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On a Polynomial Inequality of P. Erdős and T. Grünwald

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Let f be a polynomial with only real zeros having -1, $+1$ as consecutive zeros. It was proved by P. Erdős and T. Grünwald that if $f(x) > 0$ on $(-1, 1)$, then the ratio of the area under the curve to the area of the tangential rectangle does not exceed $2/3$. The main result of our paper is a multidimensional version of this result. First, we replace the class of polynomials considered by Erdős and Grünwald by the *wider* class $\mathfrak{C}$ consisting of functions of the form $f(x) := (1-x^2)\psi(x)$, where $|\psi|$ is logarithmically concave on $(-1, 1)$, and show that their result holds for all functions in $\mathfrak{C}$. More generally, we show that if $f \in \mathfrak{C}$ and $\max_{-1 \le x \le 1}|f(x)| \le 1$, then for all $p > 0$, the integral $\int_{-1}^{1} |f(x)|^p \,\mathrm{d}x$ does not exceed $\int_{-1}^{1}(1 - x^2)^p \,\mathrm{d}x$. It is this result that is extended to higher dimensions. Our consideration of the class $\mathfrak{C}$ is crucial, since, unlike the narrower one of Erdős and Grünwald, its definition does not involve the distribution of zeros of its elements; besides, the notion of logarithmic concavity makes perfect sense for functions of several variables.

Keywords: Polynomials with real zeros; Logarithmically concave functions

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1. INTRODUCTION

For any continuous function $f : [-1, 1] \longmapsto \mathbb{C}$ and any $p \in (0, \infty)$, let

$$\|f\|_p := \left(\frac{1}{2}\int_{-1}^{1} |f(x)|^p \mathrm{d}x\right)^{1/p};$$

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besides, let

$$\|f\|_\infty := \max_{-1\le x\le 1} |f(x)|.$$

If $0<p<\infty$ and f is an arbitrary polynomial, then the trivial inequality

$$\|f\|_p \le \|f\|_\infty \tag{1}$$

is also the best possible one. In fact, it becomes an equality for any constant.

For $m\ge 2$ let $\wp_m$ be the class of all polynomials of degree at most m having only real zeros and $-1, 1$ as consecutive zeros. Furthermore, let $\wp := \bigcup_{m=2}^{\infty} \wp_m$. It was proved by Erdős and Grünwald [3] that if $f\in\wp$, then

$$\|f\|_1 \le \left(\frac{1}{2}\int_{-1}^{1} |1-x^2|\,\mathrm{d}x\right)\|f\|_\infty = \frac{2}{3}\|f\|_\infty, \tag{2}$$

where equality holds if and only if $f(x)\equiv c(1-x^2), c\in\mathbb{C}$. We extend this result by proving that under the same condition

$$\|f\|_p \le \left(\frac{1}{2}\int_{-1}^{1} |1-x^2|^p\,\mathrm{d}x\right)^{1/p}\|f\|_\infty,$$

not only for $p=1$ but for all $p>0$. In fact, we shall prove more.

One of the most important properties of a polynomial f with only real zeros is that $|f|$ is logarithmically concave between two consecutive zeros. Indeed, if $f(x) := c\prod_{\mu=1}^{m}(x-x_\mu)$, then

$$\left\{\frac{f'(x)}{f(x)}\right\}' = -\sum_{\mu=1}^{m}\frac{1}{(x-x_\mu)^2}$$

is negative at each point of the real line where it is defined. We extend the class $\wp$ by considering the class $\mathfrak{C}$ of all functions of the form $f(x) := (1-x^2)\psi(x)$, where $|\psi|$ is logarithmically concave on $(-1,1)$. Note that $\wp$ is a subset of $\mathfrak{C}$. The following result holds.

THEOREM 1 *Let f belong to $\mathfrak{C}$. If $f(x)$ is not a constant multiple of $1-x^2$, then*

$$\|f\|_p < \left(\frac{1}{2}\int_{-1}^{1} |1-x^2|^p \, dx\right)^{1/p} \|f\|_\infty, \quad 0 < p < \infty. \tag{3}$$

Note It may be added that the coefficient of $\|f\|_\infty$ in this inequality is equal to

$$4\left(\frac{\Gamma(p+1)\Gamma(p+1)}{\Gamma(2p+2)}\right)^{1/p}.$$

Some Remarks

1. Not only $\wp$ is a subset of $\mathfrak{C}$ but more generally, if

$$\psi(x) := \prod_{\mu=1}^{m} (1 - xu_\mu)^{\alpha_\mu},$$

where the numbers u_μ belong to $[-1, 1]$ and the numbers α_μ are all positive, then $f(x) := c\,(1-x^2)\psi(x)$ belongs to $\mathfrak{C}$ for all $c \neq 0$. Indeed, for all $x \in (-1, 1)$, we have

$$\left\{\frac{\psi'(x)}{\psi(x)}\right\}' = -\sum_{\mu=1}^{n} \frac{\alpha_\mu u_\mu^2}{(1-xu_\mu)^2} \leq 0.$$

For the relevance of such functions see [11].

