J. of Inequal. & Appl., 1999, Vol. 4, pp. 345–373
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Landau-type Inequalities and L^p-bounded Solutions of Neutral Delay Systems

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(Received 30 November 1998; Revised 21 January 1999)

In Section 1 relations between various forms of Landau inequalities $\|y^{(m)}\|^n \leq \lambda\|y\|^{n-m}\|y^{(n)}\|^m$ and Halperin–Pitt inequalities $\|y^{(m)}\| \leq \varepsilon\|y^{(n)}\| + S(\varepsilon)\|y\|$ are discussed, for arbitrary norms, intervals and Banach-space-valued y. In Section 2 such inequalities are derived for weighted L^p-norms, Stepanoff- and Orlicz-norms.

With this, Esclangon–Landau theorems for solutions y of linear neutral delay difference-differential systems are obtained: If y is bounded e.g. in a weighted L^p- or Stepanoff-norm, then so are the $y^{(m)}$. This holds also for some nonlinear functional differential equations.

Keywords: Landau inequalities; Esclangon–Landau theorem; L^p-bounded solutions; Neutral differential-difference systems

AMS 1991 Subject Classifications: Primary: 26D10, 34C11, 34K40; Secondary: 34K25, 34D40

0 INTRODUCTION AND NOTATIONS

To prove that bounded solutions of certain linear differential equations are quasiperiodic, Esclangon [6,7] needed and demonstrated that such bounded solutions have bounded derivatives. This result was later used by Bohr and Neugebauer [4] to get the almost periodicity of bounded solutions of nth order linear equations with constant coefficients and almost periodic right-hand side.

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Landau [21] extended Esclangon's result on the boundedness of the derivatives of bounded solutions to linear differential equations with only bounded coefficients. In the following we will call such theorems "Esclangon–Landau-" or "EL-results". They have played an important role in the discussion of the asymptotic behaviour of solutions of differential equations, see e.g. Basit and Zhikov [2], Levitan and Zhikov [22, p. 95 and 97], and the references in [3, p. 596]. In [3] EL-results were obtained for difference-differential equations and the sup-norm.

For his EL-results Landau showed, under some additional assumptions and with the sup-norm, for a compact interval,

$$\|y^{(m)}\|^n \leq \lambda_n \|y\|^{n-m} \|y^{(n)}\|^m, \quad 0 < m < n \tag{0.1}$$

([20, 1913 for $n = 2$, 21, 1930 p. 182, Hilfssatz 3]); a qualitative form can be found in Hardy and Littlewood [12, p. 422, Theorem 3])

We will call results of this type Landau inequalities; a thorough discussion of the many results in this direction can be found in Chapter 1 of Mitrinović, Pečarić and Fink [25], mostly for scalar-valued y and unbounded intervals.

To get EL-results for L^p-bounded and Banach-space-valued solutions, one needs however (0.1) or related inequalities for bounded intervals and such norms, then not so much can be found in the literature.

In Section 1 we discuss first the relations between various forms of (0.1), especially the asymptotic form (for compact intervals approaching the boundary) needed later, for vector-valued y.

It turns out that a stronger version of (0.1) is a Nirenberg inequality [26, appendix],

$$\|y^{(m)}\| \leq \varepsilon^{n-m} \|y^{(n)}\| + K\varepsilon^{-m} \|y\|, \quad 0 < \varepsilon \leq \varepsilon_0 \tag{0.2}$$

a weaker variant is obtained by replacing $K\varepsilon^{-m}$ by an arbitrary function $S(\varepsilon)$ (Halperin and Pitt [11]). Again relations, also with asymptotic versions, are discussed, for general intervals and norms.

In Section 2 we obtain Nirenberg and then Landau inequalities for weighted L^p or Stepanoff-norms, arbitrary intervals and vector-valued functions. From the explicit form of the constants K and λ there (usually not optimal) we can deduce the asymptotic forms needed. For Orlicz-norms we get at least Halperin–Pitt inequalities, for bounded intervals.

This is applied in Section 3 to linear delayed neutral difference-differential equations and systems, with bounded operator-valued coefficients: For weighted L^p-norms or weighted Stepanoff S^p-norms still an EL-result is true, $1 \leq p \leq \infty$, if the weight function does not oscillate too wildly, similarly for Orlicz norms (Corollary 3.2). These results seem new and non-trivial even for bounded intervals and scalar-valued solutions. With an asymptotic Landau inequality even some non-linear functional differential equations can be treated (Proposition 3.6).

In the following X is a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $J \subset \mathbb{R}$ is an interval with endpoints α and β, $-\infty \leq \alpha < \beta \leq \infty$. For $f: J \to X$ and $M \subset J$

$$g = fM \text{ means } g = f \text{ on } M, g = 0 \text{ else in } J; \tag{0.3}$$

$|f|$ is defined by $|f|(x) := \|f(x)\|$, $x \in J$. $|I|$ is the length of the interval $I \subset \mathbb{R}$; "a.e." is with respect to Lebesgue measure on $\mathbb{R}$. Integrals are usually (Bochner-) Lebesgue integrals (Hille–Phillips [15]). A seminorm $\| \; \|$ is a norm without "$\|x\| = 0$ implies $x = 0$". For n natural

$$C^{(n)}(J, X) := \{f \in C^{n-1}(X, J): f^{(n-1)} \text{ locally absolutely continuous and } f^{(n-1)'} \text{ exists a.e. in } J\}; \tag{0.4}$$

then

$$f^{(n)}(x) := f^{(n-1)'}(x) \text{ where it exists in } J, \text{ else} := 0. \tag{0.5}$$

The L^p-spaces are spaces of measurable functions, not equivalence classes.

1 LANDAU, NIRENBERG AND HALPERIN–PITT INEQUALITIES

With the notations of the introduction we assume in the following:

$$\begin{aligned} &V \text{ linear } \subset X^J \text{ (pointwise operations)}, \| \; \| : V \to [0, \infty) \\ &\text{seminorm satisfying: if } y \in C^{(1)}(J, X),\ I \text{ compact } \subset J, yI \\ &\text{and } y'I \in V \text{ and } \|yI\| = 0, \text{ then } \|y'I\| = 0;\ n \text{ integer} \geq 2. \end{aligned} \tag{1.1}$$

Here V can be e.g. $L^p(J, X)$, with $\| \; \| = \| \; \|_p$.

DEFINITION 1.1 *We say that the strong Landau (or Kolmogorov) inequality* $L_n^s = L_n^s(\lambda) = L_n^s(\lambda, \|\ \|)$ *holds (for* V*) if* $0 \le \lambda < \infty$ *and*

$$\|y^{(m)}\|^n \le \lambda \|y\|^{n-m} \|y^{(n)}\|^m \quad \text{for } 0 < m < n \tag{1.2}$$

and all $y \in C^{(n)}(J, X)$ *with* $y^{(m)} \in V$, $0 \le m \le n$ *(see* (0.4), (0.5)*).*

The *weak Landau inequality* $L_n^w = L_n^w(\lambda, \tau, \|\ \|)$ holds if with $\lambda, \tau \in (0, \infty]$ one has (1.2) with y as after (1.2) and with the additional conditions

$$\|y^{(n)}\| > 0 \text{ and } (\|y\|/\|y^{(n)}\|)^{1/n} \le \tau. \tag{1.3}$$

The *Landau inequality* $L_n = L_n(\lambda, \tau, \|\ \|)$ holds if $\lambda, \tau \in (0, \infty]$ and for y as after (1.2) with $0 \le \|y\| \le a$, $0 \le \|y^{(n)}\| \le b$, $0 < b$, where $a, b \in [0, \infty)$, and with $(a/b)^{1/n} \le \tau$, one has

$$\|y^{(m)}\|^n \le \lambda a^{n-m} b^m, \quad 0 < m < n. \tag{1.4}$$

The *asymptotic Landau inequality* $L_n^a = L_n^a(\lambda, \|\ \|)$ holds, with $0 < \lambda < \infty$, if to each $y \in C^{(n)}(J, X)$ with $y^{(m)}\, I \in V$ for $0 \le m \le n$ and all compact $I \subset J$ and for which furthermore $\|y^{(n)} I\|$ is $\not\equiv 0$ in I there exists a compact interval $I(y) \subset J$ such that (see (0.3))

$$\|y^{(m)} I\|^n \le \lambda \|yI\|^{n-m} \|y^{(n)} I\|^m, \quad 0 < m < n, \ \ I(y) \subset I \subset J \tag{1.5}$$

(the $y^{(m)}$ need not be in V).

Landau inequalities have been introduced in [20, $n = 2$, 21, p. 182, Hilfssatz 3], where Landau showed that L_n holds for compact J, $\|f\| = \sup_J |f|$, $X = \mathbb{R}$, with $\lambda_2 = 4$, $\lambda_n = 2^{n \cdot 2^n}$, $\tau = (1/2)|J|$; this implies immediately a strong Landau inequality for unbounded J. Kolmogorov [16] determined the optimal λ_n (even $\lambda_{n,m}$) in L_n^s for $J = \mathbb{R} = X$, $\|f\| = \sup_{\mathbb{R}} |f|$, $V =$ bounded functions. The asymptotic form L_n^a can be traced back to Hardy and Littlewood [12, p. 422, Theorem 3], it was used in [3, Lemma 2.5] for general X and $\|\ \|_\infty$.

Obvious relations, for fixed J, V, $\|\ \|$, n, λ, any $\tau > 0$:

$$L_n^s(\lambda) = L_n(\lambda, \infty) = L_n^w(\lambda, \infty) \Rightarrow L_n(\lambda, \tau) \Rightarrow L_n^w(\lambda, \tau). \tag{1.6}$$

Also

$$L_n^a(\lambda) \Rightarrow L_n^s(\lambda) \quad \text{for } \|y^{(n)}\| > 0, \tag{1.7}$$

provided V satisfies

$$f \in V, I \text{ compact} \subset J \ \Rightarrow fI \in V \text{ and } \|fI\| \to \|f\| \text{ as } I \to J. \tag{1.8}$$

Even with (1.8), $L_n^s(\lambda) \Rightarrow L_n^a(\lambda+\varepsilon)$ follows only for y with $y^{(m)} \in V$; see Remarks 2.17(a) and (c).

Example 1.2 $y(t) = t + \varepsilon \sin t$ shows that already L_2^s and therefore L_2^a are false for any bounded J, any λ, any $V = L^p$ with $\| \ \|_p$, $1 \le p \le \infty$ (see, however, Proposition 2.16, but also Example 1.13).

So for bounded J for (1.2) additional conditions are necessary. We work here with Landau's condition (1.3); for other types of Landau inequalities in this situation see Gorny [9], Levitan and Zhikov [22, p. 95], Redheffer and Walter [27].

