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An Inequality for Polynomials with Elliptic Majorant

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Let $\bar{T}(x) := T_n(\xi x)$ be the transformed Chebyshev polynomial of the first kind, where $\xi = \cos(\pi/2n)$. We show here that $\bar{T}_n$ has the greatest uniform norm in $[-1, 1]$ of its k-th derivative ($k \geq 2$) among all algebraic polynomials f of degree not exceeding n, which vanish at ± 1 and satisfy the inequality $|f(x)| \leq \sqrt{1-\xi^2 x^2}$ at the points $\{\xi^{-1}\cos((2j-1)\pi/2n-2)\}_{j=1}^{n-1}$.

Keywords: Markov inequality; Duffin and Schaeffer type inequality; Chebyshev polynomials

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1. INTRODUCTION AND STATEMENT OF RESULT

Let π_n be the set of all algebraic polynomials of degree at most n with real coefficients. Throughout, $\|\cdot\|$ designates the uniform norm in $[-1, 1]$, $\|f\| := \sup_{x\in[-1,1]}|f(x)|$. We shall use the customary notation $T_n(x) = \cos n \arccos x$ for the Chebyshev polynomial of the first kind, and we denote by ξ its largest zero, i.e.,

$$\xi = \cos\frac{\pi}{2n}.$$

The transformed Chebyshev polynomial of the first kind

$$\bar{T}_n(x) := T_n(\xi x) \tag{1.1}$$

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is known as being the extremal polynomial in the classical Schur inequality [12], which is the analogue of the Markov inequality for polynomials satisfying zero boundary conditions. Precisely, Schur's inequality asserts that $\bar{T}_n$ has the greatest uniform norm of its first derivative on $[-1, 1]$ among all $f \in \pi_n$, satisfying $f(\pm 1) = 0$ and $\|f\| \le 1$.

In a recent paper Milev and Nikolov [5] have shown that the extremality of $\bar{T}_n$ persists for higher order derivatives as well. Precisely, Milev and Nikolov proved the following extension of Schur's inequality.

THEOREM A *Let $f \in \pi_n$ satisfy $f(\pm 1) = 0$ and*

$$\left| f\left(\xi^{-1} \cos \frac{j\pi}{n} \right) \right| \le 1 \quad (j = 1, \ldots, n-1). \tag{1.2}$$

Then

$$\|f^{(k)}\| \le \|\bar{T}_n^{(k)}\| \tag{1.3}$$

for $k = 2, \ldots, n$. Moreover, equality in (1.3) *is possible if and only if $f = \pm \bar{T}_n$.*

This result is in the spirit of the famous refinement of Markov's inequality, found by Duffin and Schaeffer [3]. Namely, for the validity of (1.3) it suffices to assume that $|f| \le \bar{T}_n$ not on the whole interval $[-1, 1]$, but only at the extremal points of $\bar{T}_n$. For some related results the reader may consult [1,2,13].

In [6] the author developed a technique for the derivation of inequalities of Duffin and Schaeffer type. Unlike the classical case the "checking" points are not necessarily assumed to be the extremal points of the majorant (i.e., the extremal polynomial). This technique is applied here to exhibit another extremal property of $\bar{T}_n$. Precisely, we prove the following theorem.

THEOREM 1.1 *Let $f \in \pi_n$ $(n \ge 2)$ satisfy $f(-1) = f(1) = 0$ and let*

$$|f(x)| \le \sqrt{1 - \xi^2 x^2} \quad \textit{for } x = \xi^{-1} \cos \frac{(2j-1)\pi}{2n-2} \quad (j = 1, \ldots, n-1). \tag{1.4}$$

Then

$$\|f^{(k)}\| \le \|\bar{T}_n^{(k)}\| \tag{1.5}$$

for $k=2,\ldots,n$. *Moreover,* equality in (1.5) *is possible if and only if* $f=\pm\bar{T}_n$.

The inequality (1.4) is fulfilled e.g., if the graph of $y=f(x)$ in $[-1,1]$ is contained into the interior of the ellipse $\xi^2x^2+y^2=1$ (but, evidently, this is not necessary for (1.4) to hold). Thus, Theorem 1.1 may be regarded as an inequality of Duffin-Schaeffer-Schur type for polynomials with elliptic majorant. Without any claim for completeness, we mention that inequalities for polynomials with curved majorants are obtained in [7,9,10].

2. THE PROOF

There is nothing to prove when $n=2$, since in this case $f=c\bar{T}_n$ and $|c|\le 1$ in view of (1.4). Therefore we assume in what follows that $n\ge 3$.

