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A Remark on Polynomial Norms and Their Coefficients

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This paper presents new lower bounds for the norms of 2-homogeneous real-valued polynomials on l_p spaces for $0 < p \leq \infty$ which are sharper than those recently given by the author.

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This note is concerned with the general problem of the relation between the norm of a polynomial and its coefficients. This type of problem has been studied in many contexts [1–6,8–11] over the years, because of both its relevance to non-trivial problems in mathematics and because of its our inherent interest. In this note we focus our attention on lower bounds for the norms of 2-homogeneous real-valued polynomials on l_p spaces. Recently the author [9] gave lower bounds for the norms of 2-homogeneous real-valued polynomials on l_p spaces for $0 < p \leq \infty$. We here improve them.

Let E be a real Banach space with the unit sphere S_E and $m \geq 2$, a natural number. $\mathcal{P}(^mE)$ denotes the Banach space of m-homogeneous real-valued polynomials on E, endowed with the polynomial norm

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$\|P\| = \sup_{\|x\| \leq 1} |P(x)|$. See Dineen [7] for more details about the theory of polynomials on Banach spaces.

LEMMA 1 *Let $\emptyset \neq S \subset S' \subset \mathbf{N}$ and $x, a_{ij} \in \mathbf{R}$ for $i, j \in S'$ with $i<j$. Then*

$$\max_{\epsilon_k=\pm 1,\ k\in S} \left| x + \sum_{i,j\in S,\ i<j} a_{ij}\epsilon_i\epsilon_j \right| \leq \max_{\epsilon_k=\pm 1,\ k\in S'} \left| x + \sum_{i,j\in S',\ i<j} a_{ij}\epsilon_i\epsilon_j \right|.$$

Proof Let $m \geq 2$ be a positive integer. It is enough to show that

$$\max_{\epsilon_k=\pm 1,\ 1\leq k\leq m} \left| x + \sum_{1\leq i<j\leq m} a_{ij}\epsilon_i\epsilon_j \right| \leq \max_{\epsilon_k=\pm 1, 1\leq k\leq m+1} \left| x + \sum_{1\leq i<j\leq m+1} a_{ij}\epsilon_i\epsilon_j \right|.$$

Let

$$M = \max_{\epsilon_k=\pm 1,\ 1\leq k\leq m} \left| x + \sum_{1\leq i<j\leq m} a_{ij}\epsilon_i\epsilon_j \right| = \left| x + \sum_{1\leq i<j\leq m} a_{ij}\epsilon_i'\epsilon_j' \right|$$

for some sign choices $\epsilon_1', \ldots, \epsilon_m'$. Let

$$\epsilon_{m+1}' = \operatorname{sign}\left(\sum_{1\leq i\leq m} a_{im+1}\epsilon_i' \right)$$

if $x + \sum_{1\leq i<j\leq m} a_{ij}\epsilon_i'\epsilon_j' \geq 0$ and

$$\epsilon_{m+1}' = -\operatorname{sign}\left(\sum_{1\leq i\leq m} a_{im+1}\epsilon_i' \right)$$

otherwise. Then we have

$$\max_{\epsilon_k=\pm 1,\ 1\leq k\leq m+1} \left| x + \sum_{1\leq i<j\leq m+1} a_{ij}\epsilon_i\epsilon_j \right| \geq \left| x + \sum_{1\leq i<j\leq m+1} a_{ij}\epsilon_i'\epsilon_j' \right|$$

$$= \left| x + \sum_{1\leq i<j\leq m} a_{ij}\epsilon_i'\epsilon_j' \right| + \left| \sum_{1\leq i\leq m} a_{im+1}\epsilon_i' \right| \geq M.$$

LEMMA 2 *Let $m \geq 2$ be a positive integer. Let x, a_{ij} $(1 \leq i < j \leq m) \in \mathbf{R}$. Then*

$$\max_{\epsilon_k = \pm 1,\ 1 \leq k \leq m} \left| x + \sum_{1 \leq i < j \leq m} a_{ij}\epsilon_i\epsilon_j \right| \geq |x| + \max_{1 \leq i < j \leq m} |a_{ij}|.$$

The equality holds if and only if the following conditions are satisfied. Without loss of generality, assume that $\max_{1 \leq i < j \leq m} |a_{ij}| = |a_{12}|$.

(a) $a_{ij} = 0$ *for* $3 \leq i < j \leq m$.
(b) $x a_{12} a_{1i} a_{2i} \leq 0$ *and* $|a_{1i}| = |a_{2i}|$ *for each* $3 \leq i \leq m$.
(c) $\sum_{3 \leq i \leq m} |a_{1i}| \leq \min\{|x|, |a_{12}|\}$.