Throughout each of the following lemmas, J, X, V and its seminorm $\| \ \|$ are fixed and satisfy (1.1).

LEMMA 1.3 *If L_2^a holds with $\lambda_2 \ge 1$, then L_n^a holds for $n \ge 2$, with $\lambda_n = \lambda_2^{n^3/8}$.*

Proof By induction one can show (see [18, p. 232, (2.14)]), for $\lambda = \lambda_{n,m}$ in (1.5)

$$\lambda_{n,m} \le \lambda_{2,1}^{nm(n-m)/2}, \quad 0 < m < n, \ 2 \le n. \tag{1.9}$$

LEMMA 1.4 *If L_2^s holds with $\lambda \ge 1$ and $n \ge 2$, then L_n^s holds with λ_n as in Lemma* 1.3.

Proof As for Lemma 1.3.

LEMMA 1.5 *If $L_2(\lambda_2, \tau)$ holds with $\tau > 0$, $\lambda_2 \ge 1$, and $n \ge 2$, then $L_n(\lambda_n, \tau)$ holds with $\lambda_n = \lambda_2^{n2^{n-1}}$.*

Proof This has been shown by Landau [21, pp. 182–183], for $\| \ \|_\infty$, compact J, $X = \mathbb{R}$, $\tau = (1/2)|J|$, $\lambda_2 = 4$. His ingenious proof works also in our more general situation, for the χ on p. 182 e.g. one has to use $\chi := \max(\lambda_2^{2^{n-2}}, \ \max\{\|y^{(m)}\|: 0 < m < n\})$.

Question: Can one improve this to $\lambda_n = A \cdot \lambda_2^{c \cdot n^3}$ as in Lemma 1.3? (Yes under the assumptions of Lemma 1.15 via $L_2 \Rightarrow N_2 \Rightarrow N_n \Rightarrow L_n$ of below.)

Example 1.6 If $y_k(t) = t + k^{-3} \sin kt$, $t \in J = [0,1]$, $k \in \mathbb{N}$, with $\| \ \|_p$, $1 \leq p \leq \infty$, and the nonlinear V containing just the y_k and their derivatives up to order 3, one can show that L_3^s holds, but L_2^s holds for no $\lambda < \infty$.

Do there exist such linear V as in (1.1)?

LEMMA 1.7 *Assume $V \subset L^1(J, X)$, J bounded, V containing all bounded continuous f, assume further the existence of $C_1, C_2 \in (0, \infty)$ with*

$$C_1 \|f\|_1 \leq \|f\|_V \leq C_2 \sup_J |f|, \ f \text{ bounded} \in V. \tag{1.10}$$

Then $L_2(\lambda, \tau)$ and $L_2^w(\lambda, \tau)$ are equivalent.

Examples are $L^p(J, X)$ or more general Orlicz-spaces with Lebesgue measure.

Proof With y as before (1.4) with $\|y''\| < b$ and $0 < a$ define $w_{u,v}(t) := y(t) + xu \sin vt$ with $x \in X$, $\|x\| = 1$. Then $w_{u,v}^{(j)} \in V$, with $w_v := w_{\varepsilon/(C_2 v), v}$ one has $\|w_v - y\| \leq \varepsilon$, $\|w_v' - y'\| \leq \varepsilon$, for $0 < \varepsilon < b - \|y''\|$. With a continuity argument there is $s \in (1, \infty)$ with $\|w_s''\| = b + \varepsilon(b/a)$; then $\|w_s\| / \|w_s''\| \leq (\|y\| + \varepsilon)/(b + \varepsilon(b/a)) \leq a/b \leq \tau^2$. L_2^w yields $(\|y'\| - \varepsilon)^2 \leq \|w_s'\|^2 \leq (\|y\| + \varepsilon)(b + \varepsilon(b/a))$, $\varepsilon \to 0$ gives L_2. This works for any V containing z with $\|z^{(j)}(v \cdot)\| \leq C_0$, $0 < \delta_0 \leq \|z''(v \cdot)\|$ for $v \leq 1$, $j = 0, 1, 2$, then without (1.10).

COROLLARY 1.8 *L_2^w implies L_n^w for $n \geq 2$, with λ_n of Lemma 1.5, provided V, $\| \ \|$ are as in Lemma 1.7.*

Question: Direct proof of $L_2^w \Rightarrow L_n^w$, for more general V? Characterization of V with $L_2^w \Rightarrow L_2$?

DEFINITION 1.9 *We say that a Nirenberg inequality $N_n = N_n(K, \sigma) = N_n(K, \sigma, V, \| \ \|)$ holds iff with $K, \sigma \in [0, \infty]$ for all y as after (1.2) one has*

$$\|y^{(m)}\| \leq \varepsilon^{n-m} \|y^{(n)}\| + \frac{K}{\varepsilon^m} \|y\| \quad \textit{for } 0 < \varepsilon \text{ real } \leq \sigma, \ 0 < m < n. \tag{1.11}$$

A strong N_n^s holds means $N_n(K, \infty, V, \| \|)$ is true.

LEMMA 1.10 *If $N_2(K, \sigma)$ holds with $4K \geq 1$, then $N_n(K_n, \sigma)$ holds for $n \geq 2$ with*

$$K_n = 2^{n-4}(4K)^{\binom{n}{2}}. \tag{1.12}$$

Proof by induction If N_k holds for $2 \leq k \leq n$, one has, with N_2 and $y_m := \|y^{(m)}\|$, y as after (1.2) for $n+1$,

$$y_{n-1} \leq \varepsilon y_n + K_n \varepsilon^{1-n} y_0, \quad 0 < \varepsilon \leq \sigma_n = \sigma,$$
$$y_n \leq \eta y_{n+1} + K y_{n-1}/\eta \leq \eta y_{n+1} + \frac{K\varepsilon}{\eta} y_n + \frac{KK_n}{\varepsilon^{n-1}} y_0, \quad 0 \leq \eta \leq \sigma.$$

With $\varepsilon = \eta/(2K) \leq \sigma$ one gets, with $\delta := 2\eta$

$$y_n \leq 2\eta y_{n+1} + \frac{2KK_n}{\eta\varepsilon^{n-1}} y_0 = \delta y_{n+1} + \frac{(2K)^n K_n 2^n}{\delta^n} y_0, \tag{1.13}$$

which is (1.11) for $n+1$ and $m = n$, with even $0 < \delta \leq 2\sigma$.

Substituting this in (1.11), one gets for $0 < m < n$

$$y_m \leq \varepsilon^{n-m} \delta y_{n+1} + K_n((4K)^n \varepsilon^{n-m} \delta^{-n} + \varepsilon^{-m}) y_0$$

for $0 < \varepsilon \leq \sigma$ and $0 < \delta \leq 2\sigma$. So $\delta = \varepsilon$ is possible, yielding with $(4K)^n \geq 1$

$$y_m \leq \varepsilon^{(n+1)-m} y_{n+1} + K_n(2(4K)^n)\varepsilon^{-m} y_0, \quad 0 < \varepsilon \leq \sigma_{n+1} = \sigma;$$

with (1.13), this holds for $0 < m < n+1$.

So

$$K_{n+1} = 2(4K)^n K_n = 2(4K)^n 2^{n-4}(4K)^{\binom{n}{2}} = 2^{n+1-4}(4K)^{\binom{n+1}{2}}.$$

In the above, the case $\sigma = \infty$ gives

LEMMA 1.11 *If $N_2^s(K)$ holds with $4K \geq 1$, then $N_n^s(K_n)$ holds, with* (1.12).

LEMMA 1.12 *For each $n \geq 2$, N_n^s and L_n^s are equivalent, with*

$$\lambda_n = (1 + \mathrm{e}^{1/\mathrm{e}})^n K_n^{n-1} \textit{ resp. } K_n = n^{-1/(n-1)} \lambda_n, \tag{1.14}$$

provided $K_n \geq 1$ resp. $\lambda_n \geq 1$.

Remark For $n=2$, $\lambda_2=4K_2$ by (1.15); (1.14) can be improved to $K_n=(1-(1/n))n^{-1/(n-1)}\lambda_n$, which is optimal.

Proof We show the equivalence even for each fixed m, $0<m<n$. If, with $y_j:=\|y^{(j)}\|$, (1.11) holds for all $\varepsilon>0$, the right side has its minimum for $0=(n-m)\varepsilon^{n-m-1}y_n-mK\varepsilon^{-m-1}y_0$ or ($y_n=0 \Rightarrow y_m=0 \Rightarrow$ (1.2)) $\varepsilon=(mKy_0/((n-m)y_n))^{1/n}$. This ε gives (1.2), with

$$\lambda_{n,m}=\left(\left(\frac{m}{n-m}\right)^{(n-m)/n}+\left(\frac{n-m}{m}\right)^{m/n}\right)^{n}K_{n,m}^{n-m}. \tag{1.15}$$

With $t^{1/t}\leq e^{1/e}$ for $1\leq t<\infty$ and $K_{n.m}=K_n$ this gives Part 1 of (1.14). Conversely, (1.2) for m implies for $0<\varepsilon<\infty$

$$\begin{aligned} y_m^n \leq \lambda_{n,m}y_0^{n-m}y_n^m &= \lambda_{n,m}\binom{n}{m}^{-1}\binom{n}{m}(\varepsilon^{-m}y_0)^{n-m}(\varepsilon^{n-m}y_n)^m \\ &\leq \left(\lambda_{n,m}\Big/\binom{n}{m}\right)(\varepsilon^{n-m}y_n+\varepsilon^{-m}y_0)^n. \end{aligned}$$

This gives (1.11) with

$$K_{n,m}=\left(\lambda_{n,m}\Big/\binom{n}{m}\right)^{1/(n-m)}. \tag{1.16}$$

If $\lambda_{n,m}=\lambda_n\geq 1$, this gives Part 2 of (1.14.)

Example 1.13 N_n^s trivially implies N_n; the converse is in general false:

$$J=[3,\infty),\ \ X=\mathbb{R},\ \ \|f\|=\sup\{|f(t)|/t:\ 3\leq t\},$$
$$V=\{f\in C(J,\mathbb{R}): \|f\|<\infty\}.$$

$y_\delta=t+\delta\sin t$ shows, that L_2^s is false for any $\lambda\in[0,\infty)$ – though $|J|=\infty$. One can show however that N_2 (14, 3) is true (Landau's L_2 $(4,(1/2)|I|)$ for compact I and $\|\ \|_\infty$ gives $L_2^s(4)$ for J, $\|\ \|_\infty$, then $N_2^s(1)$ by Lemma 1.12 and the above remark; apply this to $f=y/t$.) For $|J|<\infty$ a simpler example follows, with $\|\ \|_p$, from Example 1.2, Lemma 1.12 and Proposition 2.1.