Instead of dealing with $\bar{T}_n$ on the interval $[-1,1]$ we prefer to study the customary Chebyshev polynomial of the first kind T_n on the interval $[-\xi,\xi]$. Let $x_0<x_1<\cdots<x_n$ be the zeros of

$$\omega(x) := (x^2-\xi^2)T_{n-1}(x).$$

Over the new basic interval $[-\xi,\xi]$ the polynomials $p\in\pi_n$ under consideration will be assumed to satisfy

$$p(x_0)=p(x_n)=0 \tag{2.1}$$

and

$$|p(x_j)|\le\sqrt{1-x_j^2}\quad (j=1,\ldots,n-1). \tag{2.2}$$

Notice that (2.1) and (2.2) are fulfilled for $p=\pm T_n$, the second one with equality sign.

The proof of Theorem 1.1 relies on the pointwise inequality given by the following theorem.

THEOREM 2.1 *Let* $p\in\pi_n$ $(n\ge 3)$ *satisfy the conditions* (2.1) *and* (2.2). *Then for each* $k\in\{1,\ldots,n\}$ *and for every* $x\in[-\xi,\xi]$

$$|p^{(k)}(x)|\le\max\{|T_n^{(k)}(x)|,|Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) := \frac{n(n-2)}{(n-1)k\xi}\left[\left(x^2 - \frac{n+k-2}{n-2}\xi^2\right)T_{n-1}^{(k)}(x) + kxT_{n-1}^{(k-1)}(x)\right]. \tag{2.3}$$

Proof Let $\omega_\nu(x) := \omega(x)/(x-x_\nu)$, $\nu = 0,\dots,n$. For any $p \in \pi_n$ the Lagrange interpolation formula yields

$$p^{(k)}(x) = \sum_{\nu=0}^{n} \frac{p(x_\nu)}{\omega_\nu(x_\nu)}\omega_\nu^{(k)}(x). \tag{2.4}$$

In particular, for polynomials obeying the restrictions (2.1) and (2.2) formula (2.4) yields

$$|p^{(k)}(x)| \le \sum_{\nu=1}^{n-1} \left|\frac{\omega_\nu^{(k)}(x)}{\omega_\nu(x_\nu)}\right| \sqrt{1-x_\nu^2}. \tag{2.5}$$

An elegant result due to V. Markov asserts that, if two polynomials have only real and simple zeros and they interlace, then the interlacing property is inherited by the zeros of their derivatives (for a proof, see e.g., [11], Lemma 2.7.1). As is mentioned by Bojanov ([1], p. 39), for polynomials of the same degree this result indicates that the zeros of the derivative of a polynomial depend monotonically on the zeros of the polynomial. We apply this observation to the polynomials ω_ν. Since for $i > j$ the zeros of $\omega_i(x)$ are less than or equal to the corresponding zeros of $\omega_j(x)$, the same relation remains valid for the zeros of $\omega_i^{(k)}$ and $\omega_j^{(k)}$. Therefore, the j-th zeros of the polynomials $\{\omega_i^{(k)}\}_{i=1}^{n-1}$ are located between the j-th zero of $\omega_n^{(k)}$ and the j-th zero of $\omega_0^{(k)}$. Let $\{\beta_i\}_{j=1}^{n-k}$ and $\{\alpha_i\}_{i=2}^{n-k+1}$ be the zeros of $\omega_n^{(k)}$ and $\omega_0^{(k)}$, respectively, arranged in increasing order. Set $\alpha_1 := -\xi$, $\beta_{n-k+1} := \xi$, then the above reasoning implies that if $x \in [\alpha_j, \beta_j]$, then

$$\text{sign}\{\omega_\nu^{(k)}(x)\} \text{ is the same for all } \nu \in \{1,\dots,n-1\}. \tag{2.6}$$

Further, we observe that the zeros of ω and T_n interlace, and

$$T_n(x_\nu) = (-1)^{n-\nu}\sqrt{1-x_\nu^2} = \text{sign}\{\omega_\nu(x_\nu)\}\sqrt{1-x_\nu^2}, \quad \nu = 1,\dots,n-1.$$

Therefore, putting $p = T_n$ in (2.4), for $x \in [\alpha_j, \beta_j]$ ($j \in \{1, \ldots, n-k+1\}$) we obtain

$$|T_n^{(k)}(x)| = \left| \sum_{\nu=1}^{n-1} \frac{\omega_\nu^{(k)}(x)}{|\omega_\nu(x_\nu)|} \sqrt{1 - x_\nu^2} \right| = \sum_{\nu=1}^{n-1} \left| \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(x_\nu)} \right| \sqrt{1 - x_\nu^2}. \qquad (2.7)$$