2. Let

$$\psi(z) := \prod_{\mu=1}^{m} (z - x_\mu - \mathrm{i}y_\mu) \qquad \big(|y_\mu| \leq |x_\mu| - 1, \quad \mu = 1, \ldots, m\big).$$

Setting $\Psi(x) := |\psi(x)|$, we see that for $-1 \leq x \leq 1$,

$$\left\{\frac{\Psi'(x)}{\Psi(x)}\right\}' = -\sum_{\mu=1}^{m} \frac{(x-x_\mu)^2 - {y_\mu}^2}{|x - z_\mu|^2} \leq 0,$$

i.e. $|\psi|$ is logarithmically concave on $(-1, 1)$. Hence $f(z) := c(1-z^2)\psi(z)$ belongs to $\mathfrak{C}$ for all $c \neq 0$, and Theorem 1 applies to such polynomials as well. Note that these polynomials may have complex coefficients, except that they are required to have all their zeros in $E = E_1 \cup E_2$, where

$$E_1 := \{z = x + \mathrm{i}y:\ |y| \le x - 1,\ x \ge 1\},$$
$$E_2 := \{z = x + \mathrm{i}y:\ |y| \le 1 - x,\ x \le -1\}.$$

Here, it may be added that, in order to obtain a meaningful improvement on (1) it is not enough to assume that $f(-1) = f(1) = 0$ and that $f(z) \neq 0$ for $|z| < 1$. In fact, the supremum of $\|f\|_p/\|f\|_\infty$ over all such polynomials is 1 as the example $f(z) := 1 - z^{2m}$, $m \in \mathbb{N}$, shows.

3. An entire function is said to belong to the Laguerre–Pólya class, $\mathcal{L}\!-\!\mathcal{P}$ for short, if it is the local uniform limit in $\mathbb{C}$ of a sequence of polynomials with only real zeros [7, pp. 174–177, 10]. Let us denote by $(\mathcal{L}\!-\!\mathcal{P})_1$, the set of all functions in $\mathcal{L}\!-\!\mathcal{P}$ which have $x = -1$, $x = 1$ as consecutive zeros. A function f in $(\mathcal{L}\!-\!\mathcal{P})_1$ can be written as $f(z) = (1 - z^2)\psi(z)$, where

$$\psi(z) := c\mathrm{e}^{-az^2+bz} \prod_{\nu=1}^{\infty} (1 - t_\nu z)\mathrm{e}^{t_\nu z}, \quad (c \neq 0,\ a \ge 0,\ b \in \mathbb{R}),$$

and $-1 \le t_\nu \le 1$ for $\nu = 1, 2, 3, \ldots$ such that $\sum_{\nu=1}^{\infty} t_\nu^2 < \infty$. Note that $(\mathcal{L}\!-\!\mathcal{P})_1 \subseteq \mathfrak{C}$. So, Theorem 1 certainly applies to all functions in $(\mathcal{L}\!-\!\mathcal{P})_1$.

Extensions of Theorem 1 to Higher Dimensions

A priori, it is not clear what kind of functions of several variables correspond to polynomials in one variable having only real zeros. The observation that the modulus of such a polynomial is logarithmically concave between two consecutive zeros does, however, provide a clue. In view of Theorem 1, it seems natural to consider functions whose moduli are logarithmically concave on an appropriate region in $\mathbb{R}^n$, like the open n-dimensional cube $C_n := (-1, 1) \times \cdots \times (-1, 1)$ or the hypersphere

$$B_n := \left\{ (x_1, \ldots, x_n):\ \sum_{\nu=1}^{n} x_\nu^2 < 1 \right\}.$$

As an analogue of $\mathfrak{C}$, we introduce the class $\mathfrak{C}_n$ of functions in n variables $x_1, \ldots, x_n$ which are of the form

$$f(x_1, \ldots, x_n) := (1 - x_1^2) \cdots (1 - x_n^2)\psi(x_1, \ldots, x_n),$$

where $|\psi|$ is logarithmically concave on C_n. As a direct generalization of the problem considered and solved by Erdős and Grünwald, we ask the following question. How large can

$$\left(2^{-n} \int \cdots \int_{C_n} |f(x_1, \ldots, x_n)|^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_n\right)^{1/p}$$

be if f belongs to $\mathfrak{C}_n$ and $|f(\mathbf{x})| \leq 1$ for all $\mathbf{x} = (x_1, \ldots, x_n)$ in C_n. The next result contains the answer.

THEOREM 2 *Let C_n and $\mathfrak{C}_n$ be as above. If $f \in \mathfrak{C}_n$, then for all $p > 0$,*

$$\left(2^{-n} \int \cdots \int_{C_n} |f(x_1, \ldots, x_n)|^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_n\right)^{1/p}$$
$$< \left(2^{-1} \int_{-1}^{1} |1 - x^2|^p \, \mathrm{d}x\right)^{n/p} \sup_{\mathbf{x} \in C_n} |f(\mathbf{x})|, \tag{4}$$

unless $f(x_1, \ldots, x_n)$ is a constant multiple of $(1 - x_1^2) \cdots (1 - x_n^2)$.