LEMMA 1.14 *$N_n(K,\sigma)$ implies $L_n(\lambda(\tau),\tau)$ for each real $\tau > 0$ and $n \geq 2$, with*

$$\lambda(\tau) = (K + \varrho^{-n})^n \max(\varrho^n, \varrho^{n(n-1)}), \quad \varrho := \frac{\tau}{\sigma}. \tag{1.17}$$

Proof If $y_m := \|y^{(m)}\| \leq \varepsilon^{n-m} y_n + K_{n,m}\varepsilon^{-m} y_0$ for $0 < \varepsilon \leq \sigma$, $0 \leq y_n \leq b$, $y_0 \leq a < \infty$, $0 < b$, then if $\varepsilon := -(a/b)^{1/n}$, $\leq \sigma$, one gets

$$y_m^n \leq \left(\left(\frac{\sigma}{\tau}\right)^{n-m} + K_{n,m}\left(\frac{\tau}{\sigma}\right)^m\right)^n a^{n-m} b^m \quad \text{if } \left(\frac{a}{b}\right)^{1/n} \leq \tau, \tag{1.18}$$

which is L_n, with (1.17).

LEMMA 1.15 *If J, V, $\| \;\|$ are as in Lemma* 1.7 *with* (1.10), *then for any $\lambda, \tau, \sigma \in (0,\infty)$, $L_2^w(\lambda,\tau)$ implies $N_2(K,\sigma)$ with*

$$K = \max\left\{\frac{\lambda}{4}, \sigma\frac{C_2}{C_1}(\tau^{-2} + 4/|J|^2)\right\}. \tag{1.19}$$

Proof If, with $y_m := \|y^{(m)}\|$, $y_0/y_2 \leq \tau^2$ with $y_2 > 0$, then $y_1^2 \leq \lambda y_0 y_2 \leq (\lambda/4)(\varepsilon y_2 + (1/\varepsilon)y_0)^2$ implies even $N_2(\lambda/4, \infty)$.

If $0 \leq y_2 < y_0\tau^{-2}$, (2.8) of Section 2 and (1.10) give

$$y_1 \leq c\left(y_2 + \frac{4}{|J|^2} y_0\right), \quad c := C_2/C_1. \tag{1.20}$$

So

$$y_1 \leq c(y_0\tau^{-2} + 4|J|^{-2}y_0) \leq \varepsilon y_2 + \sigma c(\tau^{-2} + 4|J|^{-2})y_0/\sigma \leq \varepsilon y_2 + \frac{K}{\varepsilon} y_0$$

if $0 < \varepsilon \leq \sigma$, with $K = \sigma c(\tau^{-2} + 4|J|^{-2})$.

Remark Lemmas 1.15 and 1.14 give a new proof of $L_2^w(\lambda,\tau) \Rightarrow L_2(\tilde{\lambda},\tau)$ of Lemma 1.7, but only with $\tilde{\lambda} > \lambda$ in general, even for optimal σ.

Question: Can one extend Lemma 1.15 to more general norms resp. to $L_n^w \Rightarrow N_n$, $n \geq 2$? (For norms as in proposition 2.1, N_n always holds.)

DEFINITION 1.16 *We say that a Halperin–Pitt inequality* $H_n = H_n(S) = H_n(S, V, \| p)$ *holds if with* $S:(0,\infty)\to[0,\infty)$ *one has for all* y *as after* (1.2)

$$\|y^{(m)}\| \le \varepsilon\|y^{(n)}\| + S(\varepsilon)\|y\|, \quad 0<m<n, \ 0<\varepsilon. \tag{1.21}$$

An asymptotic Halperin–Pitt inequality $H_n^a(S)$ *holds if for each* $y \in C^{(n)}(J,X)$ *with* $y^{(m)}I \in V$ *for* $0 \le m \le n$ *and all compact intervals* $I \subset J$ *there is a compact* $I(y) \subset J$ *such that*

$$\begin{aligned}&\|y^{(m)}I\| \le \varepsilon\|y^{(n)}I\| + S(\varepsilon)\|yI\|, \quad 0<m<n, \ 0<\varepsilon,\\ &I(y) \subset \textit{compact } I \subset J.\end{aligned} \tag{1.22}$$

The pointwise $H_n^{\bullet a}$ *is defined as* H_n^a, *but with* S *depending on* y, *similarly for* $H_n^\bullet$.

Remark If (1.21) holds only for $0<\varepsilon\le$ some $\sigma<\infty$, with $S(\varepsilon):=S(\sigma)$ for $\varepsilon>\sigma$ it holds for all $\varepsilon>0$, we can assume $\sigma=\infty$, $H_n^s \equiv H_n$.

Such inequalities seem to have been considered first by Halperin and Pitt [11, Theorem 1, (2.1.2), Theorems 3 and 4] in their study of the closedness of ordinary differential operators and their adjoints in L^p.

LEMMA 1.17 *For any* $n \ge 2$, H_2 *implies* H_n *with suitable* S.

Proof Similar as for Lemma 1.10, with

$$S_{n+1}(\varepsilon) := S_n\Big(\frac{\varepsilon}{2}\Big) + 2S_2\Big(\frac{\varepsilon}{2}\Big)S_n\Big(1\Big/\Big(2S_2\Big(\frac{\varepsilon}{2}\Big)\Big)\Big) \quad \text{if } 0<\varepsilon\le 1/2. \tag{1.23}$$

Also similarly as Lemma 1.3, with (1.23), one gets

LEMMA 1.18 *For* $n \ge 2$, H_2^a *implies* H_n^a *with suitable* S.

LEMMA 1.19 *For* $n \ge 2$, $H_2^{\bullet a}$ *implies* $H_n^{\bullet a}$.

Collecting some of the above results, one has for $n \ge 2$

$$\begin{aligned}&L_2^a \Rightarrow L_n^a \iff N_n^a \Rightarrow^{(*)} L_n^s \iff N_n^s \Rightarrow N_n \Rightarrow L_n \Rightarrow L_n^w\\ &N_n \Rightarrow H_n \equiv H_n^s \Rightarrow H_n^\bullet, \quad N_n^a \Rightarrow H_n^a \Rightarrow H_n^{\bullet a} \Rightarrow^{(*)} H_n^\bullet,\end{aligned} \tag{1.24}$$

where N_n^a is defined as L_n^a, H_n^a with $\sigma = \infty$ and $\|y^{(n)}I\| \neq 0$; for $^{(*)}$ the assumption (1.8) is needed, and only (1.7) holds;

$$L_2 \iff L_2^w \iff N_2 \text{ if (1.10) holds.} \tag{1.25}$$

Question: For what V, $\|\ \|$ is $L_n^w \Rightarrow H_n$ true, at least for $n = 2$?

2 INEQUALITIES FOR WEIGHTED L^P-NORMS

In this section J, X are as in the introduction, $w: J \to (0, \infty)$ is a Lebesgue measurable weight function with

$$C_\delta := \sup\left\{\frac{w(s)}{w(t)}: s, t \in J,\ |s - r| \le \delta\right\}, \quad 0 < \delta \le \infty, \tag{2.1}$$

$$\|f\|_{p,w} := \left(\int_J |f|^p w \,\mathrm{d}t\right)^{1/p} \quad \text{resp. } \mu_{\mathrm{L}} - \sup_J w|f| \tag{2.2}$$

for Bochner–Lebesgue measurable $f: J \to X$, $1 \le p \le \infty$, $\mu_{\mathrm{L}} =$ Lebesgue measure; $\|\ \|_p := \|\ \|_{p,1}$.

Proposition 2.1 *If $1 \le p \le \infty$ and w is a weight function with $C_{\delta_0} < \infty$ for some $0 < \delta_0 \le \infty$, then $\|\ \|_{p,w}$ satisfies an asymptotic Nirenberg inequality, i.e. for any J, $y \in C^{(2)}(J, X)$, I compact interval $\subset J$ one has*

$$\|y'I\|_{p,w} \le \varepsilon\|y''I\|_{p,w} + \frac{K}{\varepsilon}\|yI\|_{p,w} \quad \text{for } 0 < \varepsilon \le \sigma \tag{2.3}$$

with σ, K given by (2.10) *resp.* (2.13), (2.14).

Independent of p and $|I| \ge 1$ one can use

$$K = 32C_{\delta_0}^2, \quad \sigma = (1/2)\min(\delta_0, |I|), \quad 1 \le p \le \infty. \tag{2.4}$$

The case $X = \mathbb{C}$, $p = 2$, $w \equiv 1$ is due to Nirenberg [26, p. 671, (1)], also for functions of several variables; see also [25, p. 11 and p. 22] for $p = 2$, and [25, pp. 30–33 and p. 37], recalling (1.25).

Remark 2.2 (a) In (2.3) the $\|y''I\|_{p,w}$ can be ∞ if $p > 1$ (see Corollaries 2.5/6). Also, $\|y''I\| > 0$ is not needed.

(b) For $0 \notin \bar{J}$ unbounded and ω real, $w = t^\omega$ or $\mathrm{e}^{\omega t}$ we have finite C_δ. Here $C_\delta \to 1$ has $\delta \to 0$, so $K \to 2^{4-2/p}$ resp. 1 if $n_I \to \infty$ resp. $\delta_0 \to 0$.

(c) In proposition 2.1 bounded J or $\delta_0 = \infty$ are also admissible; but then $C_{\delta_0} < \infty$ implies $0 < \inf_J w \leq \sup_J w < \infty$, one can assume $w \equiv 1$. See example 2.3.

(d) For $p = \infty$, $w \equiv 1$ the $K = 1$ of (2.10) cannot be improved by Remark 2.9. See also [25, p. 11 and p. 22].

(e) For $p = \infty$ and $J = [\alpha, \beta)$ with $\beta \leq \infty$, proposition 2.1 can be extended to arbitrary decreasing $w : J \to (0, \infty)$ and $I = [\alpha, x)$, $\alpha + \delta \leq x \leq \beta$, with $\sigma = (1/2)C\delta\omega$, $K = (C\omega)^2$, $\omega := w(\alpha)/w(\alpha + \delta)$, $\delta \in (0, |J|)$, $C \geq 1$.

True also for $p < \infty$?

Example 2.3 (2.3) becomes false for $J = [0, 1)$, $w = 1/(1-t)$, $y = 1 - t + \eta \sin t$, $1 \leq p \leq \infty$: $C_\delta = \infty$. See Remark 2.2(e).

Example 2.4 For general norms Proposition 2.1 becomes false:

For any interval $J, X = \mathbb{R}$, $V =$ piecewise continuous bounded functions: $J \to \mathbb{R}$ one can construct $f_n \in C^2(J, \mathbb{R})$ with compact support and $c_n \in (0, 2^{-n}]$ such that with $\|f\| := \sum_1^\infty c_n |f(r_n)|$, $r_n =$ rationals $\in J$, one has (1.1), (1.8), $\|f_n\| \to 0$, $\|f_n''\| \to 0$, $\|f_n'\| = 1$, and $(\|f_n\|/\|f_n''\|)^{1/2} \to 0$. So even L_2^w, H_2, and therefore $L_2^a, L_2^s, L_2, N_2^a, N_2^s, N_2, H_2^a$ are here false, for any finite $\lambda, \tau, K, \sigma, S$. See Example 3.5.