Thus, we conclude on the basis of (2.5) and (2.7) that, for every $p \in \pi_n$ satisfying (2.1) and (2.2) we have

$$|p^{(k)}(x) \leq |T_n^{(k)}(x)| \quad \text{for all } x \in \bigcup_{j=1}^{n-k+1} [\alpha_j, \beta_j]. \qquad (2.8)$$

Theorem 2.1 will be proved if we succeed in showing that under the same assumptions for p

$$|p^{(k)}(x)| \leq |Z_{n,k}(x)| \quad \text{for all } x \in \bigcup_{j=1}^{n-k} (\beta_j, \alpha_{j+1}). \qquad (2.9)$$

For $j = 2, \ldots, n-1$ the j-th zero of T_n is located between the j-th zero of ω_n and the j-th zero of ω_0, while the first zeros of T_n and ω_n coincide as well as the last zeros of T_n and ω_0. That is to say, the zeros of T_n interlace with both the zeros of ω_0 and ω_n. Then, according to Markov's result every interval (β_j, α_{j+1}) ($j = 1, \ldots, n-k$) contains exactly one zero of T_n. This implies

$$\operatorname{sign}\{T_n^{(k)}(\alpha_j)\} = \operatorname{sign}\{T_n^{(k)}(\beta_j)\} = (-1)^{n+1-k-j} \quad (j = 1, \ldots, n-k). \qquad (2.10)$$

Further, we make use of the identity

$$T_n^{(k)}(x) = \frac{n}{n-1} [x T_{n-1}^{(k)}(x) + (n+k-2) T_{n-1}^{(k-1)}(x)]$$

to obtain the following relations between T_n and $Z_{n,k}$.

$$Z_{n,k}(x) - T_n^{(k)}(x) = \frac{n(n-2)}{(n-1)k\xi} \left(x - \frac{n+k-2}{n-2} \xi \right) \omega_n^{(k)}(x), \qquad (2.11)$$

$$Z_{n,k}(x) + T_n^{(k)}(x) = \frac{n(n-2)}{(n-1)k\xi} \left(x + \frac{n+k-2}{n-2} \xi \right) \omega_0^{(k)}(x). \qquad (2.12)$$

Hence

$$Z_{n,k}(x) = \begin{cases} -T_n^{(k)}(x) & \text{for } x = \alpha_j \ (j = 2, \ldots, n-k+1), \\ T_n^{(k)}(x) & \text{for } x = \beta_j \ (j = 1, \ldots, n-k). \end{cases} \qquad (2.13)$$

As it has been already mentioned, if $p \in \pi_n$ satisfies the assumptions of Theorem 2.1, then $|p^{(k)}| \le |T_n^{(k)}|$ at the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$, and consequently

$$|p^{(k)}(x)| \le |Z_{n,k}(x)| \quad \text{for } x = \alpha_{j+1} \text{ and } x = \beta_j \ (j = 1, \dots, n-k). \tag{2.14}$$

Therefore, in view of (2.13) and (2.10), each of the polynomials $Z_{n,k} + p^{(k)}$ and $Z_{n,k} - p^{(k)}$ has at least one zero in each of the intervals $[\alpha_j, \beta_j]$, $j = 2, \dots, n-k$. Moreover,

$$\operatorname{sign}\{(Z_{n,k} \pm p^{(k)})(\alpha_{n-k+1})\} = -\operatorname{sign}\{T_n^{(k)}(\alpha_{n-k+1})\} = -1,$$

while $Z_{n,k}(x) \pm p^{(k)}(x) > 0$ for x large enough, since these polynomials have a positive leading coefficient. Therefore, each of the polynomials $Z_{n,k} \pm p^{(k)}$ has at least one zero located to the right of α_{n-k+1}. Similar reasonings show that $Z_{n,k} \pm p^{(k)}$ must have also at least one zero located to the left of β_1.

Since the polynomials $Z_{n,k} \pm p^{(k)}$ are of exact degree $n-k+1$, we conclude that each of them has maximal number of zeros, and that these zeros lie outside the set $\cup_{j=1}^{n-k}(\beta_j, \alpha_{j+1})$. This means that $Z_{n,k} \pm p^{(k)}$ do not change the sign on this set, and then (2.9) follows by virtue of (2.14). Theorem 2.1 is proved.

For the proof of Theorem 1.1 we need some auxiliary propositions. We formulate as a lemma a special case of a well-known expansion property of ultraspherical polynomials (see e.g., Szegö [14], Eq. (4.9.19), or Rivlin [11], p. 158, Remark 1).