In the case where $p = 1$ and $n = 2$, this theorem says that if $f \in \mathfrak{C}_2$ and $f(x, y) > 0$ for $-1 < x, y < 1$, then the ratio of the volume under the surface $z = f(x, y)$ and the volume of the tangential parallelopiped does not exceed 4/9. The analogy of this result with that of Erdős and Grünwald is obvious. Instead of assuming the square $\{(x, y) \in \mathbb{R}^2 : -1 < x, y < 1\}$ to be the domain of definition of the function f we may consider functions on other regions in $\mathbb{R}^2$. We shall only look at the class $\mathcal{F}_2$ consisting of functions of the form $f(x, y) := (1 - x^2 - y^2)\psi(x, y)$, where $|\psi|$ is logarithmically concave on $B_2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. The answer to the corresponding question in this case is contained in the following result.

THEOREM 3 *Let B_2 and $\mathcal{F}_2$ be as above. If $f \in \mathcal{F}_2$, then for all $p > 0$,*

$$\left(\frac{1}{\pi} \iint_{B_2} |f(x, y)|^p \, \mathrm{d}x\mathrm{d}y\right)^{1/p}$$
$$< \left(\frac{1}{\pi} \iint_{B_2} (1 - x^2 - y^2)^p \, \mathrm{d}x\mathrm{d}y\right)^{1/p} \sup_{x^2 + y^2 < 1} |f(x, y)|, \tag{5}$$

unless $f(x, y)$ is a constant multiple of $1 - x^2 - y^2$.

2. PROOF OF THEOREM 1

Without loss of generality we assume $f(x)$ to be *positive* on $(-1,1)$. Thus, $f(x):=(1-x^2)\psi(x)$, where $\log\psi(x)$ is concave on $(-1,1)$. Because of concavity, $\log\psi(x)$ is not only continuous on $(-1,1)$ but also bounded *above*. Consequently, f is continuous on $[-1,1]$. It follows that the supremum of $|f(x)|$ on $[-1,1]$ is finite and cannot be attained at -1 or $+1$. For simplicity, let $\|f\|_\infty=1$.

We claim that $\|f\|_\infty$ is attained at only one point of $(-1,1)$, which we denote by ξ. For this observe that ξ satisfies the equation

$$\log\frac{1}{1-x^2}=\log\psi(x).$$

The function $\log(1/(1-x^2))$ being strictly convex on $(-1,1)$, the line

$$L_\xi:\ y=\log\frac{1}{1-\xi^2}+\frac{2\xi}{1-\xi^2}(x-\xi)$$

meets the curve $y=\log(1/(1-x^2))$ if and only if $x=\xi$. For all other x it lies below the curve. Now it suffices to note that no point of the curve $y=\log\psi(x)$ lies above the line L_ξ. Suppose $(t,\log\psi(t))$ lies above L_ξ for some $t<\xi$. The line segment joining the point $(t,\log\psi(t))$ to the point $(\xi,\log\psi(\xi))$ intersects the curve $y=\log(1/(1-x^2))$ at a point $(s,\log(1/(1-s^2)))$. It is clear that $\log(1/(1-x^2))<\log\psi(x)$ for $s<x<\xi$. Hence $\|f\|_\infty$ cannot be 1, which is a contradiction. The case $t>\xi$ can be treated similarly.

We conclude that if f belongs to $\mathfrak{C}$, then for some ξ belonging to $(-1,1)$ we have

$$|f(x)|\le M_\xi(x):=\frac{1-x^2}{1-\xi^2}\exp\left(\frac{2\xi(x-\xi)}{1-\xi^2}\right)\quad(-1\le x\le 1).$$

Let us look for the supremum of the quantity

$$I_p(\xi):=\left(\frac{1}{2}\int_{-1}^{1}(M_\xi(x))^p\mathrm{d}x\right)^{1/p}\quad(0<p<\infty),$$

as ξ is allowed to vary in $(-1,1)$ and hope that it is attained for $\xi=0$.

If x is any given number in $(-1,1)$, then $(\mathrm{M}_\xi(x))^p$ tends to 0 as $\xi \downarrow -1$, i.e. ξ tends to -1 from the right or as $\xi \uparrow 1$, i.e. ξ tends to $+1$ from the left. Since $0 \le M_\xi(x) \le 1$ for all $x \in [-1,1]$, we may apply the dominated convergence theorem of Lebesgue to conclude that

$$\lim_{\xi\downarrow -1}\left(I_p(\xi)\right)^p = \lim_{\xi\downarrow -1}\frac{1}{2}\int_{-1}^{1}(M_\xi(x))^p\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1}\lim_{\xi\downarrow -1}(M_\xi(x))^p\,\mathrm{d}x = 0,$$
$$\lim_{\xi\uparrow +1}\left(I_p(\xi)\right)^p = \lim_{\xi\uparrow +1}\frac{1}{2}\int_{-1}^{1}(M_\xi(x))^p\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1}\lim_{\xi\uparrow +1}(M_\xi(x))^p\,\mathrm{d}x = 0.$$