Proof of Proposition 2.1 With the fundamental theorem of calculus for vector-valued functions ([15, Theorem 3.8.6, p. 88]) one shows for $y \in C^{(2)}(J, X)$

$$\begin{aligned} &y(u) = y(x) + (u - x)y'(x) + \int_x^u \int_x^t y'' \, \mathrm{d}s \, \mathrm{d}t, \\ &u, x \in I := [b - a, b + a] \subset J. \end{aligned} \tag{2.5}$$

With $v \in I$ one gets

$$y(v) - y(u) = (v - u)y'(x) + \int_u^v \int_x^t y''(s) \, \mathrm{d}s \, \mathrm{d}t. \tag{2.6}$$

If $v = b + z$, $u = b - z$, $0 \leq z \leq a$, integration with respect to z over $[0, a]$ yields

$$\begin{aligned} &\int_b^{b+a} y \, \mathrm{d}s - \int_{b-a}^b y \, \mathrm{d}s = a^2 y'(x) + \int_0^a \int_{b-z}^{b+z} \int_x^t y''(s) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}z, \\ &\|y'(x)\| \leq \frac{4}{|I|^2} \int_I |y| \, \mathrm{d}s + \int_I |y''| \, \mathrm{d}s, \quad x \in I \text{ compact } \subset J \end{aligned} \tag{2.7}$$

(see Brown and Hinton [5], with 9 instead of 4 and $X = \mathbb{R}$).

If $v = b + a$, $u = b - a$ in (2.6), one gets, for $x \in I$,

$$2a\|y'(x)\| \leq 2\|yI\|_\infty + \int_{b-a}^{b+a} |t - x|\,\mathrm{d}t\|y''I\|_\infty \leq 2\|yI\|_\infty + (a^2 + (x - b)^2)\|y''I\|_\infty$$

or

$$\|y'(x)\| \leq \frac{|I|}{2}\|y''I\|_\infty + \frac{2}{|I|}\|yI\|_\infty, \quad x \in I \subset J. \tag{2.8}$$

Case $p = \infty$: For compact intervals M, I with $M \subset I \subset J$ and $|I| \leq \delta_0$, (2.8) gives, on M

$$\begin{aligned} w|y'| &\leq \frac{|M|}{2}\left\|\left(\sup_M w\right)y''M\right\|_\infty + \frac{2}{|M|}\left\|\left(\sup_M w\right)yM\right\|_\infty \\ &\leq C_{|M|}\left(\frac{|M|}{2}\|y''M\|_{\infty,w} + \frac{2}{|M|}\|yM\|_{\infty,w}\right) \\ &\leq C_{|I|}\left(\frac{|M|}{2}\|y''M\|_{\infty,w} + \frac{2}{|M|}\|yM\|_{\infty,w}\right). \end{aligned}$$

Since this holds for any such $M \subset I$, one gets, with $\varepsilon = (1/2)C \cdot |M|$, and now any compact interval $I \subset J$

$$\|y'I\|_{\infty,w} \leq \varepsilon\|y''I\|_{\infty,w} + \frac{K}{\varepsilon}\|yI\|_{\infty,w}, \quad 0 < \varepsilon \leq \sigma, \tag{2.9}$$

$$K = C^2, \quad \sigma = \tfrac{1}{2}C\delta, \quad \delta := \min(|I|, \delta_0), \ \ C_\delta \leq C \text{ arbitrary } < \infty. \tag{2.10}$$

Case $1 \leq p < \infty$: Since $(u + v)^p \leq 2^{p-1}(u^p + v^p)$ for $u, v \geq 0$, (2.7) implies on I with Hölder

$$\begin{aligned} w|y'|^p &\leq 2^{p-1}\left(4^p|I|^{-2p}\left(\int_I |y|\,\mathrm{d}s\right)^p + \left(\int_I |y''|\,\mathrm{d}s\right)^p\right) \cdot \sup_I w \\ &\leq 2^{p-1}\left(4^p|I|^{-2p}\int_I |y|^p \sup_I w\,\mathrm{d}s + \int_I |y''|^p \sup_I w\,\mathrm{d}s\right) \\ &\quad \cdot |I|^{p(1-1/p)}, \end{aligned}$$

$$\int_I |y'|^p w\,\mathrm{d}s \le 2^{p-1}C_{|I|}\left(4^p|I|^{-p}\int_I |y|^p w\,\mathrm{d}s + |I|^p\int_I |y''|^p w\,\mathrm{d}s\right) \quad (2.11)$$

provided $|I| \le \delta_0$; (2.11) holds also for $p=1$.

If now M is any compact interval in J, subdivide it into n intervals I_j of length $|M|/n \le \delta_0$. Adding the inequalities (2.11) for these $I=I_j$, writing I instead of M and using $(u+v)^{1/p} \le u^{1/p} + v^{1/p}$, one gets

$$\|y'I\|_{p,w} \le 2^{1-1/p}C_{|I|/n}^{1/p}\frac{|I|}{n}\|y''I\|_{p,w} + 2^{3-1/p}C_{|I|/n}^{1/p}\frac{n}{|I|}\|yI\|_{p,w}. \quad (2.12)$$

Define

$$\sigma = 2C^{1/p}\frac{|I|}{n_I}, \quad \text{with } n_I \in \mathbb{N},\ C_{|I|/n_I} \le 2C < \infty \quad (2.13)$$

($C_{\delta_0} < \infty$ implies $C_\delta < \infty$ for any $0<\delta<\infty$, so everything above is defined).

Then if $0<\varepsilon\le\sigma$, there is $m \ge n_I$ with $n_I/(m+1) < \varepsilon \le n_I/m$, so $n=m+1$ and $C_{|I|/n}$ replaced by $2C$ in (2.12) yields

$$\|y'I\|_{p,w} \le \varepsilon\|y''I\|_{p,w} + \frac{K}{\varepsilon}\|yI\|_{p,w} \quad \text{if } 0<\varepsilon<\sigma,$$

with

$$K = 16C^{2/p}\left(1+\frac{1}{n_I}\right), \quad 1 \le p < \infty. \quad (2.14)$$

If one chooses n_I with $|I|/n_I \le \delta_0$, one gets (2.4) from (2.13) and (2.14), resp. (2.10).

Special case $w\equiv 1$: Then $\delta_0=\infty$, $C_\delta=1$, $n_I=1$; $\sigma=|I|$ and $K=32$ are possible by (2.10), (2.13), (2.14) for $1\le p\le\infty$, for $p=1$ even $K=8$, and $K=1$ for $p=\infty$ (see Remark 2.2(d)).

Proposition 2.1 yields, with $2C \ge C_{\min(|J|,\delta_0)} \ge C_{\min(|I|,\delta_0)}$ (no continuity of C_δ in δ is needed)

COROLLARY 2.5 *In Proposition* 2.1 *one can omit the I in* (2.3), *with K, σ of* (2.10) *if* $p=\infty$, *resp.* (2.4), *and* $|I|$ *replaced by* $|J|$ (*also if* δ_0 *or* $|J|=\infty$).

Special case $|J| = \delta_0 = \infty$, i.e. $w \equiv 1 \equiv C_\delta$: Then $\sigma = \infty$, so even N_2^s and with Lemmas 1.11, 1.12 all N_n^s, L_n^s are true, $n \geq 2$, $\| \, \| = \| \, \|_p$, $1 \leq p \leq \infty$, $V = L^p(J, X)$. For $J = \mathbb{R} = X$ optimal $\lambda = \lambda_{n,m}$ for L_n^s have been determined by Kolmogorov [16] for $p = \infty$, they are upper bounds for the $\lambda_{n,m}$ by Stein [28, Theorem 2], for $1 \leq p < \infty$. However even for monotone decreasing w the L_2^s is in general false by Example 1.13.

COROLLARY 2.6 *For any interval* J, $1 \leq p \leq \infty$, w *as in Proposition* 2.1 *or Remark* 2.2(e), $n \geq 2$ *and* $y \in C^{(n)}(J, X)$, *if* y *and* $y^{(n)}$ *belong to* $L_w^p :=$ $\{f$ *Bochner–Lebesgue measurable*: $J \to X \mid \|f\|_{p,w} < \infty\}$, *then* $y^{(m)} \in L_w^p$, $0 < m < n$.

Proof Corollary 2.5 and Lemma 1.10.

For $X = \mathbb{C}$ and $w \equiv 1$ this has been shown by Halperin and Pitt [11, Theorems 1 and 3], $p = \infty = |J|$ already by Hardy and Littlewood [12, p. 422, Theorem 3(a)], and Esclangon [7]. $J = \mathbb{R}$, $1 \leq p < \infty$, $X = \mathbb{C}$ and $w \equiv 1$ can also be found in Stein [28, Theorem 3].

There are two ways of getting Landau inequalities from Proposition 2.1: either $N_2 \Rightarrow N_n \Rightarrow L_n$, or $N_2 \Rightarrow L_2 \Rightarrow L_n$ (Lemmas 1.10, 1.14, 1.5).

The second method gives nicer formulas, we prefer the first, it gives in general better λ_n:

PROPOSITION 2.7 *For* J, p, w *as in Proposition* 2.1, $n \geq 2$, *and any* $0 < \tau < \infty$ *one has*

$$\|y^{(m)} I\|_{p,w}^n \leq \lambda_n(\tau) \|yI\|_{p,w}^{n-m} b^m, \quad 0 < m < n \tag{2.15}$$

for any $y \in C^{(n)}(J, X)$, *I compact* $\subset J$, $0 \leq \|y^{(n)} I\|_{p,w} \leq b$ *with* $0 < b < \infty$,

$$\left(\|yI\|_{p,w}/b\right)^{1/n} \leq \tau, \tag{2.16}$$

$$\lambda_n(\tau) = \left(2^{n-4}(4K)^{\binom{n}{2}} + \left(\frac{\sigma}{\tau}\right)^n\right)^n \max\left(\left(\frac{\tau}{\sigma}\right)^n, \left(\frac{\tau}{\sigma}\right)^{n(n-1)}\right), \tag{2.17}$$

$$K = K(p, w, I) = \begin{cases} 32C^{2/p}, & \text{if } 1 \leq p < \infty, \\ 4C^2, & \text{if } p = \infty, \end{cases} \text{ with} \tag{2.18}$$
$$2C \geq C_\delta, \quad \delta := \min(|I|, \delta_0),$$

$$\sigma = \sigma(p, w, I) = \begin{cases} C^{1/p}\delta, & \text{if } 1 \leq p < \infty, \\ C\delta, & \text{if } p = \infty, \end{cases} \delta,\ C \text{ as in (2.18)}. \tag{2.19}$$

Proof Since with $V := L^p_w(I, X)$ of Corollary 2.6 any $y \in C^{(2)}(I, X)$ with $y^{(j)} \in V$ for $0 \le j \le 2$ can be extended to an $z \in C^{(2)}(J, X)$, Proposition 2.1 gives $N_2(K, \sigma)$ for this V and $\| \|_{p,w}$ restricted to I, with K, σ of (2.18), (2.19). So Lemma 1.10 gives N_n, then Lemma 1.14 the L_n, with (2.15), (2.16). Here (2.13), (2.14), for minimal n_I with $|I|/n_I \le \delta$ and $2C \ge C_\delta$ one gets $2|I|/n_I \ge \delta := \min(|I|, \delta_0)$ for $p < \infty$, i.e. (2.19).

