\Delta^2 u_2] \,\mathrm{d}x \end{aligned}$$

Integration by parts leads to

$$\Phi'(t) = -2\epsilon_1 \int_D (\Delta w)^2 \,\mathrm{d}x - 2\int_D \Delta w[w + (\epsilon_1 - \epsilon_2)\Delta u_2] \,\mathrm{d}x$$

and an application of the arithmetic–geometric mean inequality gives the inequality

$$\Phi'(t) \le \frac{(1+\alpha)}{2\epsilon_1}\int_D w^2 \,\mathrm{d}x + \frac{(1+\alpha)}{2\alpha\epsilon_1}(\epsilon_1 - \epsilon_2)^2 \int_D (\Delta u_2)^2 \,\mathrm{d}x \tag{2.4}$$

for an arbitrary positive constant α. Thus, we have

$$\Phi'(t) \le \frac{1+\alpha}{2\epsilon_1}\Phi + \frac{1+\alpha}{2\alpha\epsilon_1}(\epsilon_1 - \epsilon_2)^2 \int_D (\Delta u_2)^2 \,\mathrm{d}x. \tag{2.5}$$

Integration of this inequality results in the bound

$$\Phi(t) \leq \left(\frac{1+\alpha}{2\alpha\epsilon_1}\right)(\epsilon_1-\epsilon_2)^2 \int_0^t \int_D \exp\left\{\frac{(1+\alpha)}{2\epsilon_1}(t-\eta)\right\}(\Delta u_2)^2\,\mathrm{d}x\,\mathrm{d}\eta. \tag{2.6}$$

We proceed by multiplying the equation for u_2 by $u_2\exp\{(1+\alpha)(t-\eta)/2\epsilon_1\}$ and integrating the result.
This gives

$$\begin{aligned}
&\frac{1}{2}\int_D u_2^2\,\mathrm{d}x - \frac{1}{2}\exp\left\{\frac{(1+\alpha)t}{2\epsilon_1}\right\}\int_D f^2\,\mathrm{d}x \\
&\quad + \frac{1+\alpha}{4\epsilon_1}\int_0^t \int_D \exp\left\{\frac{(1+\alpha)(t-\eta)}{2\epsilon_1}\right\}u_2^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad + \int_0^t \int_D \exp\left\{\frac{(1+\alpha)(t-\eta)}{2\epsilon_1}\right\}u_2\Delta u_2\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad + \epsilon_2\int_0^t \int_D \exp\left\{\frac{(1+\alpha)}{2\epsilon_1}(t-\eta)\right\}(\Delta u_2)^2\,\mathrm{d}x\,\mathrm{d}\eta = 0.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\frac{1+\alpha}{4\epsilon_1}\int_0^t \int_D \exp\left\{\frac{(1+\alpha)(t-\eta)}{2\epsilon_1}\right\}u_2^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad + \epsilon_2\int_0^t \int_D \exp\left\{\frac{(1+\alpha)(t-\eta)}{2\epsilon_1}\right\}(\Delta u_2)^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad \leq \frac{\beta}{2\epsilon_2}\int_0^t \int_D \exp\left\{\frac{(1+\alpha)}{2\epsilon_1}(t-\eta)\right\}u_2^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad + \frac{\epsilon_2}{2\beta}\int_0^t \int_D \exp\left\{\frac{(1+\alpha)}{2\epsilon_1}(t-\eta)\right\}(\Delta u_2)^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad + \frac{1}{2}\exp\left\{\frac{(1+\alpha)t}{2\epsilon_1}\right\}\int_D f^2\,\mathrm{d}x
\end{aligned} \tag{2.7}$$

for an arbitrary positive constant β. The choices $\beta=1$, $\alpha=(2\epsilon_1-\epsilon_2)/\epsilon_2$ lead to the inequality

$$\epsilon_2\int_0^t\int_D \exp\left\{\frac{(1+\alpha)}{2\epsilon_1}(t-\eta)\right\}(\Delta u_2)^2\,\mathrm{d}x\,\mathrm{d}\eta \le \mathrm{e}^{t/\epsilon_2}\int_D f^2\,\mathrm{d}x. \tag{2.8}$$