So, the supremum of $\sigma_p(\xi) := 2(I_p(\xi))^p$ on $(-1,1)$ is attained at one or several points in $(-1,1)$. At any such point $\sigma_p'(\xi)$ must vanish. Elementary calculations give

$$\sigma_p'(\xi) = 2p\frac{1+\xi^2}{(1-\xi^2)^{p+2}}\exp\left(-\frac{2p\xi^2}{1-\xi^2}\right)\left\{\int_{-1}^{1}(1-x^2)^p\exp\left(\frac{2p\xi}{1-\xi^2}x\right)x\,\mathrm{d}x\right.$$
$$\left.-\xi\int_{-1}^{1}(1-x^2)^p\exp\left(\frac{2p\xi}{1-\xi^2}x\right)\mathrm{d}x\right\}.$$

Setting

$$\tau(\xi) := 2p\frac{1+\xi^2}{(1-\xi^2)^{p+2}}\exp\left(-\frac{2p\xi^2}{1-\xi^2}\right),$$
$$\omega(\xi,x) := (1-x^2)^p\exp\left(\frac{2p\xi}{1-\xi^2}x\right)$$

we obtain

$$\sigma_p'(\xi) = \tau(\xi)\left\{\int_{-1}^{1}\omega(\xi,x)\times x\,\mathrm{d}x - \xi\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x\right\}.$$

Let us check the sign of $\sigma_p''(\xi)$ at a point ξ where $\sigma_p'(\xi) = 0$, i.e. at a point ξ where

$$\int_{-1}^{1}\omega(\xi,x)\times x\,\mathrm{d}x - \xi\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x = 0. \tag{6}$$

At any point ξ satisfying (6) we have $\sigma_p''(\xi)/\tau(\xi) = \Omega(\xi)$, where

$$\Omega(\xi) := \frac{2p(1+\xi^2)}{(1-\xi^2)^2}\int_{-1}^{1}\omega(\xi,x)\times x^2\,\mathrm{d}x - \int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x \\ - \frac{2p\xi(1+\xi^2)}{(1-\xi^2)^2}\int_{-1}^{1}\omega(\xi,x)\times x\,\mathrm{d}x.$$

Since $\tau(\xi) > 0$, the sign of $\sigma_p''(\xi)$ at a critical point of σ_p agrees with that of $\Omega(\xi)$. Now, we note that

$$\int_{-1}^{1}\omega(\xi,x)\times x^2\,\mathrm{d}x = \int_{-1}^{1}\left\{(1-x^2)^p - (1-x^2)^{p+1}\right\}\exp\left(\frac{2p\xi}{1-\xi^2}x\right)\mathrm{d}x \\ = \int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x - \frac{p+1}{p}\frac{1-\xi^2}{\xi}\int_{-1}^{1}\omega(\xi,x)\times x\,\mathrm{d}x.$$

Hence at a critical point of σ_p, i.e. at a point ξ satisfying (6), we have

$$\Omega(\xi) := 2p\frac{1+\xi^2}{(1-\xi^2)^2}\left(\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x - \frac{p+1}{p}\frac{1-\xi^2}{\xi}\int_{-1}^{1}\omega(\xi,x)\times x\mathrm{d}x\right) \\ - \int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x - 2p\frac{\xi(1+\xi^2)}{(1-\xi^2)^2}\int_{-1}^{1}\omega(\xi,x)\times x\,\mathrm{d}x \\ = 2p\frac{1+\xi^2}{(1-\xi^2)^2}\left(\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x\right. \\ \left.-\frac{p+1}{p}(1-\xi^2)\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x\right) - \int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x \\ - 2p\frac{\xi^2(1+\xi^2)}{(1-\xi^2)^2}\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x \\ = -\frac{3+\xi^2}{1-\xi^2}\int_{-1}^{1}\omega(\xi,x)\times 1\,\mathrm{d}x < 0.$$

This means that every critical point of σ_p is a point of local maximum. Since two consecutive local maxima must be separated by a point of local minimum, we conclude that σ_p has only one local maximum in $(-1,1)$. It is easily seen that σ_p is an even function of ξ and so its unique local maximum in $(-1,1)$ must occur for $\xi = 0$. Hence, for all $p \in (0,\infty)$

and $\xi \in (-1,1)\backslash\{0\}$, we have

$$\sigma_p(\xi) < \sigma_p(0) = 2^{2p+1}B(p+1, p+1),$$

where $B(.,.)$ is the *beta function*. From this (3) follows.

3. THE LIMITING CASE $p=0$ OF (3)

It is known (see for example [6, §6.8]) that if S belongs to $L^p(-1,1)$ for some $p>0$, then $((1/2)\int_{-1}^{1}|S(x)|^p\mathrm{d}x)^{1/p}$ tends to the limit

$$\exp\left(\frac{1}{2}\int_{-1}^{1}\log|S(x)|\,\mathrm{d}x\right)$$

as $p\to 0$. This is exactly the value given to the functional $\|S\|_p$ when $p=0$. From Theorem 1 it follows that if f belongs to $\mathfrak{C}$, then

$$\|f\|_0 \le \exp\left(\frac{1}{2}\int_{-1}^{1}\log|1-x^2|\,\mathrm{d}x\right)\|f\|_\infty.$$

Although the inequality is sharp the argument we have just used to obtain it does not allow us to identify the extremal functions. However, as an addendum to Theorem 1 we prove the following result.