COROLLARY 2.8 *If* $\delta_0 < \infty$ *or* $|J| < \infty$, *Proposition* 2.7 *remains true if there everywhere I is replaced by J.*

If $|J| = \infty$ *and* $w \equiv 1$, (2.15) *holds with* $I = J$ *and* y *as there, but with* $\tau = \infty$ (*i.e. without* (2.16)) *and*

$$\lambda = \lambda_n(\infty) = n\left(\frac{n}{n-1} 2^{n-4} (4K)^{\binom{n}{2}}\right)^{n-1}, \quad K \text{ of (2.18)}, \quad C = \frac{1}{2}. \tag{2.20}$$

Proof The first part follows as Corollary 2.5. For the second part one can take $\tau = t|I|$ with fixed $t \in (0, \infty)$; $I \to J$ gives then, for any y as before (2.16) and $\delta_0 = \infty$, inequality (2.15) without I (also if some terms are ∞, with $0 \cdot \infty := 0$); instead of $\lambda_n(\tau)$ one gets, with suitable K', $\lambda = (K' + 1/s)^n$ $\max(s, s^{n-1})$, $s := (2^{1/p} t)^n$ if $p < \infty$ resp. $(2t)^n$ if $p = \infty$. The minimum with respect to $s \in (0, \infty)$ gives (2.20).

Remark 2.9 (a) The variable in τ in (2.16) gives less flexibility than might appear: $L_n(\lambda_0, \tau_0)$ already implies $L_n(\lambda_0 \cdot \max(1, (\tau/\tau_0)^{n(n-1)}), \tau)$ for any $\tau > 0$.

(b) In Landau's case $p = \infty$, $w \equiv 1$, $\tau = (1/2)|I|$, $X = \mathbb{R}$, for $n = 2$ our (2.17)–(2.19) give $\lambda_2 = 4$, which is optimal by Landau [20, Satz 2]; for $n \ge 3$ our λ_n are much smaller than the $\lambda_n = 2^{n2^n}$ of Landau [21, Hilfssatz 3].

(c) Even for $n = 2$, $p = \infty$, $w \equiv 1$ Proposition 2.7 is more general than Landau's result: $y''(t)$ need not exist everywhere, and in (2.15) and $\|y'' I\|_\infty \le b$ only the μ_L − sup is used.

(d) Corollary 2.8 says that $L_n(\lambda(\tau), \tau)$ is true for any J, τ and $\| \|_{p,w}$ with $\delta_0 < \infty$ or $|J| < \infty$, and $\lambda(\tau)$ independent of $|J| \ge \delta_0$; however $\lambda_n(\tau) \to \infty$ as $|J| \to 0$ as it should: Example 2.10.

(e) For $|J| = \infty$ and $w \equiv 1$, Corollary 2.8 gives even the strong $L^s_n(\lambda_n(\infty))$ with explicit λ for $\| \|_p$, $1 \le p \le \infty$ (also "Special case" after Corollary 2.5); for $p = \infty$ one has $\lambda_2(\infty) = 4$, which is optimal by Matorin [24] for $J = [0, \infty)$; for $p = \infty$ and $J = \mathbb{R}$, $\lambda = 2$ is optimal by

Hadamard and Kolmogorov, $\lambda = 1$ for $p = 2$ by Hardy–Littlewood–Polya [25, p. 5]. More can be found in [10, 18, p. 229, 25, pp. 2–7, 28, 30, p. 4 and p. 9].

For increasing w, $|J| = \infty$ and $1 \le p < \infty$ a strong L_2^s has been shown by Goldstein–Kwong–Zettl [8, p. 23, 25, p. 37 (84.3)]; for $w = t^\beta$ see [5, 18, p. 238 Theorem 4, 25, p. 51 no. 102]; for decreasing w this is false by Example 1.13. See also Remarks 2.17(c) and (f), Corollaries 2.11, 2.19, Examples 1.2, 2.3, 2.10.

(f) Proposition 2.7 and the first part of Corollary 2.8 hold also for $p = \infty$ and J, w as in Remark 2.2(e), with suitable λ.

(g) If a strong L_n^s holds for J and the seminorm $\|\ \|$ (as in (e)), then for non-negative integer m the L_n^s is also true (with the same λ) for the seminorm $\|f\|_{[m]} := \sum_{j=0}^{n} \|f^{(j)}\|$ (Hölder, $p = n/(n-k)$); special cases have been treated by Upton [30]. The same holds for the later asymptotic L_n^a of Proposition 2.16.

Example 2.10 For no $n \ge 2$, $1 \le p \le \infty$ and fixed λ, τ a $L_n(\lambda, \tau)$ holds for arbitrary J: $J = [0, \varepsilon]$, $y = t\,(+\eta \sin t)$ if $n = 2$.

For applications to differential equations, we need asymptotic Landau inequalities; under additional assumptions one gets one already from Proposition 2.7:

COROLLARY 2.11 *For J, p, w, n as in Proposition 2.1 or Remark 2.9(f), to each $y \in C^{(n)}(J, X)$ with $\|y\|_{p,w} < \infty$ and $y^{(n)} \not\equiv 0$ there exist $\lambda(y) < \infty$ and a compact $I(y)$ such that*

$$\|y^{(m)}I\|_{p,w}^{n} \le \lambda(y)\|yI\|_{p,w}^{n-m}\|y^{(n)}I\|_{p,w}^{m}, \quad I(y) \subset I \subset J, \ 0 < m < n. \tag{2.21}$$

If $|J| = \infty$ and $w \equiv 1$, then $\|yI\|_p = O(|I|^n)$ suffices for (2.21); *if even $\|yI\|_p = o(|I|^n)$, then any $\lambda(y) > \lambda_n(\infty)$ of* (2.20) *is possible in* (2.21), *independent of y.*

This follows from Proposition 2.7 with $\tau(y) = (\|y\|_{p,w}/b_0)^{1/n}$ with $b_0 = \|y^{(n)}I(y)\|_{p,w} > 0$ for some compact $I(y)$, $0 < \delta_0 \le |I(y)|$.

If $1 < p < \infty$, one gets e.g.

$$\lambda(y) = \left(2^{n-4}(4K)^{\binom{n}{2}} + \frac{1}{s}\right)^{n} \cdot \max(s, s^{n-1}), \text{ with } s \ge s(y) := \left(\frac{\tau(y)2^{1/p}}{\delta_0 C_{\delta_0}^{1/p}}\right)^{n},$$

with K of (2.18) with $C = C_{\delta_0}/2$. This works also in the case $\|yI\| = O(|I|^n)$, $\delta_0 = \infty$. For the case $o(|I|^n)$ one can argue as in the proof of the second part of Corollary 2.8.

The last statement of Corollary 2.11 follows for $p = \infty$ also (except for the explicit $\lambda_n(\infty)$) from results of Gorny [9, 25, p. 7], or Redheffer and Walter [27].

For $f \in L^p_{\mathrm{loc}}(J, X)$ and w as before (2.1) the *weighted Stepanoff norm* is defined by

$$\|f\|_{S^p_w} := \sup\{\|fI\|_{p,w}\colon |I| = 1, \text{ interval } I \subset J\}, \ 1 \le p < \infty. \quad (2.22)$$

This definition and the above results yield

COROLLARY 2.12 *If* $|J| = \infty$ *and* $1 \le p < \infty$, *then Corollary* 2.5, *Corollary* 2.6 *and Proposition* 2.7 (*with* $|I| = 1$ *in* (2.4), (2.10), (2.13), (2.18)) *hold also for* $\| \ \|_{S^p_w}$ *instead of* $\| \ \|_{p,w}$ (*also in* (2.16)).

A strong Landau inequality L^s_n for Stepanoff-norms can be found in Upton [30], for $J = \mathbb{R}$, $X = \mathbb{C}$, $w \equiv 1$.

COROLLARY 2.13 *For* $J = [\alpha, \infty)$ *resp.* $\mathbb{R}$, $1 \le p < \infty$, *w as in Proposition* 2.1, *to* $y \in C^{(n)}(J, X)$ *with* $y^{(n)} \not\equiv 0$ *and Stepanoff-norm* $\|y\|_{S^p_w} < \infty$, *there exist* $\lambda(y) < \infty$ *and a compact interval* $I(y) \subset J$ *such that*

$$\|y^{(m)}I\|^n_{S^p_w} \le \lambda(y)\|yI\|^{n-m}_{S^p_w}\|y^{(n)}I\|^m_{S^p_w}, \quad I(y) \subset I \subset J, \ 0 < m < n. \quad (2.23)$$

Proof By assumption there is t_0 with $b_0 := \|y^{(n)}I_{t_0}\|_{S^p_w} > 0$, $I_t := [t, t+1]$, so $\|y^{(n)}I\|_{S^p_w} \ge b_0$ if $I \supset I_{t_0} =: I(y)$. Furthermore $\|yI_t\|_{p,w} \le \|y\|_{S^p_w} =: a_0 < \infty$ for any $t \in J$. With $b(t) := \max(\|y^{(n)}I_t\|_{p,w}, b_0)$ one has $(\|yI_t\|_{p,w}/b(t))^{1/n} \le (a_0/b_0)^{1/n} =: \tau_0$ for any $t \in J$.

So Proposition 2.7 gives, with $\delta := \min(1, \delta_0)$, $\|y^{(m)}I_t\|^n_{p,w} \le \lambda_n(\tau_0)$ $\|yI_t\|^{n-m}_{p,w} b(t)^m \le \lambda_n(\tau_0)\|yI\|^{n-m}_{S^p_w} \max(\|y^{(n)}I\|^m_{S^p_w}, b_0^m)$ for any $t \in J$, $I_t \subset I$. For $I(y) \subset I$ this yields (2.23), with $\lambda(y) = \lambda_n(\tau_0)$ only depending on τ_0, p, δ.