LEMMA 2.1 *For every $k, l \in \mathbb{N}$ $(n \ge k \ge l)$ there holds*

$$T_n^{(k)}(x) = \sum_{m=l}^{n-k+l} a_m(k,l) T_m^{(l)}(x) \quad \textit{with nonnegative } a_m(k,l).$$

Next, we denote by $\|\cdot\|_\star$ the uniform norm in the interval $[-\xi, \xi]$, *i.e.*,

$$\|g\|_\star = \max_{x \in [-\xi, \xi]} |g(x)|.$$

The next lemma can be found in ([5], Lemma 2.4) (its proof follows easily from Lemma 2.1).

LEMMA 2.2 *For every natural k, m, n $(1 \le k \le m \le n)$ there holds*

$$\|T_m^{(k)}\|_\star = T_m^{(k)}(\xi).$$

Moreover, if $k < m$ and $|x| < \xi$, then $|T_m^{(k)}(x)| < T_m^{(k)}(\xi)$.

With the help of these two lemmas we prove

LEMMA 2.3 *For every $k \in \{2, \dots, n\}$*

$$\|Z_{n,k}\|_\star < \|T_n^{(k)}\|_\star.$$

Proof We write

$$Z_{n,k}(x) = c[u_{n,k}(x) - v_{n,k}(x)], \tag{2.15}$$

where

$$c = \frac{n(n-2)}{(n-1)k\xi},$$

$$u_{n,k}(x) = (x^2 - 1)T_{n-1}^{(k)}(x) + kxT_{n-1}^{(k-1)}(x),$$

and

$$v_{n,k}(x) = \left(\frac{n+k-2}{n-2}\xi^2 - 1\right)T_{n-1}^{(k)}(x).$$

Note that $(n+k-2)\xi^2/(n-2) - 1 > 0$ for all natural numbers k and $n \ge 3$. Further, from (2.15) and Lemma 2.2 we obtain

$$\begin{aligned}
\|Z_{n,k}\|_\star &\le c[\|u_{n,k}\|_\star + \|v_{n,k}\|_\star] \\
&= c[v_{n,k}(\xi) + \|u_{n,k}\|_\star] \\
&= -Z_{n,k}(\xi) + c[u_{n,k}(\xi) + \|u_{n,k}\|_\star] \\
&= T_n^{(k)}(\xi) - c\left(2 + \frac{k}{n-2}\right)k\xi T_{n-1}^{(k-1)}(\xi) + c[u_{n,k}(\xi) + \|u_{n,k}\|_\star]
\end{aligned} \tag{2.16}$$

(for the last equality we have used Eq. (2.12)). Clearly, Lemma 2.3 will be proved if we succeed in showing that

$$u_{n,k}(\xi) + \|u_{n,k}\|_\star < \left(2 + \frac{k}{n-2}\right) k\xi T_{n-1}^{(k-1)}(\xi). \tag{2.17}$$

We consider separately the cases $k=2$ and $k \geq 3$.

Case $k=2$ Using the differential equation for T_{n-1}, we express $u_{n,2}$ as

$$u_{n,2}(x) = (n-1)^2 T_{n-1}(x) + xT'_{n-1}(x).$$

This representation and Lemma 2.2 yield

$$\|u_{n,2}\|_\star \leq (n-1)^2 + \xi T'_{n-1}(\xi).$$

Consequently, we obtain

$$u_{n,2}(\xi) + \|u_{n,2}\|_\star \leq (n-1)^2\left(1 + \sin\frac{\pi}{2n}\right) + 2\xi T'_{n-1}(\xi).$$

After some simple manipulations we conclude that (2.17) will hold with $k=2$ if

$$(n-1)(n-2)\left(1 + \sin\frac{\pi}{2n}\right) \sin\frac{\pi}{2n} < 2n\cos^2\frac{\pi}{2n}.$$

The validity of this last inequality is easily verified.

Case $k \geq 3$ In this case we shall show that

$$\|u_{n,k}\|_\star = u_{n,k}(\xi). \tag{2.18}$$

Having established (2.18), we will obtain that (2.17) is equivalent to the inequality

$$2(1-\xi^2)T_{n-1}^{(k)}(\xi) + \frac{k^2}{n-2}\xi T_{n-1}^{(k-1)}(\xi) > 0,$$

which is obviously true. Thus, it remains to prove (2.18). For $k=3$ we have

$$u_{n,3}(x) = (x^2-1)T'''_{n-1}(x) + 3xT''_{n-1}(x) = (n-2)nT'_{n-1}(x),$$

where for the last equality we have used the differential equation for ultraspherical polynomials (note that $T'_{n-1} = (n-1)P_{n-2}^{(1)}$). In view of