PROPOSITION 1 *Let f belong to $\mathfrak{C}$. If $f(x)$ is not a constant multiple of $1-x^2$, then*

$$\|f\|_0 < \exp\left(\frac{1}{2}\int_{-1}^{1}\log|1-x^2|\,\mathrm{d}x\right)\|f\|_\infty = \left(\frac{2}{e}\right)^2\|f\|_\infty. \tag{7}$$

Proof We have to determine $\sup\{I_0(\xi): -1<\xi<1\}$, where

$$I_0(\xi) := \exp\left(\frac{1}{2}\int_{-1}^{1}\log(1-x^2)\,\mathrm{d}x\right)\left\{\frac{1}{(1-\xi^2)\exp[2\xi^2/(1-\xi^2)]}\right\}^{1/2}.$$

Since

$$(1-\xi^2)\exp\left(\frac{2\xi^2}{1-\xi^2}\right) \ge (1-\xi^2)\left\{1+\frac{2\xi^2}{1-\xi^2}+\frac{2\xi^4}{(1-\xi^2)^2}\right\}$$

for $-1<\xi<1$ we conclude that $I_0(\xi)$ tends to zero as $\xi \downarrow -1$ or $\xi \uparrow +1$. Hence the desired supremum of $I_0(\xi)$ is attained at one or several points in $(-1,1)$. Any such point must be a critical point of the function $(I_0(\xi))^2$. It is easily checked that $\xi=0$ is the only critical point of I_0^2 in $(-1,1)$. Hence for $0<|\xi|<1$

$$I_0(\xi) < I_0(0) = \exp\left(\frac{1}{2}\int_{-1}^{1} \log|1-x^2|\,\mathrm{d}x\right) = \left(\frac{2}{e}\right)^2,$$

which proves (7).

4. PROOF OF THEOREM 2 AND A REMARK

Without loss of generality we assume that

$$f(\mathbf{x}) := (1-x_1^2)\cdots(1-x_n^2)\psi(\mathbf{x}) \quad (\mathbf{x} := (x_1,\ldots,x_n)),$$

where $\ln|\psi(\mathbf{x})|$ is not only concave but $\psi(\mathbf{x})>0$ for all $\mathbf{x}\in C_n$ and that $\max_{\mathbf{x}\in C_n} f(\mathbf{x}) = 1$. Take an arbitrary f satisfying these conditions and let $(\xi_1,\ldots,\xi_n)$ be a point in C_n such that $f(\xi_1,\ldots,\xi_n)=1$. This means that

$$\ln\psi(\mathbf{x}) \le -\sum_{\nu=1}^{n} \ln(1-x_\nu^2) \tag{8}$$

with equality for $\mathbf{x}=(\xi_1,\ldots,\xi_n)$.

Using well known criteria [5, p. 58], it can be seen that the function $-\sum_{\nu=1}^{n} \ln(1-x_n^2)$ is (strictly) convex on C_n. Hence the set

$$K_1 := \left\{(\mathbf{x},x_{n+1}) \in \mathbb{R}^{n+1}:\ \mathbf{x}\in C_n,\ x_{n+1} \ge -\sum_{\nu=1}^{n}\ln(1-x_n^2)\right\}$$

is convex. It is, clearly, supported by the hyperplane H defined by

$$x_{n+1} = -\sum_{\nu=1}^{n}\ln(1-\xi_\nu^2) + \sum_{\nu=1}^{n}\frac{2\xi_\nu}{1-\xi_\nu^2}(x_\nu-\xi_\nu),$$

and because of the strict convexity of K_1, the two meet only in the point $(\xi_1,\ldots,\xi_n)$. The set

$$K_2 := \left\{(\mathbf{x},x_{n+1}) \in \mathbb{R}^{n+1}:\ x_{n+1} \le \ln\psi(\mathbf{x})\right\}$$

is also convex and must be *supported by H*, otherwise (8) would be contradicted. It follows that for all $\mathbf{x} \in C_n$,

$$\ln \psi(\mathbf{x}) \leq -\sum_{\nu=1}^{n} \ln(1-\xi_\nu^2) + \sum_{\nu=1}^{n} \frac{2\xi_\nu}{1-\xi_\nu^2}(x_\nu - \xi_\nu),$$

and so

$$f(\mathbf{x}) \leq \frac{1-x_1^2}{1-\xi_1^2} \cdots \frac{1-x_n^2}{1-\xi_n^2} \exp\left\{\sum_{\nu=1}^{n} \frac{2\xi_\nu}{1-\xi_\nu^2}(x_\nu - \xi_\nu)\right\}.$$

Hence for all $p > 0$,

$$\begin{aligned}&\int \cdots \int_{C_n} |f(x_1, \ldots, x_n)|^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &\quad \leq \prod_{\nu=1}^{n} \int_{-1}^{1} \left(\frac{1-x_\nu^2}{1-\xi_\nu^2}\right)^p \exp\left(\frac{2p\xi_\nu}{1-\xi_\nu^2}(x_\nu - \xi_\nu)\right) \mathrm{d}x_\nu.\end{aligned}$$