LEMMA 2.14 *If* $1 \le p \le \infty$, $w\colon J \to (0, \infty)$ *with* $w(s) \ge w(t)$ *if* $s \le t$, $J = [\alpha, \beta)$, $y \in C^{(2)}(J, X)$, $y_j(x) := \|y^{(j)}[\alpha, x]\|_{p,w}$ *for* $x \in J$, *and* $y_2 \not\equiv 0$, *one has for* $\beta = \infty$

$$\overline{\lim_{x\to\infty}} \frac{y_0(x)}{x^2 y_2(x)} \le \begin{cases} (2p)^{-1/p}, & \text{if } 1 < p < \infty, \\ \dfrac{1}{2} + \dfrac{\|y'(\alpha)\| w(\infty)}{2y_2(\infty)}, & \text{if } p = 1, \\ \dfrac{1}{2}, & \text{if } p = \infty. \end{cases} \quad (2.24)$$

If $\beta < \infty$, *at least* $\overline{\lim}_{x\to\beta}\, y_0/y_2 \le \chi = \chi(p, |J|, w, y) < \infty$.

Proof Since w and y_2 are monotone, the limits $w(\infty)\geq 0$ and $0<y_2(\infty)\leq\infty$ are defined. We prove only the case $w\equiv 1$ needed below.

If $p<\infty$, to $\varepsilon>0$ there is $C\ (=2((1+\varepsilon)^{1/p}-1)^{-p})$ with

$$(u+v)^p \leq (1+\varepsilon)u^p + Cv^p, \quad u,v\in[0,\infty). \tag{2.25}$$

This and (2.5) with $x=\alpha$ yields, with $A:=\|y(\alpha)\|$, $B:=\|y'(\alpha)\|$

$$\begin{aligned}\|y(x)\|^p &\leq C(A+(x-\alpha)B)^p + (1+\varepsilon)(x-\alpha)^p\left(\int_\alpha^x |y''|\,\mathrm{d}t\right)^p\\ &\leq C(A+(x-\alpha)B)^p + (1+\varepsilon)(x-\alpha)^{p+p/q}\int_\alpha^x |y''|^p\,\mathrm{d}t. \end{aligned}\tag{2.26}$$

Integrating, one gets for $x\in J$ and $1\leq p<\infty$

$$y_0(x) \leq C^{1/p}(A+(x-\alpha)B)(x-\alpha)^{1/p} + \left(\frac{1+\varepsilon}{2p}\right)^{1/p}(x-\alpha)^2 y_2(x). \tag{2.27}$$

Since ε is arbitrary, one gets (2.24) for $p<\infty$.

$p=\infty$ has been shown in [3, (2.13)].

If $\beta<\infty$, (2.27) resp. (2.26) gives at least $\overline{\lim}\, y_0/y_2<\infty$, $1\leq p\leq\infty$.

Remark 2.15 (a) At least for $p=1$ and ∞ the constants "1/2" in (2.24) cannot be improved; see also (2.13) in [3].

(b) Lemma 2.14 becomes false for increasing w, any p (see Remark 2.17(e)).

(c) For $J=\mathbb{R}$ one can show that (2.24) still is true, with $\alpha=0$ and $y_j(x):=\|y^{(j)}[-x,x]\|_{p,w}$.

(d) For Stepanoff-norms one has $\overline{\lim}\, y_0(x)/(x^2y_2(x))\leq 1/2$ for $1\leq p<\infty$, w as in Lemma 2.14, with $I_x:=[\alpha,x]$ resp. $[-x,x]$ and $y_j(x):=\|y^{(j)}I_x\|_{S^p_w}$.

Proposition 2.16 *If $J=[\alpha,\beta]$, $n\geq 2$, $1\leq p\leq\infty$, $y\in C^{(n)}(J,X)$ with $y^{(n)}\not\equiv 0$, $y_j:=\|y^{(j)}[\alpha,x]\|_p$, $\varepsilon>0$, there exist $x_{\varepsilon,y}\in J$ with*

$$y_m(x)^n \leq (\lambda+\varepsilon)y_0(x)^{n-m}y_2(x)^m, \quad x_{\varepsilon,y}\leq x<\beta,\ 0<m<n. \tag{2.28}$$

Here for $\beta=\infty$

$$\lambda=\lambda_n(p)=(\lambda_2(p))^{n^3/8}, \tag{2.29}$$

$$\lambda_2(\infty) = \frac{9}{2}, \quad \lambda_2(p) = \left(32\left(\frac{1}{8p}\right)^{1/(2p)} + \left(\frac{p}{2}\right)^{1/(2p)}\right)^2$$

$$\textit{for } 1 < p < \infty, \tag{2.30}$$

$$\lambda_2(1) = (64(1 + A(y)))^2/A(y),$$

$$A(y) := 2 + 2\|y'(\alpha)\|/y_2(\infty).$$

For $\beta < \infty$ *one has* (2.28) *only with some* $\lambda = \lambda(n, p, |J|, y(\alpha), y'(\alpha), y_2(\infty)) < \infty$, $1 \le p \le \infty$.

The proof follows for $n = 2$ from Proposition 2.7 with $w \equiv 1$, $C = 1/2$, $\delta_0 = \infty$, $I = [\alpha, x]$, $\sigma = (1/2)^{1/p}|I|$ resp. $(1/2)|I|$, $b = y_2(x)$, using $\tau = \sqrt{\chi(p) + \eta}|I|$ with $\chi(p) =$ right-hand side of (2.24) if $\beta = \infty$, $\eta > 0$; by the assumptions, $y_2(\infty)$ is defined $\in (0, \infty]$; for $|J| < \infty$, $\tau = \sqrt{\chi + \eta}$ independent of I, $|I| \ge x_{\varepsilon,y} - \alpha > 0$ gives $\lambda < \infty$. Lemma 1.3 gives the general case.

Remark 2.17 (a) Proposition 2.16 says that for unbounded J and $V = L^p(J, X)$ with $1 < p \le \infty$ an asymptotic Landau inequality L_n^a is true. For $p = \infty$ this generalizes Lemma 2.5 of [3]. For $p = 1$ or bounded J only a "pointwise $L_n^{\bullet a}$" holds, the λ depends on y; in all these cases exist $y \in C^\infty$ with arbitrarily large λ. These examples and Corollary 2.8 show also that $L_n^s \Rightarrow L_n^a$ is false for $p = 1$, $|J| = \infty$, $n = 2$.

(b) (2.30) yields $\lambda_2(1+) = 144\frac{1}{2} < \lambda_2(p) < \lambda_2(\infty-) = 33^2$ for $1 < p < \infty$.

(c) Corollary 2.11 gives more general (pointwise) asymptotic Landau inequalities if y is bounded in some way; for example if $y_0(x) = o(x^2)$ in Proposition 2.16, then $\lambda = \lambda_n(\infty)$ of (2.20) is possible in (2.28), which is in general better than the λ of (2.29). See Remarks 2.9 and after (1.8), and (1.7).

(d) Proposition 2.16 and the remarks hold also for $J = \mathbb{R}$ with e.g. $y_j(x) := \|y^{(j)}[-x, x]\|_p$, and the same $\lambda_n(p)$ if $1 < p \le \infty$. This, (a), (1.7) and Lemma 1.4 give again L_n^s (for $1 < p \le \infty$) of Remark 2.9(e).

(e) In all four cases J bounded or unbounded and w decreasing or increasing, there exist w and y showing that for no $\lambda < \infty$ and no p an asymptotic Landau inequality L_2^a is true for $V = L_w^p(J, \mathbb{R})$.

(f) The examples (e) show also that for no λ and p a strong Landau inequality L_2^s holds for general $\| \ \|_{p,w}$, except in the case $|J| = \infty$ and w increasing (the $y^{(j)}$ of (e) are $\in L_w^p$; except: [8], Remark 2.9(e)).

For the following Halperin–Pitt inequality we assume:
J interval $\subset \mathbb{R}$, V $\mathbb{K}$-vectorspace $\subset X^J$ with monotone seminorm $\|\ \|$, i.e. $\|f\| \le \|g\|$ if $f, g \in V$ with $|f| \le |g|$, and with $1I \in V$ for I compact interval $\subset J$,

$$\|1I\| \to 0 \quad \text{as } |I| \to 0, \tag{2.31}$$

$$C_1 \int_I |f|\,\mathrm{d}x \le \|fI\| \quad \text{if } fI \in V \cap L^1,\ 0 < C_1 \text{ independent of } f, I. \tag{2.32}$$

Proposition 2.18 *For $J, V, \|\ \|$ as above and $0 < r < \infty$ there exists $S : (0,\infty) \to (0,\infty)$ such that for any compact interval $I \subset J$ with $|I| \ge r$ and $y \in C^{(2)}(J, X)$ with $y^{(j)}\tilde{I} \in V$, $0 \le j \le 2$ (see (0.3)), $\tilde{I}$ compact $\subset J$, one has*

$$\|y'I\| \le \varepsilon\|y''I\| + S(\varepsilon)\|yI\|,\ 0 < \varepsilon < \infty,\ |I| \ge r. \tag{2.33}$$

Proof To $\varepsilon > 0$ choose δ_ε with $\|1M\| \le C_1\varepsilon$ if $|M| \le \delta_\varepsilon$, M compact interval $\subset J$, then n minimal $\in \mathbb{N}$ with $|I|/n < \delta := \min(\delta_\varepsilon, r)$, and compact intervals I_j with $|I_j| = |I|/n$, $I = \bigcup I_j$. With (2.7) one gets for $y \in C^{(2)}(J, X)$ with $y^{(j)}\tilde{I} \in V$

$$\begin{aligned}
\|y'I\| &\le \left\|\sum |y'|I_j\right\| \le \sum \|y'I_j\| \le \sum \|y'I_j\|_\infty \|1I_j\| \le \sum \|y'I_j\|_\infty C_1\varepsilon \\
&\le C_1\varepsilon \sum\left(\int_{I_j} |y''|\,\mathrm{d}x + 4(|I|/n)^{-2}\int_{I_j} |y|\,\mathrm{d}x\right) \\
&\le C_1\varepsilon\left(\int_I |y''|\,\mathrm{d}x + 4(\delta/2)^{-2}\int_I |y|\,\mathrm{d}x\right) \le \varepsilon\|y''I\| + S(\varepsilon)\|yI\|,
\end{aligned}$$

with

$$S(\varepsilon) := \frac{16\varepsilon}{(\min(\delta_\varepsilon, r))^2}, \quad \text{with } \|1M\| \le C_1\varepsilon \text{ if } |M| \le \delta_\varepsilon, \quad M \text{ compact interval} \subset J. \tag{2.34}$$

Special case $V = L^1$: Then one gets Proposition 2.1, $p = 1$, $w \equiv 1$, with $K = 16$, $\sigma = r = |I|$.