Consequently, (2.6) becomes

$$\Phi(t) \le \frac{(\epsilon_1-\epsilon_2)^2}{\epsilon_2(2\epsilon_1-\epsilon_2)}\left\{\int_D f^2\,\mathrm{d}x\right\}\mathrm{e}^{t/\epsilon_2} \tag{2.9}$$

which is the desired continuous dependence result. We note that if we choose $\beta=\epsilon_2(1+\alpha)/(2\epsilon_1)$ and then choose α appropriately, we can obtain a symmetric version of (2.9), namely

$$\Phi(t) \le \frac{1}{2}\left\{\frac{\mathrm{e}^{t/\epsilon_1}}{\epsilon_1(2\epsilon_2-\epsilon_1)}+\frac{\mathrm{e}^{t/\epsilon_2}}{\epsilon_2(2\epsilon_1-\epsilon_2)}\right\}(\epsilon_1-\epsilon_2)^2\int_D f^2\,\mathrm{d}x. \tag{2.10}$$

3. PSEUDO-PARABOLIC REGULARIZATION

For Problem 2, suppose u and v are two solutions corresponding to the parameters ϵ_1 and ϵ_2, respectively, where $\epsilon_2>\epsilon_1$. Then the difference $w=v-u$ satisfies the initial-boundary value problem

$$\begin{aligned} &w_{,t}+\Delta w-\epsilon_2\Delta w_{,t}=(\epsilon_2-\epsilon_1)\Delta u_{,t} && \text{in } D\times(0,T)\\ &w=0 && \text{on } \partial D\times[0,T]\\ &w(x,0)=0, && x\in D. \end{aligned} \tag{3.1}$$

We now define a functional

$$\Phi(t)=\int_0^t\int_D (w^2+\epsilon_2 w_{,i}w_{,i})\,\mathrm{d}x\,\mathrm{d}\eta, \tag{3.2}$$

which we show satisfies a first order differential inequality. Differentiating (3.2), we have

$$\Phi'(t) = 2\int_0^t \int_D (ww,_\eta + \epsilon_2 w,_i w,_{i\eta})\,\mathrm{d}x\,\mathrm{d}\eta. \tag{3.3}$$

Substitution of the differential equation in (3.1) and integration by parts leads to

$$\Phi'(t) = 2\int_0^t \int_D w,_i w,_i\,\mathrm{d}x\,\mathrm{d}\eta - 2(\epsilon_2 - \epsilon_1)\int_0^t \int_D w,_i u,_{i\eta}\,\mathrm{d}x\,\mathrm{d}\eta.$$

Then an application of the arithmetic–geometric mean inequality gives

$$\Phi'(t) \leq \left(\frac{2+\alpha}{\epsilon_2}\right)\Phi + \frac{1}{\alpha}(\epsilon_2 - \epsilon_1)^2 \int_0^t \int_D u,_{i\eta} u,_{i\eta}\,\mathrm{d}x\,\mathrm{d}\eta \tag{3.4}$$

for a positive constant α. We now need to bound the second term on the right side of (3.4) in terms of data.

Multiplying the differential equation in (1.2) by $u,_\eta$ and then integrating over D and with respect to η, we see that

$$\epsilon_1 \int_0^t \int_D u,_{i\eta} u,_{i\eta}\,\mathrm{d}x\,\mathrm{d}\eta = -\int_0^t \int_D u,_\eta^2\,\mathrm{d}x\,\mathrm{d}\eta + \int_0^t \int_D u,_{i\eta} u,_i\,\mathrm{d}x\,\mathrm{d}\eta. \tag{3.5}$$

Application of Schwarz's inequality and the arithmetic–geometric mean inequality to the second term in this expression leads to

$$\int_0^t \int_D u,_{i\eta} u,_{i\eta}\,\mathrm{d}x\,\mathrm{d}\eta \leq \frac{1}{\epsilon_1^2}\int_0^t \int_D u,_i u,_i\,\mathrm{d}x\,\mathrm{d}\eta. \tag{3.6}$$