Proof Use induction on m. If $m = 2$, then the lemma is true because $\max\{|x + a_{12}|, |x - a_{12}|\} = |x| + |a_{12}|$. Suppose that the lemma is true for $2, 3, \ldots, m-1$. Without loss of generality, we may assume that $\max_{1 \leq i < j \leq m} |a_{ij}| = |a_{34}|$. Put

$$M = \max_{\epsilon_k = \pm 1,\ 1 \leq k \leq m} \left| x + \sum_{1 \leq i < j \leq m} a_{ij}\epsilon_i\epsilon_j \right|.$$

Substituting $\epsilon_1 = \pm 1$, we get

$$\max_{\epsilon_k = \pm 1,\ 2 \leq k \leq m} \left| \left(x + \epsilon_2 \left(\sum_{3 \leq j \leq m} a_{2j}\epsilon_j \right) + \sum_{3 \leq i < j \leq m} a_{ij}\epsilon_i\epsilon_j \right) + \left(a_{12}\epsilon_2 + \sum_{3 \leq j \leq m} a_{1j}\epsilon_j \right) \right| \leq M \tag{1}$$

and

$$\max_{\epsilon_k = \pm 1,\ 2 \leq k \leq m} \left| \left(x + \epsilon_2 \left(\sum_{3 \leq j \leq m} a_{2j}\epsilon_j \right) + \sum_{3 \leq i < j \leq m} a_{ij}\epsilon_i\epsilon_j \right) - \left(a_{12}\epsilon_2 + \sum_{3 \leq j \leq m} a_{1j}\epsilon_j \right) \right| \leq M. \tag{2}$$

By adding (1) and (2) and the triangle inequality, we get

$$\max_{\epsilon_k=\pm 1,\ 2\le k\le m}\left|x+\epsilon_2\left(\sum_{3\le j\le m} a_{2j}\epsilon_j\right)+\sum_{3\le i<j\le m} a_{ij}\epsilon_i\epsilon_j\right|\le M. \qquad (3)$$

Again, by substituting $\epsilon_2=\pm 1$ into (3) and adding each other and the triangle inequality, we get

$$\max_{\epsilon_k=\pm 1,\ 3\le k\le m}\left|x+\sum_{3\le i<j\le m} a_{ij}\epsilon_i\epsilon_j\right|\le M. \qquad (4)$$

By induction hypothesis and (4), we have

$$\begin{aligned}|x|+\max_{1\le i<j\le m}|a_{ij}| &= |x|+|a_{34}|=|x|+\max_{3\le i<j\le m}|a_{ij}|\\ &\le \max_{\epsilon_k=\pm 1,\ 3\le k\le m}\left|x+\sum_{3\le i<j\le m} a_{ij}\epsilon_i\epsilon_j\right|\le M.\end{aligned}$$

Suppose that conditions (a)–(c) are satisfied. From now on, we will assume that

$$\max_{1\le i<j\le m}|a_{ij}|=|a_{12}|.$$

Then by (a),

$$\begin{aligned}M=\max_{\epsilon_k=\pm 1,\ 3\le k\le m}\Bigg\{&|x+a_{12}|+\left|\sum_{3\le i\le m}(a_{1i}+a_{2i})\epsilon_i\right|,\\ &|x-a_{12}|+\left|\sum_{3\le i\le m}(a_{1i}-a_{2i})\epsilon_i\right|\Bigg\}. \qquad (5)\end{aligned}$$

Without loss of generality, assume that $xa_{12}\ge 0$. Then by (b),

$$|x+a_{12}|+\left|\sum_{3\le i\le m}(a_{1i}+a_{2i})\epsilon_i\right|=|x+a_{12}|=|x|+|a_{12}|$$

for any sign choices $\epsilon_3, \dots, \epsilon_m$ and, by (b) and (c),

$$\begin{aligned} |x - a_{12}| + \left| \sum_{3 \le i \le m} (a_{1i} - a_{2i})\epsilon_i \right| &\le |x - a_{12}| + 2 \sum_{3 \le i \le m} |a_{1i}| \\ &\le |x - a_{12}| + 2\min\{|x|, |a_{12}|\} \\ &= |x| + |a_{12}| \end{aligned} \tag{6}$$

for any sign choices $\epsilon_3, \dots, \epsilon_m$. Thus $M = |x| + |a_{12}|$. Let us prove the necessary condition. First we will prove it when $m = 4$. Some computation shows that

$$\begin{aligned} &\max_{\epsilon_k = \pm 1,\ 1 \le k \le 4} \left| x + \sum_{1 \le i < j \le 4} a_{ij}\epsilon_i\epsilon_j \right| \\ &= \max\{|x + a_{12} \pm a_{34}| + |(a_{13} + a_{23}) \pm (a_{14} + a_{24})|, \\ &\qquad |x - a_{12} \pm a_{34}| + |(a_{13} - a_{23}) \pm (a_{14} - a_{24})|\}. \end{aligned}$$