Corollary 2.19 *Proposition 2.18 holds for $\|\ \|_{p,w}$, $1 \le p < \infty$ and the weightfunction w satisfying $\inf_J w > 0$ and w integrable over J, with $|J| < \infty$.*

Proposition 2.18 can be applied to Orlicz-norms (see [17,23,31]): A $\Phi : [0,\infty) \to [0,\infty]$ will be called an *Orlicz-function* (OF) iff $\Phi(0)=0$, $\Phi \not\equiv 0$, $\Phi \not\equiv \infty$ on $(0,\infty)$, and Φ is convex. Then for a measure space (Y, Ω, μ), $L^\Phi(\mu, X) := \{f: Y \to X \mid f$ Bochner – μ – measurable, $\int \Phi(t|f|)\,\mathrm{d}\mu \leq 1$ for some $t \in (0,\infty)\}$, Orlicz–Luxenburg norm $\|f\|_\Phi := \inf\{s>0: \int \Phi(|f|/s)\,\mathrm{d}\mu \leq 1\}$.

For OF Φ, L^Φ is a $\mathbb{K}$-vectorspace and $\| \ \|_\Phi$ a monotone seminorm on L^Φ, with $\|f\|_\Phi = 0$ iff $f=0$ μ-a.e. For any OF Φ, $\Psi(t) := \sup\{st - \Phi(t): 0 \leq s < \infty\}$ defines a "conjugate" OF such that for $f \in L^\Phi(\mu, X)$, $g \in L^\Psi(\mu, \mathbb{K})$ one has $fg \in L^1(\mu, X)$ (usual L^1) and $\int_Y |fg|\,\mathrm{d}\mu \leq 2\|f\|_\Phi \|g\|_\Psi$.

For $Y=$ interval $J \subset \mathbb{R}$, $\Omega=$ Lebesgue measurable sets $\subset J$ and $\mu=$ Lebesgue measure μ_{L} we write $L^\Phi = L^\Phi(J,X) := L^\Phi(\mu_L \mid J, X)$, then $0 < \|1J\|_\Psi < \infty$ if $0 < |J| < \infty$, so (2.32) holds with $C_1 = 1/(2\|1J\|_\Psi)$. (2.31) is true if $\Phi(t) < \infty$ for $0 < t < \infty$:

COROLLARY 2.20 *If Φ is an OF with $\Phi(t) < \infty$ for $0 < t < \infty$ and $|J| < \infty$, then Proposition* 2.18 *is true for* $\| \ \| = \| \ \|_\Phi$, $V = L^\Phi(J, X)$.

Question: Is such an asymptotic Halperin–Pitt inequality also true for $|J| = \infty$? By Ha Huy Bang [10] at least a strong Landau inequality holds for $J=\mathbb{R}$, $X=\mathbb{C}$ and Orlicz-norms.

3 ESCLANGON–LANDAU THEOREMS FOR NEUTRAL SYSTEMS

In the following, we consider neutral delay differential-difference systems

$$\sum_{k=0}^{n} \sum_{j=1}^{m} a_{jk}(t) y^{(k)}(t - t_j) = f(t); \tag{3.1}$$

here $n \geq 1$, $m \geq 1$, $J = [\alpha, \beta)$ with $-\infty < \alpha < \beta \leq \infty$, $t_1 = 0 < t_j \leq \tau < \infty$ for $1 < j \leq m$, $J' := [\alpha - \tau, \beta)$, $f: J \to X$ Banach space over $\mathbb{K}$, $a_{jk}: J \to L(X) := \{$continuous linear operators$: X \to X\}$ with operator norm. t_{jk} are not more general.

y is called a *solution* of (3.1) on J if $y \in C^{(n)}(J', X)$ and (3.1) holds a.e. on J, with $y^{(n)}$ of (0.5).

Systems of such equations are included: $a_{jk} = r \times r$-matrix $(a_{jk,uv})$, $y=$ column vector $(y_1, \ldots, y_r)$.

Furthermore we assume that V is a $\mathbb{K}$-linear space $\subset X^J$ with monotone seminorm $\| \ \|$ satisfying

$$f, g \in V, \ |f| = |g| \text{ a.e. implies } \|f\| = \|g\|, \tag{3.2}$$

there is $D_\tau < \infty$ with $\|g_t I\| \leq D_\tau \|g I_t\|$ for $0 \leq t \leq \tau$, I compact (3.3)
interval with $I, I_t \subset J$ and $g : J' \to X$ with $g_t I \mid J$ and $g I_t \mid J \in V$,

where $g_t(s) := g(s-t)$, $I_t := \{s - t : s \in I\}$.

THEOREM 3.1 *Assume $m, n, t_j, J, X, V, \| \ \|$ as above with* (3.2), (3.3); *assume further that the coefficients a_{jk} in* (3.1) *are bounded on J, with*

$$D_\tau \cdot \sum_{j=2}^{m} \sup_J |a_{jn}| < 1, \quad a_{1n} = 1. \tag{3.4}$$

Assume finally that a pointwise asymptotic Halperin–Pitt inequality $H_2^{\bullet a}$ holds for V, $\| \ \|$ (Definition 1.16). *If then y is a solution of* (3.1) *on J, with $fI, y_{t_j}^{(k)} I \mid J \in V$ for all compact intervals $I \subset J$, $0 \leq k \leq n$, $1 \leq j \leq m$, such that $\|f I_x\|$ and $\|y I_x | J\|$ are $O(\Psi(x))$ for $x \to \beta$ with some non-decreasing $\Psi > 0$ for $I_x = [\alpha, x]$, then also $\|y^{(k)} I_x \mid J\| = O(\Psi(x))$, $0 < k \leq n$.*

Proof With (3.2) and $\|g\| \leq \|h\| = \|(|h|)\|$ if $|g| \leq |h|$ on J, $g, h \in V$ one gets for $\alpha \leq x < \beta$, if $\|a_{jk}(t)\| \leq A$ for $t \in J$ (measurability of the a_{jk} is not needed)

$$\begin{aligned}
\|(y^{(n)} I_x) \mid J\| &= \left\| f I_x - \left(\sum_{j=2}^{m} a_{jn} y_{t_j}^{(n)} + \sum_{k=0}^{n-1} \sum_{j=1}^{m} a_{jk} y_{t_j}^{(k)} \right) I_x \,\middle|\, J \right\| \\
&\leq K_1 \Psi(x) + \sum_{j=2}^{m} \sup_J |a_{jn}| \cdot \|y_{t_j}^{(n)} I_x \mid J\| \\
&\quad + mnA \max\{\|y_{t_j}^{(k)} I_x \mid J\| : \ 1 \leq j \leq m, \ 1 \leq k < n\}.
\end{aligned}$$

(3.3) and the monotonicity of $\| \ \|$ give, for $1 \leq j \leq m$, $0 \leq k \leq n$

$$\|(y_{t_j}^{(k)} I_x) \mid J\| \leq \|(y_{t_j}^{(k)} [\alpha, \alpha + \tau]) \mid J\| + D_\tau \|(y^{(k)} I_x) \mid J\|.$$

So if all $\|(y_{t_j}^{(k)} [\alpha, \alpha + \tau]) \mid J\| \leq B$, $< \infty$ by assumption, with suitable K_1 one gets for $x \in J$

$$\begin{aligned}
\|(y^{(n)} I_x) \mid J\| &\leq K_1 \Psi(x) + \left(\sum_{2}^{m} \sup_J |a_{jn}| \right) (B + D_\tau \|(y^{(n)} I_x) \mid J\|) \\
&\quad + mnA(B + \max\{\|(y^{(k)} I_x) \mid J\| : \ 0 \leq k < n\}).
\end{aligned}$$

By assumption $2\eta := 1 - D_\tau \sum_2^m \sup_J |a_{jn}| > 0$, one has for $x \in J$

$$2\eta \|(y^{(n)} I_x) \mid J\| \leq m(n+1)AB + K_1 \Psi(x) + mnA \max_{k<n} \|(y^{(k)} I_x) \mid J\|. \tag{3.5}$$

$H_2^{\bullet a}$ and Lemma 1.19 imply $H_n^{\bullet a}$, so to the given y there exist $c(y) \in J$ and $S : (0, \infty) \to [1, \infty)$ with

$$\|(y^{(k)} I_x) \mid J\| \leq \varepsilon \|(y^{(n)} I_x) \mid J\| + S(\varepsilon) \|(y I_x) \mid J\|,$$
$$c(y) \leq x < \beta, \ \ 0 < k < n, \ \ 0 < \varepsilon.$$

Choosing $\varepsilon = \eta/(mnA)$, (3.5) yields

$$\|(y^{(n)} I_x) \mid J\| \leq m(n+1)AB + K_1 \Psi(x) + mnA \cdot S(\varepsilon) \|(y I_x) \mid J\|,$$
$$c(y) \leq x < \beta.$$

Since by assumption $\|(y I_x) \mid J\| = O(\Psi(x))$ and Ψ is non-decreasing $\geq \Psi(\alpha) > 0$, one gets with suitable $K_2 < \infty$

$$\|(y^{(n)} I_x) \mid J\| \leq K_2 \Psi(x), \quad x \in J.$$

$H_n^{\bullet a}$ gives the same for $y^{(k)}$, $0 < k < n$.

COROLLARY 3.2 *Theorem 3.1 holds in the following cases, with* m, n, J, t_j, X, (3.1), (3.4) *as there*

(*a*) $\| \ \| = \| \ \|_{p,w}$ *of* (2.2), $1 \leq p \leq \infty$, *w as in Proposition* 2.1, *and only* $(y_{t_j}^{(n)} I_0) \mid J \in V = L_w^p(J, X)$; *here* $D_\tau = C_\tau$ *of* (2.1), $I_0 = [\alpha, \alpha + \min_{2 \leq j} t_j]$ *resp.* $\emptyset$.

(*b*) $\| \ \| =$ *Stepanoff-norm* $\| \ \|_{S_w^p}$ *of* (2.22), $1 \leq p < \infty$, $J = [\alpha, \infty)$, *w and* D_τ *as in* (*a*).

(*c*) $\| \ \|$ *as in Proposition* 2.18 *with* (3.3); *special case: Orlicz-norm* $\| \ \|_\Phi$ *with Lebesgue measure as in Corollary* 2.20, $|J| < \infty$. (*Then* $D_\tau = 1$ *in* (3.4).)

(*d*) $\| \ \|_{\infty,w}$, *w decreasing*, (3.4) *with* $D_\tau = 1$.

(*e*) $J = \mathbb{R}$ *and* $m = 1$, *with* $I_x = [-x, x]$ (*all* $t_j = 0$, *ordinary differential system* (3.1), *so* $D_\tau = 1$ *in* (3.4)); *especially for seminorms as in* (*a*) *and* (*b*).