Integration of the identity

$$0 = \int_0^t \int_D u[u,_\eta + \Delta u - \epsilon_1 \Delta u,_\eta]\,\mathrm{d}x\,\mathrm{d}\eta$$

results in

$$\epsilon_1 \int_0^t \int_D u_{,i} u_{,i\eta} \, \mathrm{d}x \, \mathrm{d}\eta = -\frac{1}{2} \int_D u^2 \Big|_0^t \mathrm{d}x + \int_0^t \int_D u_{,i} u_{,i} \, \mathrm{d}x \, \mathrm{d}\eta. \tag{3.7}$$

If we now set

$$G = \int_0^t \int_D u_{,i} u_{,i} \, \mathrm{d}x \, \mathrm{d}\eta$$

then (3.7) yields the inequality

$$\epsilon_1 \frac{\mathrm{d}G}{\mathrm{d}t} \leq Q + 2G \tag{3.8}$$

where the data term $Q = \int_D u^2(0) \, \mathrm{d}x$. We integrate (3.8) to find that

$$G \leq \frac{1}{2} Q \mathrm{e}^{2t/\epsilon_1} \tag{3.9}$$

and in view of this bound and (3.6) we have

$$\int_0^t \int_D u_{,i\eta} u_{,i\eta} \, \mathrm{d}x \, \mathrm{d}\eta \leq \frac{Q}{2\epsilon_1^2} \mathrm{e}^{2t/\epsilon_1}. \tag{3.10}$$

Thus, we obtain from (3.4) the differential inequality

$$\Phi'(t) \leq \left(\frac{2+\alpha}{\epsilon_2} \right) \Phi + \frac{(\epsilon_2 - \epsilon_1)^2}{2\alpha\epsilon_1^2} Q \mathrm{e}^{2t/\epsilon_1} \tag{3.11}$$

which, upon integration gives

$$\Phi(t) \leq \frac{Q\epsilon_2(\epsilon_2 - \epsilon_1)^2}{2\alpha\epsilon_1[(2+\alpha)\epsilon_1 - 2\epsilon_2]} \left\{ \mathrm{e}^{((2+\alpha)/\epsilon_2)t} - \mathrm{e}^{2t/\epsilon_1} \right\} \tag{3.12}$$

where we have assumed that α is chosen so that $(2+\alpha)\epsilon_1 > 2\epsilon_2$. Inequality (3.12) is the desired continuous dependence result. We note that in the limit as α tends to $2(\epsilon_2 - \epsilon_1)/\epsilon_1$, (3.12) becomes

$$\Phi(t) \leq \frac{Q(\epsilon_2 - \epsilon_1)t\mathrm{e}^{2t/\epsilon_1}}{4\epsilon_1}. \tag{3.13}$$

4. HYPERBOLIC REGULARIZATION

To handle Problem 3, we again assume that u and v are two solutions corresponding to the parameters ϵ_1 and ϵ_2, respectively. Again, let us assume $\epsilon_2 > \epsilon_1$. We then introduce two new functions u^* and v^* defined as

$$u^* = \int_0^t u \, \mathrm{d}\eta, \quad v^* = \int_0^t v \, \mathrm{d}\eta. \tag{4.1}$$

These functions satisfy the differential equations

$$u^*_{,t} + \Delta u^* - \epsilon_1 u^*_{,tt} = u(0) - \epsilon_1 u_{,t}(0) \tag{4.2}$$

and

$$v^*_{,t} + \Delta v^* - \epsilon_2 v^*_{,tt} = v(0) - \epsilon_2 v_{,t}(0). \tag{4.3}$$

If we now set $w = v^* - u^*$, then w satisfies

$$w_{,t} + \Delta w - \epsilon_2 w_{,tt} = (\epsilon_1 - \epsilon_2)u_{,t}(0) + (\epsilon_2 - \epsilon_1)u^*_{,tt}. \tag{4.4}$$

Consider the functional

$$\Phi(t) = \int_D (w_{,i}w_{,i} + \epsilon_2 w_{,t}^2) \, \mathrm{d}x. \tag{4.5}$$