By some calculation, we get

(a) $a_{34} = 0$.
(b) $x a_{12} a_{1i} a_{2i} \le 0$ and $|a_{1i}| = |a_{2i}|$ for $i = 3, 4$.
(c) $\sum_{3 \le i \le 4} |a_{1i}| \le \min\{|x|, |a_{12}|\}$.

Let $m \ge 4$. Suppose that $M = |x| + |a_{12}|$. Let $3 \le i_0 < j_0 \le m$ be fixed. Let σ be the permutation on $\{1, 2, \dots, m\}$ such that

$$\sigma(3) = i_0, \quad \sigma(4) = j_0, \quad \sigma(i_0) = 3, \quad \sigma(j_0) = 4.$$

Define $b_{ij} = a_{\sigma(i)\sigma(j)}$ for each $1 \le i < j \le m$. By Lemma 1,

$$\max_{\epsilon_k = \pm 1,\ 1 \le k \le 4} \left| x + \sum_{1 \le i < j \le 4} b_{ij}\epsilon_i\epsilon_j \right| \le \max_{\epsilon_k = \pm 1,\ 1 \le k \le m} \left| x + \sum_{1 \le i < j \le m} b_{ij}\epsilon_i\epsilon_j \right| = M$$

and by the first claim of Lemma 2,

$$\begin{aligned} \max_{\epsilon_k = \pm 1,\ 1 \le k \le 4} \left| x + \sum_{1 \le i < j \le 4} b_{ij}\epsilon_i\epsilon_j \right| &\ge |x| + \max_{1 \le i < j \le 4} |b_{ij}| \\ &= |x| + |a_{12}| = M, \end{aligned}$$

so

$$\max_{\epsilon_k=\pm 1,\ 1\le k\le 4}\left|x+\sum_{1\le i<j\le 4} b_{ij}\epsilon_i\epsilon_j\right| = |x| + \max_{1\le i<j\le 4}|b_{ij}|.$$

By the above argument for $m=4$ case, we have $0=b_{34}=a_{i_0j_0}$ and $xb_{12}b_{13}b_{24}=xa_{12}a_{1i_0}a_{1j_0}\le 0$ and $|a_{1i_0}|=|b_{13}|=|b_{24}|=|a_{1j_0}|$, showing (a) and (b). Suppose that (c) is not true. By (a), (b), (5) and the triangle inequality,

$$\begin{aligned} M &\ge \max_{\epsilon_k=\pm 1,\ 3\le k\le m}|x-a_{12}| + 2\left|\sum_{3\le i\le m} a_{1i}\epsilon_i\right| \\ &= |x-a_{12}| + 2\sum_{3\le i\le m}|a_{1i}| > |x-a_{12}| + 2\min\{|x|,|a_{12}|\} \\ &= |x| + |a_{12}| = M, \end{aligned}$$

a contradiction. Therefore we complete the proof.

Remark Lemma in [9] can be improved as follows. Let E be a normed space over a field (**C** or **R**) and $m\ge 2$, a natural number. Let x, a_{ij} $(1\le i<j\le m)\in E$. Then

$$\sum_{\epsilon_k=\pm 1,\ 1\le k\le m}\left\|x+\sum_{1\le i<j\le m}\epsilon_i\epsilon_j a_{ij}\right\| \ge 2^m \max_{1\le i<j\le m}\{\|x\|, \|a_{ij}\|\}.$$

Using Lemma 2, we obtain the main result of this paper.

THEOREM 3 *Let* $P(x)=\sum_{i\le j} b_{ij}x_ix_j \in \mathcal{P}(^2 l_p)$, $b_{ij}\in \mathbf{R}, 0<p\le\infty$. *Then we have*

$$\|P\| \ge \sup_{m\in\mathbf{N},\ (w_1,w_2,\ldots,w_m,0,\ldots)\in S_{l_p}}\left\{\left|\sum_{1\le i\le m} b_{ii}w_i^2\right| + \max_{1\le i<j\le m}|b_{ij}w_iw_j|\right\}.$$

Proof It follows from Lemma 2 because

$$\begin{aligned}\|P\| &\geq |P(\epsilon_1 w_1, \ldots, \epsilon_m w_m, 0, 0, \ldots)| \\ &= \left| \sum_{1\leq i\leq m} b_{ii} w_i^2 + \sum_{1\leq i<j\leq m} b_{ij} w_i w_j \epsilon_i \epsilon_j \right|\end{aligned}$$

for any $(w_1, w_2, \ldots, w_m, 0, \ldots) \in S_{l_p}$, $x = \sum_{1\leq i\leq m} b_{ii} w_i^2$ and $a_{ij} = b_{ij}\epsilon_i\epsilon_j$ for any sign choices $\epsilon_1, \ldots, \epsilon_m$.