COROLLARY 3.3 *With m, n, t_j, J, X, (3.1), (3.4) as in Theorem 3.1 and y a solution of (3.1) on J with $y^{(n)} \mid [\alpha-\tau, \alpha] \in L^p$, if f and $y \mid J \in V = L^p_w(J, X)$, then also $y^{(k)} \mid J \in V$, $0 < k \leq n$, with p, w as in Corollary 3.2(a). A corresponding result holds for $V = \{g : J \to X$ Bochner–Lebesgue measurable: $\|g\|_{S^p_w} < \infty\}$ (as in Corollary 3.2(b), resp. Orlicz-space $L^\Phi(J, X)$ with $|J| < \infty$ (Corollary 3.2(c)), resp. $J = \mathbb{R}$ etc. as in Corollary 3.2(e).*

Proof of Corollary 3.2(a)

(a) Since in Proposition 2.1 the K and σ are independent of $I \subset J$ with $|I| \geq$ some $\delta_0 > 0$ (and of y), one has even an asymptotic N_2^a and therefore H_2^a. For (3.3), (3.4) one can use $D_\tau = C_\tau$ defined by (2.1), $< \infty$ if some $C_{\delta_0} < \infty$. Since the $y^{(k)}$ are continuous for $0 \leq k < n$, automatically all these $(y_{t_j}^{(k)} I) \mid J \in L^p_w$ or equivalently $\in L_p$ (w and $1/w$ are locally bounded); for $p = 1$ also $(y_{t_j}^{(n)} I) \mid J \in L^p_w$ by the definition (0.4) of $C^{(n)}(J, X)$.

(b) Follows as (a) with Corollary 2.12, case Corollary 2.5, for $\| \ \|_{S^p_w}$ and $I = I_x$, $\alpha + 1 \leq x < \infty$.

(c) Proposition 2.18 gives an asymptotic Halperin–Pitt inequality H_2^a since in (2.33/34) the $S(\varepsilon)$ does not depend on I (and y); this implies the pointwise $H_2^{\bullet a}$. H_2^a holds especially for Orlicz-norms $\| \ \|_\Phi$ by Corollary 2.20, here the definition of $\| \ \|_\Phi$ gives equality in (3.3) with $D_\tau = 1$.

(d) Remark 2.2(e); (3.3) holds with $D_\tau = 1$ since w is decreasing.

(e) Use the transformation $t \to -t$, $J = [0, \infty)$, in (a)–(d).

For Corollary 3.3, $(y^{(n)}[\alpha, \alpha + k\rho]) \mid J \in L^p_w$ follows by induction on k with (3.1) if $\rho := \min\{t_2, \ldots, t_m\} > 0$; if $V =$ Orlicz-space L^Φ and $m > 1$, the starting assumption needed is $(y_{t_j}^{(n)}[\alpha, \alpha + \rho]) \mid J \in L^\Phi$ for $2 \leq j \leq m$.

Remark 3.4

(a) The special case $p = \infty$, $m = 1$, $w \equiv 1$, $X = \mathbb{R}$ is the classical Esclangon–Landau theorem of [21, Satz 1]; in [3] this has been extended to $m > 1$ and general X.

(b) Without (3.4) Theorem 3.1 and the corollaries are in general false: Example 5.3 and Remarks 5.8 in [3].

(c) $a_{1n} = 1$ in (3.4) can be replaced by "a_{1n} uniformly invertible", i.e. $\|a_{1n}(t)v\| \geq \eta_0 \|v\|$ for $v \in X$, $t \in J$, with some $\eta_0 > 0$, and D_τ/η_0 in (3.4) instead of D_τ.

(d) Usually the condition "$y_{t_j}^{(k)} I \mid J \in V$ for all j, k, compact I" can be weakened to $(y_{t_j}^{(n)}[\alpha, \alpha + \rho]) \mid J \in V, \rho = \min\{t_2, \ldots, t_m\}$, so for $m = 1$ (or $V = L_w^1$) it can be omitted entirely: see Corollaries 3.2(a), 3.3.
(e) Corollaries 3.2 and 3.3 are even for bounded J (and $m = 1$, $X = \mathbb{R}$) non-trivial if $p < \infty$. For $p = \infty$ see Remark 2.9(b) of [3]; then essentially $w \equiv 1$ by Remark 2.2(c) here, except in Corollary 3.2(d).
(f) For $p = \infty$ one can admit arbitrary variable $t_j : J \to [0, \tau]$ in corollary 3.2(a) and (d), provided $y^{(n)}$ is continuous.
(g) For decreasing w as in Corollary 3.2(d) and $1 \leq p < \infty$, one can get at least the boundedness of $w(x) \cdot \int_\alpha^x |y^{(k)}|^p \, dt$, $0 < k \leq n$, if $\|yI_x \mid J\|_{p,w}$ and $\|fI_x\|_{p,w}$ are bounded: $\Psi = w^{-1/p}$, $\| \|_p$ in Theorem 3.1.
(h) In Theorem 3.1 one can also use $\|g\| = \int_1^\infty \|g\|_p \, d\mu(p)$, μ Borel measure on $[1, \infty)$ with e.g. compact support.
(i) An analogue, where "bounded" is replaced by "uniformly continuous", can be found in [3, Corollary 3.3(a), Theorem 4.1].

Example 3.5 By glueing together f_n as in example 2.4, to any $-\infty < \alpha < \beta \leq \infty$ one can construct $f \in C^2(\mathbb{R}, \mathbb{R})$ and a monotone norm on the piecewise continuous bounded functions: $J \to \mathbb{R}$ with (1.8), such that $\|f\| \leq 1$, $\|f''\| \leq 1$, but $\|f'I\| \to \infty$ as I compact $\to J = [\alpha, \beta)$, and $f \equiv 0$ on some $[\alpha, \alpha + \varepsilon]$.

Complementing example 2.4, this shows that here the pointwise asymptotic $L_2^{\bullet a}$, $N_2^{\bullet a}$ and even $S_2^{\bullet a}$ are false. It shows further that already for the equation $y'' = f$ Theorem 3.1 becomes false without $S_2^{\bullet a}$.

With an asymptotic Landau inequality one gets Esclangon–Landau results even for some non-linear functional differential equations/inequalities of Landau type [21, p. 179]:

In the following we assume $m, n, t_j, J, X, V, \| \|$ with (3.2), (3.3) and a_{jn} with (3.4) as in Theorem 3.1, $y \in C^{(n)}(J', X)$, $y_{t_j}(t) := y(t - t_j)$, all $(y_{t_j}^{(k)} I \mid J \in V$ for compact $I \subset J$, $1 \leq j \leq m$, $0 \leq k \leq n$, with $y = 0$ on I if $\|yI \mid J\| = 0$. Then calculations similar as for Theorem 3.1 yield

PROPOSITION 3.6 *If a.e. on J*

$$\sum_{j=1}^{m} a_{jn}(t) y_{t_j}^{(n)}(t) = F(t, \ldots, y_{t_j}^{(k)} \mid [\alpha, t], \ldots), \tag{3.6}$$

if with finitely many constant real $c_\gamma > 0$, $\gamma = (\gamma_{j,k})$ multiindex with $1 \leq j \leq m$, $0 \leq k \leq n - 1$ and $\gamma_{jk} \in \mathbb{R}$ one has (with $0^0 := 1$) for all compact *intervals* $I \subset J$

$$\|F(\cdot, \ldots, y_{t_j}^{(k)}|[\alpha, \cdot], \ldots)I\| \le \sum_{\gamma} c_\gamma \prod_{j,k} \|y_{t_j}^{(k)} I \,|\, J\|^{\gamma_{j,k}}, \tag{3.7}$$

where $n - \sum_{j,k} k\gamma_{jk} \ge \varepsilon > 0$ *if* $c_\gamma \neq 0$, *and if a pointwise asymptotic Landau inequality* $L_2^{\bullet a}$ *holds for* V, $\|\;\|$, *then as* $x \to \beta$, $0 < k \le n$, *with* $\delta := \max\{\sum_{j,k}(n-k)\gamma_{jk}\colon c_\gamma \neq 0\}$, $I_x := [\alpha, x]$

$$\|(y^{(k)} I_x) \,|\, J\| = O\big(\|(yI_x) \,|\, J\|^{1+k((\delta/\varepsilon)-(1/n))}\big). \tag{3.8}$$

This can be applied to systems (3.6) with $y =$ column vector $(y_1, \ldots, y_r)$ with values in X^r, a_{jn} matrix valued with components $a_{jn,uv}(t) \in L(X)$, with $a_{1n} \equiv$ unit matrix, $X =$ Banach algebra, $\gamma = (\gamma_{jku})$ with γ_{jku} integers ≥ 0, and the νth component of F a polynomial with boundet $a_{\gamma,\nu}$ of the form

$$F_\nu(t, \ldots, y_{t_j}^{(k)} \mid [\alpha, t], \ldots) = \sum_{\gamma} a_{\gamma,\nu}(t) \prod_{j,k,u} \Big(y_{u,t_j}^{(k)}(t)\Big)^{\gamma_{jku}} \tag{3.9}$$

with $n - \sum_{j,k,u} k\gamma_{jku} \ge \varepsilon_0 > 0$ for all γ with some $a_{\gamma,\nu} \not\equiv 0$: With $\|\;\| = \|\;\|_\infty = \mu_{\mathrm{L}} - \sup_J$ and X^r instead of X in Proposition 3.6, with Proposition 2.16 one gets an extension of Theorem 2.8 of [3].

Another application is $p = 1$, $\|\;\|_1$, $X = \mathbb{C}$ (or Banach algebra), F as in (3.9) and the products in $\prod_{jku}$ being convolution ($\alpha = 0$)

$$(f * g)(t) := \int_0^t f(s) g(t-s)\,\mathrm{d}s, \quad t \ge 0;$$

then (3.7) follows with $\|(f*g)I\|_1 \le \|fI\|_1 \cdot \|gI\|_1$, $L_n^{\bullet a}$ holds by Proposition 2.16.

Let as finally remark also additional terms/variables of the form $\int_{t-h}^t g(s)\,\mathrm{d}s$ or $\sup_{[t-h,t]} g$, with $g = y_{t_j}^{(k)}$ or $|y_{t_j}^{(k)}|$ and $0 < h \le \tau$, are possible, since

$$\left\| \frac{1}{h} \int_{t-h}^t f\,\mathrm{d}s \right\|_p \le \|f\|_p, \quad f \in L^p(\mathbb{R}, X), \;\; 1 \le p \le \infty, \;\; 0 < h < \infty. \tag{3.10}$$

(Almost) Periodic solutions of such equations have been considered in Bantsur and Trofimchuk [1] and the references there.

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