From the proof of Theorem 1, we see that

$$\int_{-1}^{1} \left(\frac{1-x_\nu^2}{1-\xi_\nu^2}\right)^p \exp\left(\frac{2p\xi_\nu}{1-\xi_\nu^2}(x_\nu - \xi_\nu)\right) \mathrm{d}x_\nu \leq \int_{-1}^{1} (1-x_\nu^2)^p \, \mathrm{d}x_\nu,$$

where equality holds only if $\xi_\nu = 0$. Hence

$$\int \cdots \int_{C_n} |f(x_1, \ldots, x_n)|^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \leq \left(\int_{-1}^{1} (1-x^2)^p \, \mathrm{d}x\right)^n, \tag{9}$$

with equality only if $(\xi_1, \ldots, \xi_n) = \mathbf{0}$. However, if $f(\mathbf{x})$ is not identically equal to $(1-x_1^2) \cdots (1-x_n^2)$ and $(\xi_1, \ldots \xi_n) = \mathbf{0}$, then $0 < \psi(\mathbf{x}) < 1$ except for $\mathbf{x} = \mathbf{0}$, that is, $|f(\mathbf{x})| < (1-x_1^2) \cdots (1-x_n^2)$ for $\mathbf{x}$ different from $\mathbf{0}$. Hence, equality holds in (9), only if $f(\mathbf{x}) := (1-x_1^2) \cdots (1-x_n^2)$. This completes the proof of Theorem 2.

Remark 4 More generally, we may consider functions of the form

$$f(x_1, \ldots, x_n) := (a_1^2 - x_1^2) \cdots (a_n^2 - x_n^2) \psi(x_1, \ldots, x_n),$$

where $|\psi|$ is logarithmically concave on the parallelopiped

$$P_n := (-a_1, a_1) \times \cdots \times (-a_n, a_n).$$

It can be easily proved, as above, that for all $p > 0$, the ratio

$$\left(\frac{2^{-n}}{a_1 \cdots a_n} \int \cdots \int_{P_n} |f(x_1, \ldots, x_n)|^p \mathrm{d}x_1 \cdots \mathrm{d}x_n\right)^{1/p} \Big/ \sup_{\mathbf{x} \in P_n} |f(\mathbf{x})|$$

is maximized by the function $(a_1^2 - x_1^2) \cdots (a_n^2 - x_n^2)$.

5. PROOF OF THEOREM 3

Without loss of generality we assume that

$$f(x, y) := (1 - x^2 - y^2)\psi(x, y),$$

where $\ln|\psi(x, y)|$ is not only concave but $\psi(x, y) > 0$ for $(x, y) \in B_2$ and that $\sup\{f(x, y)\colon (x, y) \in B_2\} = 1$. Let (ξ, η) be a point in B_2 such that $f(\xi, \eta) = 1$. As in the proof of Theorem 2, we can show that in the present case

$$|\psi(x, y)| \leq -\ln(1 - \xi^2 - \eta^2) + \frac{2\xi}{1 - \xi^2 - \eta^2}(x - \xi) + \frac{2\eta}{1 - \xi^2 - \eta^2}(y - \eta)$$

for $(x, y) \in B_2$. Hence, for all $p > 0$, we have

$$\iint_{B_2} |f(x, y)|^p \, \mathrm{d}x \, \mathrm{d}y \leq \Phi_p(\xi, \eta),$$

where

$$\Phi_p(\xi, \eta) = \iint_{B_2} \left(\frac{1 - x^2 - y^2}{1 - \xi^2 - \eta^2}\right)^p \exp\left(2p \frac{\xi(x - \xi) + \eta(y - \eta)}{1 - \xi^2 - \eta^2}\right) \mathrm{d}x \, \mathrm{d}y.$$

Here it is more convenient to use polar coordinates. Writing

$$x = \rho \cos\theta, \quad y = \rho \sin\theta, \quad \xi = r\cos\phi, \quad \eta = r\sin\phi,$$

we see that

$$\begin{aligned}
&\Phi_p(r\cos\phi, r\sin\phi)\exp\left(\frac{2pr^2}{1-r^2}\right)\\
&= \int_0^{2\pi}\int_0^1 \left(\frac{1-\rho^2}{1-r^2}\right)^p \exp\left(\frac{2pr}{1-r^2}\rho\cos(\theta-\phi)\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\\
&= \int_0^{2\pi}\int_0^1 \left(\frac{1-\rho^2}{1-r^2}\right)^p \exp\left(\frac{2pr}{1-r^2}\rho\cos\theta\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta.
\end{aligned}$$

The last integral is, obviously, independent of ϕ. So, we denote it by $F_p(r)$ which will be our alternative notation for $\Phi_p(r\cos\phi, r\sin\phi)$.