Upon differentiation, we find that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\int_D (w_{,it}w_{,i} + \epsilon_2 w_{,t}w_{,tt}) \, \mathrm{d}x$$

and substitution of eq. (4.4) leads to the expression

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\int_D w_{,t}^2\,\mathrm{d}x - 2(\epsilon_2 - \epsilon_1)\int_D w_{,t}u^*_{,tt}\mathrm{d}x - 2(\epsilon_1 - \epsilon_2)\int_D w_{,t}u_{,t}(0)\,\mathrm{d}x. \tag{4.6}$$

Application of the arithmetic–geometric mean inequality yields

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \leq \frac{(2+\alpha+\beta)}{\epsilon_2}\Phi + \frac{(\epsilon_2-\epsilon_1)^2}{\alpha}\int_D [u^*_{,tt}]^2\,\mathrm{d}x + \frac{(\epsilon_2-\epsilon_1)^2}{\beta}\int_D [u_{,t}(0)]^2\,\mathrm{d}x \tag{4.7}$$

where α and β are arbitrary positive constants. We now must establish a bound on $\int_D u^*_{,tt}\,\mathrm{d}x$. Since

$$\int_D [u^*_{,tt}]^2\,\mathrm{d}x = \int_D u_{,t}^2\,\mathrm{d}x,$$

let us consider the identity

$$0 = \int_0^t\int_D u_{,\eta}(u_{,\eta} + \Delta u - \epsilon_1 u_{,\eta\eta})\,\mathrm{d}x\,\mathrm{d}\eta \tag{4.8}$$

to help determine such a bound. Integration of (4.8) results in the inequality

$$\epsilon_1\int_D u_{,t}^2\,\mathrm{d}x \leq \epsilon_1\int_D u_{,t}^2(0)\,\mathrm{d}x + \int_D u_{,i}(0)u_{,i}(0)\,\mathrm{d}x + 2\int_0^t\int_D u_{,\eta}^2\,\mathrm{d}x\,\mathrm{d}\eta. \tag{4.9}$$

If we let

$$G = \int_0^t\int_D u_{,\eta}^2\,\mathrm{d}x\,\mathrm{d}\eta, \quad \hat{Q} = \epsilon_1\int_D u_{,t}^2(0)\,\mathrm{d}x + \int_D u_{,i}(0)u_{,i}(0)\,\mathrm{d}x,$$

then (4.9) is the differential inequality

$$\epsilon_1 G' \leq 2G + \hat{Q}. \tag{4.10}$$

Integration of (4.10) leads to the bound

$$G \leq \frac{1}{2}\hat{Q}[e^{2t/\epsilon_1} - 1]$$

from which it follows that

$$\int_D u_{,t}^2 \, dx \leq \frac{\hat{Q}}{\epsilon_1} e^{2t/\epsilon_1}. \tag{4.11}$$

We thus obtain from (4.7) the inequality

$$\frac{d\Phi}{dt} \leq \gamma\Phi + Ae^{2t/\epsilon_1} + B \tag{4.12}$$

where

$$\gamma = \frac{2+\alpha+\beta}{\epsilon_2}, \quad A = \frac{(\epsilon_2 - \epsilon_1)^2 \hat{Q}}{\alpha\epsilon_1}, \quad B = \frac{(\epsilon_2 - \epsilon_1)^2}{\beta} \int_D [u_{,t}(0)]^2 \, dx. \tag{4.13}$$

If we integrate (4.12) we find

$$\Phi \leq \frac{B}{\gamma}(e^{\gamma t} - 1) + \frac{A\epsilon_1}{2 - \gamma\epsilon_1}(e^{2t/\epsilon_1} - e^{\gamma t}). \tag{4.14}$$