COROLLARY 4 (a) *Let* $P(x) = \sum_{i\leq j} b_{ij} x_i x_j \in \mathcal{P}(^2 l_p)$, $b_{ij} \in \mathbf{R}, 0 < p < \infty$. *Then we have*

$$\|P\| \geq \sup_{m\in\mathbf{N}} \left\{ 1/m^{2/p} \left(\left| \sum_{1\leq i\leq m} b_{ii} \right| + \max_{1\leq i<j\leq m} |b_{ij}| \right) \right\}.$$

(b) *Let* $P(x) = \sum_{i\leq j} b_{ij} x_i x_j \in \mathcal{P}(^2 l_\infty)$, $b_{ij} \in \mathbf{R}$. *Then we have*

$$\|P\| \geq \sup_{m\in\mathbf{N}} \left\{ \left| \sum_{1\leq i\leq m} b_{ii} \right| + \max_{1\leq i<j\leq m} |b_{ij}| \right\}.$$

Proof (a) follows by taking $w_k = 1/m^{1/p}$ for $k = 1, 2, \ldots, m$.

(b) follows by taking $w_k = 1$ for $k = 1, 2, \ldots, m$.

PROPOSITION 5 (a) *Let* $P(x) = \sum_{i_1,\ldots,i_n} a_{i_1\cdots i_n} x_{i_1} \cdots x_{i_n} \in \mathcal{P}(^n l_2)$, $a_{i_1\cdots i_n} \in \mathbf{K}$. *Then we have*

$$\|P\| \leq \left(\sum_{i_1,\ldots,i_n} |a_{i_1\cdots i_n}|^2 \right)^{1/2}.$$

(b) *If*

$$\sum_{i_1,\ldots,i_n} |a_{i_1\cdots i_n}|^2 < \infty,$$

then

$$P(x) = \sum_{i_1,\ldots,i_n} a_{i_1\cdots i_n} x_{i_1} \cdots x_{i_n} \in \mathcal{P}(^n l_2).$$

Proof Use induction on n.

Case $n=2$. For $x=(x_k)\in S_{l_2}$,

$$\left|\sum_{i,j} a_{ij}x_ix_j\right| \le \sum_k \left|\sum_j a_{kj}x_kx_j\right| = \sum_k |x_k|\left|\sum_j a_{kj}x_j\right|$$
$$\le \sum_k |x_k|\left(\sum_j |a_{kj}|^2\right)^{1/2}\left(\sum_j |x_j|^2\right)^{1/2}$$
(by the Hölder inequality)
$$\le \sum_k |x_k|^2\left(\sum_{k,j} |a_{kj}|^2\right)^{1/2} \quad \text{(by the Hölder inequality)}$$
$$= \left(\sum_{k,j} |a_{kj}|^2\right)^{1/2}.$$

Suppose that for $n\le k$, the proposition is true. For $x=(x_k)\in S_{l_2}$,

$$\left|\sum_{i_1,\dots,i_{k+1}} a_{i_1\cdots i_{k+1}}x_{i_1}\cdots x_{i_{k+1}}\right|$$
$$\le \sum_{i_{k+1}} |x_{i_{k+1}}|\left|\sum_{i_1,\dots,i_k} a_{i_1\cdots i_k}x_{i_1}\cdots x_{i_k}\right|$$
$$\le \sum_{i_{k+1}} |x_{i_{k+1}}|\left(\sum_{i_1,\dots,i_k} |a_{i_1\cdots i_k}|^2\right)^{1/2} \quad \text{(by the induction hypothesis)}$$
$$\le \left(\sum_{i_{k+1}} |x_{i_{k+1}}|^2\right)^{1/2}\left(\sum_{i_1,\dots,i_{k+1}} |a_{i_1\cdots i_{k+1}}|^2\right)^{1/2} \quad \text{(by the Hölder inequality)}$$
$$= \left(\sum_{i_1,\dots,i_{k+1}} |a_{i_1\cdots i_{k+1}}|^2\right)^{1/2}.$$

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