In order to determine the supremum of F_p we differentiate it with respect to r. We obtain

$$F_p{}'(r) = 2p\frac{1+r^2}{(1-r^2)^{p+2}}\exp\left(-\frac{2pr^2}{1-r^2}\right)\left(V_p(r) - rU_p(r)\right),$$

where

$$\begin{aligned}
U_p(r) &:= \int_0^{2\pi}\int_0^1 \rho(1-\rho^2)^p\exp\left(\frac{2pr}{1-r^2}\rho\cos\theta\right)\,\mathrm{d}\rho\,\mathrm{d}\theta,\\
V_p(r) &:= \int_0^{2\pi}\int_0^1 \rho^2(1-\rho^2)^p\exp\left(\frac{2pr}{1-r^2}\rho\cos\theta\right)\cos\theta\,\mathrm{d}\rho\,\mathrm{d}\theta.
\end{aligned}$$

A simple calculation shows that

$$V_p(r) - rU_p(r) = -\frac{2\pi}{(p+1)(p+2)}r + O(r^3) \quad (r \to 0).$$

There exists, therefore, a positive number r_0 such that $F_p(r)$ is strictly decreasing on $(0, r_0)$. Because of the continuity of F_p, it follows that $F_p(r) < F_p(0)$ for $0 < r < r_0$. We claim that r_0 may be taken to be 1. This will follow if we show that F_p cannot have a local minimum in $(0, 1)$. So, we may simply check the sign of $F_p''(r)$ at the points in $(0, 1)$ where $F_p'(r) = 0$. We shall see that it can only be negative. Hence, F_p cannot have a local minimum in $(0, 1)$; and, in fact, not a local maximum either.

Assume that r is a critical point of F_p in $(0, 1)$. It is easily seen that the sign of $F_p''(r)$ is the same as that of

$$G_p(r) := 2p\frac{1+r^2}{(1-r^2)^2}W_p(r) - U_p(r) - 2p\frac{r+r^3}{(1-r^2)^2}V_p(r),$$

where $U_p(r)$, $V_p(r)$ are as above and

$$W_p(r) := \int_0^1\int_0^{2\pi}\rho^3(1-\rho^2)^p\exp\left(\frac{2pr}{1-r^2}\rho\cos\theta\right)\cos^2\theta\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

Note that $F_p'(r) = 0$ if and only if $V_p(r) = rU_p(r)$. Hence, in order to determine the sign of $G_p(r)$ at a critical point of F_p in $(0, 1)$, it is desirable to find an expression for $W_p(r)$ in terms of $U_p(r)$ and $V_p(r)$. For this we write

$$W_p(r) = I_{p,1}(r) - I_{p,2}(r),$$

where

$$I_{p,1}(r) := \int_0^{2\pi}\int_0^1\rho(1-\rho^2)^p\exp\left(\frac{2pr}{1-r^2}\rho\cos\theta\right)\cos^2\theta\,\mathrm{d}\rho\,\mathrm{d}\theta$$

and

$$I_{p,2}(r) := \int_0^{2\pi}\int_0^1\rho(1-\rho^2)^{p+1}\exp\left(\frac{2pr}{1-r^2}\rho\cos\theta\right)\cos^2\theta\,\mathrm{d}\rho\mathrm{d}\theta.$$

In $I_{p,1}(r)$, we replace $\cos^2\theta$ by $1-\sin^2\theta$, and integrate by parts with respect to θ to obtain

$$\begin{aligned}I_{p,1}(r) &= U_p(r) + \int_0^1(1-\rho^2)^p\int_0^{2\pi}\sin\theta\exp\left(\frac{2pr\rho}{1-r^2}\cos\theta\right)(-\rho\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\rho\\ &= U_p(r) - \frac{1-r^2}{2pr}\int_0^{2\pi}\int_0^1(1-\rho^2)^p\exp\left(\frac{2pr\rho}{1-r^2}\cos\theta\right)\cos\theta\,\mathrm{d}\rho\,\mathrm{d}\theta.\end{aligned}$$

Since $(1-\rho^2)^p = \rho^2(1-\rho^2)^p + (1-\rho^2)^{p+1}$, we see that

$$I_{p,1}(r) = U_p(r) - \frac{1-r^2}{2pr} V_p(r)$$
$$- \frac{1-r^2}{2pr} \int_0^{2\pi} \int_0^1 (1-\rho^2)^{p+1} \exp\left(\frac{2pr\rho}{1-r^2}\cos\theta\right) \cos\theta \,\mathrm{d}\rho\,\mathrm{d}\theta.$$

Now we look for a similar representation for $I_{p,2}(r)$. Integrating by parts with respect to ρ we obtain

$$I_{p,2}(r) = -\frac{1-r^2}{2pr} \int_0^{2\pi} \int_0^1 (1-\rho^2)^{p+1} \exp\left(\frac{2pr\rho}{1-r^2}\cos\theta\right) \cos\theta \,\mathrm{d}\rho\,\mathrm{d}\theta$$
$$+ \frac{(p+1)(1-r^2)}{pr} V_p(r).$$

Hence

$$W_p(r) := I_{p,1}(r) - I_{p,2}(r) = U_p(r) - \frac{(2p+3)(1-r^2)}{2pr} V_p(r),$$

which in turn gives us

$$G_p(r) = \left(\frac{2p(1+r^2)}{(1-r^2)^2} - 1\right) U_p(r) - \frac{2p(1+r^2)}{(1-r^2)^2}\left(\frac{(2p+3)(1-r^2)}{2pr} + r\right) V_p(r).$$

It follows that if r is a critical point of F_p in (0,1), i.e. if $V_p(r) = rU_p(r)$, then

$$G_p(r) = -2\frac{2+r^2}{1-r^2} U_p(r).$$

Since $U_p(r)$ is positive, we conclude that $G_p(r)$ is negative at any point in $(0, 1)$ where F_p vanishes. Hence, so is $F_p''(r)$.