Provided we choose α and β so that $2 - \gamma\epsilon_1 \neq 0$, we can surmise from (4.14) the continuous dependence inequality

$$\begin{aligned}\Phi \leq (\epsilon_2 - \epsilon_1)^2 \bigg\{ &\frac{\epsilon_2}{\beta(2+\alpha+\beta)} e^{(2+\alpha+\beta)t/\epsilon_2} \int_D [u_{,t}(0)]^2 \, dx \\ &+ \frac{\hat{Q}\epsilon_2}{\alpha[2\epsilon_2 - (2+\alpha+\beta)\epsilon_1]} [e^{2t/\epsilon_1} - e^{(2+\alpha+\beta)t/\epsilon_2}] \bigg\}.\end{aligned} \tag{4.15}$$

If we take $\alpha=\beta$ and then let $\beta \to (\epsilon_2-\epsilon_1)/\epsilon_1$, there results

$$\Phi \leq (\epsilon_2-\epsilon_1)\mathrm{e}^{2t/\epsilon_1}\left\{\frac{\epsilon_1^2}{2}\int_D [u,_t(0)]^2 \mathrm{d}x + \hat{Q}t\right\}.$$

5. QUASI-BOUNDARY-VALUE APPROXIMATION

Consider two solutions (u_1, ϵ_1) and (u_2, ϵ_2) to Problem 4 and set $w = u_1 - u_2$. Then w satisfies the problem

$$\begin{aligned} &\Delta w + w,_t = 0 && \text{in } D\times(0,T),\\ &w = 0 && \text{on } \partial D\times[0,T],\\ &w(x,0)+\epsilon_1 w(x,T) = -(\epsilon_1-\epsilon_2)u_2(x,T), && x\in D. \end{aligned} \tag{5.1}$$

Defining

$$\Phi(t) = \int_D w^2\,\mathrm{d}x \tag{5.2}$$

we proceed to show that Φ satisfies a first order differential inequality. Differentiation of (5.2) leads to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\int_D ww,_t\,\mathrm{d}x = 2\int_D w,_i w,_i\,\mathrm{d}x \tag{5.3}$$

upon substituting the differential equation and integrating by parts. It follows from Poincaré's inequality that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \geq 2\lambda\Phi \tag{5.4}$$

where λ is the first eigenvalue for the fixed membrane problem,

$$\begin{aligned} &\Delta v + \lambda v = 0 && \text{in } \Omega,\\ &v = 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.5}$$

Integrating the differential inequality (5.4), we obtain

$$\Phi(t) \leq \Phi(T)\mathrm{e}^{-2\lambda(T-t)}. \tag{5.6}$$

We next need to find a bound for $\Phi(T)$. We have

$$\Phi(T) = \int_D w^2(x,T)\,\mathrm{d}x = -\frac{1}{\epsilon_1}\int_D w(x,T)[w(x,0) + (\epsilon_1 - \epsilon_2)u_2(x,T)]\,\mathrm{d}x$$

upon using the initial condition in (5.1). Recalling the Lagrange identity for the backward heat equation [9] we see that

$$\Phi(T) = -\frac{1}{\epsilon_1}\int_D w^2(x,T/2)\,\mathrm{d}x - \frac{(\epsilon_1 - \epsilon_2)}{\epsilon}\int_D w(x,T)u_2(x,T)\,\mathrm{d}x. \tag{5.7}$$

Dropping the first term on the right side of (5.7) and using Schwarz's inequality, it follows that

$$\Phi(T) \leq \frac{(\epsilon_1 - \epsilon_2)^2}{\epsilon_1^2}\int_D u_2^2(x,T)\,\mathrm{d}x. \tag{5.8}$$

Now a similar argument leads to a bound for $\int_D u_2^2(x,T)\,\mathrm{d}x$ in terms of data, namely

$$\int_D u_2^2(x,T)\,\mathrm{d}x \leq \frac{1}{\epsilon_2^2}\int_D f^2\,\mathrm{d}x. \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.6), we arrive at the continuous dependence inequality

$$\Phi(t) \leq \frac{(\epsilon_1 - \epsilon_2)^2}{\epsilon_1^2\epsilon_2^2}\mathrm{e}^{-2\lambda(T-t)}\int_D f^2\,\mathrm{d}x. \tag{5.10}$$

Remark We note that our continuous dependence results for each of the four problems considered lose their validity when ϵ_1 or ϵ_2 tends to zero.

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