Lemma 2.2, (2.18) holds for $k=3$. Moreover, since $u_{n,3}$ increases monotonically to the right of ξ, we conclude that

$$\|u_{m,3}\|_\star = u_{m,3}(\xi) \quad \text{for } m = 3, \ldots, n. \tag{2.19}$$

For $k > 3$ Lemma 2.1 implies

$$T_{n-1}^{(k-1)}(x) = \sum_{m=3}^{n-k+3} b_m T''_{m-1}(x) \quad \text{with nonnegative } b_m.$$

Finally, the application of Lemma 2.2 and (2.19) yields

$$\begin{aligned}
\|u_{n,k}\|_\star &= \|(x^2-1)T_{n-1}^{(k)}(x) + 3xT_{n-1}^{(k-1)}(x) + (k-3)xT_{n-1}^{(k-1)}(x)\|_\star \\
&= \left\| \sum_{m=3}^{n-k+3} b_m u_{m,3}(x) + (k-3)xT_{n-1}^{(k-1)}(x) \right\|_\star \\
&\leq \sum_{m=3}^{n-k+3} b_m \|u_{m,3}\|_\star + (k-3)\xi T_{n-1}^{(k-1)}(\xi) \\
&= \sum_{m=3}^{n-k+3} b_m u_{m,3}(\xi) + (k-3)\xi T_{n-1}^{(k-1)}(\xi) = u_{n,k}(\xi).
\end{aligned}$$

Lemma 2.3 is proved.

Proof of Theorem 1.1 Let $f \in \pi_n$ satisfy the assumptions of Theorem 1.1. Then $p(x) = f(x/\xi)$ will satisfy (2.1) and (2.2). Theorem 2.1 and Lemma 2.3 imply that for every $x \in [-\xi, \xi]$ and for $k \geq 2$

$$|p(x)| \leq \max\{|T_n^{(k)}(x)|, |Z_{n,k}(x)|\} \leq \|T_n^{(k)}\|_\star = T_n^{(k)}(\xi). \tag{2.20}$$

Turning back to the $\|\cdot\|$ norm, we conclude that $f(x) = p(\xi x)$ will satisfy

$$\|f^{(k)}\| = \xi^k \|p^{(k)}\|_\star \leq \xi^k T_n^{(k)}(\xi) = \|\bar{T}_n^{(k)}\|, \tag{2.21}$$

whence the inequality of Theorem 1.1 follows. It remains to clarify the cases in which equality holds. From Lemmas 2.2 and 2.3 we deduce that the second inequality in (2.20) becomes equality only if either $x = \xi$ or $k = n$. In the latter case we may assume again that $x = \xi$. Then careful examination of the proof of Theorem 2.1 shows that for $x=\xi$ (2.5) is fulfilled with equality sign if and only if $p=\pm T_n$. Consequently, the first inequality in (2.20) becomes equality only if $p=\pm T_n$, and therefore (2.21) holds with equality sign if and only if $f=\pm\bar{T}_n$. Theorem 1.1 is proved.

Remark 1 Unfortunately, our method of proof does not work in the case $k=1$. The reason is that Lemma 2.3 is not true for $k=1$ and $n>7$. A direct verification shows that $\|Z_{n,1}\|_\star < T_n'(\xi)$ for $3\le n\le 7$, which guarantees the validity of Theorem 1.1 for $k=1$ when $2\le n\le 7$. The real situation when $k=1$ and $n\ge 8$ is not known.

Remark 2 The inequalities of Theorem A and Theorem 1.1 remain valid also for polynomials with complex coefficients. This follows from the fact that if f is a polynomial with complex coefficients satisfying the assumptions of Theorem A (or of Theorem 1.1), then these assumptions will be satisfied by $\operatorname{Re} f$ and $e^{i\theta} f$ ($\theta\in\mathbb{R}$) as well. Indeed, let p be the extremal polynomial amongst the polynomials into consideration but allowed to have complex coefficients, and let

$$\sup_f\{\|f^{(k)}\|\} = |p^{(k)}(\xi)| = e^{i\gamma}p^{(k)}(\xi), \quad \xi\in[-1,1],$$

with some $\gamma\in\mathbb{R}$. Then the polynomial $g(x)=\operatorname{Re}\{e^{i\gamma}p(x)\}$ also belongs to the class into consideration and satisfies $g^{(k)}(\xi)=|p^{(k)}(\xi)|$. Thus, we found another extremal polynomial which, in addition, has only real coefficients.

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