6. CONNECTION WITH LINEAL FUNCTIONS

A polynomial of n variables $z_1, \dots, z_n$, which can be expressed as a product of the form $c\prod_{\mu=1}^{m}(1 + \sum_{\nu=1}^{n} \alpha_{\mu\nu} z_\nu)$ is called *lineal* [9]. It is said to be *really lineal* if c and $\alpha_{\mu\nu}$ $(1 \le \mu \le m,\ 1 \le \nu \le n)$ are all real.

The special determinant called *circulant* [4, p. 23], with the variables $z_1, \ldots, z_n$ as the elements of its first row is a lineal polynomial. In the case $n = 1$, every polynomial is lineal, but not so if $n \geq 2$. By definition, a transcendental entire function of n variables is (*really*) lineal if it is the local uniform limit in $\mathbb{C}^n$ of a convergent sequence of (*really*) lineal polynomials. The class of really lineal entire functions of one variable is the same as the Laguerre–Pólya class $\mathcal{L}$–$\mathcal{P}$ mentioned in the Introduction. The study of really lineal entire functions of several variables was started by Motzkin and Schoenberg, who found the following characterization [9, Theorem 2] for such functions.

THEOREM A* *An entire function is really lineal if and only if it admits a representation of the form*

$$f(z_1, \ldots, z_n) = \exp\left(-\sum_{\mu,\nu=1}^{n} \gamma_{\mu\nu} z_\mu z_\nu + \sum_{\nu=1}^{n} \delta_\nu z_\nu\right) \prod_{\mu=1}^{m} \left(\sum_{\nu=1}^{n} c_{\mu\nu} z_\nu\right)$$
$$\times \prod_{k=1}^{\infty} \left(1 + \sum_{\nu=1}^{n} \delta_{k\nu} z_\nu\right) e^{\left(-\sum_{\nu=1}^{n} \delta_{k\nu} z_\nu\right)},$$

where $\gamma_{\mu\nu}$, δ_ν, $c_{\mu\nu}$, $\delta_{k\nu}$ are real, the series $\sum_{k=1}^{\infty} \sum_{\nu=1}^{n} \delta_{k\nu}^2$ converges, while the quadratic form $\sum \gamma_{\mu\nu} z_\mu z_\nu$ is positive semi-definite.

For further developments see [2] and [8, Chapter 4] along with some of the references given there; also [8, p. 203] for a letter of I.J. Schoenberg to friends.

Let $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\sum_{\nu=1}^{n} |\alpha_\nu| \leq 1$ and consider the function $g(\mathbf{x}) := \ln(1 - \sum_{\nu=1}^{n} \alpha_\nu x_\nu)$ on the cube C_n defined in the Introduction. Then

$$g_{jk}(\mathbf{x}) := \frac{\partial^2 g}{\partial x_j \partial x_k}(\mathbf{x})$$
$$= \begin{cases} -\alpha_j{}^2 \Big/ \left(1 - \sum_{\nu=1}^{n} \alpha_\nu x_\nu\right)^2 & \text{if } j = k, \\ -\alpha_j \alpha_k \Big/ \left(1 - \sum_{\nu=1}^{n} \alpha_\nu x_\nu\right)^2 & \text{if } j \neq k. \end{cases}$$

Hence, the principal minor determinants of the matrix $(-g_{jk}(\mathbf{x}))$ are all non-negative. It follows (see [1, pp. 140, 147] or [5, p. 58]) that the function $1 - \sum_{\nu=1}^{n} \alpha_\nu x_\nu$ is logarithmically concave. The same can therefore be said about the real lineal polynomial

$$\psi(\mathbf{x}) := \prod_{\mu=1}^{m}\left(1 - \sum_{\nu=1}^{n} \alpha_{\mu\nu} x_\nu\right) \quad \left(\sum_{\nu=1}^{n} |\alpha_{\mu\nu}| \leq 1, \mu = 1, \ldots, n\right). \tag{10}$$

Thus, Theorem 2 holds for functions of the form

$$f(\mathbf{x}) := c(1 - x_1^2) \cdots (1 - x_n^2)\psi(\mathbf{x})$$

with ψ as in (10). It is clear that more general functions of the form

$$f(\mathbf{x}) := c(1 - x_1^2) \cdots (1 - x_n^2) \prod_{\mu=1}^{m}\left(1 - \sum_{\nu=1}^{n} \alpha_{\mu\nu} x_\nu\right)^{\beta_\mu},$$

where the numbers $\alpha_{\mu\nu}$ are as above and $\beta_\mu \geq 0$ for $\mu = 1, \ldots, m$, are also admissible.

From Theorem A* it follows that Theorem 2 applies to all functions of the form $f(\mathbf{x}) := c(1 - x_1^2) \cdots (1 - x_n^2)\psi(\mathbf{x})$, where ψ is a really lineal entire function different from zero on C_n. An analogous remark can be made about Theorem 3.

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