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Singular Solutions of a Singular Differential Equation

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An attempt is made to study the problem of existence of singular solutions to singular differential equations of the type

$$(|y'|^{-\alpha})' + q(t)|y|^{\beta} = 0, \tag{$*$}$$

which have never been touched in the literature. Here α and β are positive constants and $q(t)$ is a positive continuous function on $[0, \infty)$. A solution with initial conditions given at $t = 0$ is called singular if it ceases to exist at some finite point $T \in (0, \infty)$. Remarkably enough, it is observed that the equation $(*)$ may admit, in addition to a usual blowing-up singular solution, a completely new type of singular solution $y(t)$ with the property that

$$\lim_{t \to T-0} |y(t)| < \infty \quad \text{and} \quad \lim_{t \to T-0} |y'(t)| = \infty.$$

Such a solution is named a black hole solution in view of its specific behavior at $t = T$. It is shown in particular that there does exist a situation in which all solutions of $(*)$ are black hole solutions.

Keywords: Nonlinear differential equation; Singular differential equation; Singular solution; Blowing-up solution; Black hole solution

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0. INTRODUCTION

This paper is concerned with the singular differential equation

$$(|y'|^{-\alpha})' + q(t)|y|^{\beta} = 0, \quad t \geq 0, \tag{A}$$

where α and β are positive constants, and $q:[0,\infty) \to (0,\infty)$ is a continuous function. There has been an increasing interest in the qualitative study of singular differential equations (see e.g. [1–4]), but nothing is known about singular equations with singularities in the principal differential operators, of which (A) is a prototype.

Our purpose here is to make a detailed analysis of the behavior of solutions of (A), thereby demonstrating that (A) may have a class of strange solutions, called *black hole solutions*, which have never appeared in the literature.

Let $y_0 \in \mathbb{R}$ and $y_1 \in \mathbb{R}\backslash\{0\}$ be any given constants and let $y(t)$ denote the solution of (A) satisfying the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1. \tag{1}$$

It is easy to see that $y(t)$ is uniquely determined at least locally and is strictly monotone as long as it exists. Let $[0, T_y)$ denote the maximal interval of existence of $y(t)$. The solution $y(t)$ is called *proper* or *singular* according as $T_y = \infty$ or $T_y < \infty$. There are two possible cases for singular solutions $y(t)$ of (A): Either

$$\lim_{t \to T_y - 0} |y(t)| = \infty, \qquad \lim_{t \to T_y - 0} |y'(t)| = \infty, \tag{2}$$

or

$$\lim_{t \to T_y - 0} |y(t)| < \infty, \qquad \lim_{t \to T_y - 0} |y'(t)| = \infty. \tag{3}$$

A singular solution satisfying (2) or (3) is referred to as a *blowing-up* solution or a *black hole* solution, respectively. The latter is a new type of solution totally unknown in the textbooks or research articles.

The existence of these two types of singular solutions of (A) is established in Section 2, which is preceded by an analysis of proper solutions of (A) made in Section 1. In Section 3 an attempt is made to extend the results for (A) to a slightly more general singular equations of the form

$$(p(t)|y'|^{-\alpha})' + q(t)|y|^{\beta} = 0, \quad t \geq 0, \tag{B}$$

where $p:[0,\infty) \to (0,\infty)$ is a continuous function.

1. PROPER SOLUTIONS

Our first task is to discuss the problem of existence of proper solutions for the Eq. (A).

THEOREM 1 *The Eq.* (A) *has a proper solution if and only if*

$$\int_0^{\infty} t^{\beta} q(t)\,\mathrm{d}t < \infty. \tag{4}$$

Proof (The "only if" part) Suppose that (A) has a proper solution $y(t)$. Since $|y'(t)|$ is increasing on $[0,\infty)$ by (A), $y(t)$ is either increasing and eventually positive or decreasing and eventually negative. It suffices to deal with the former case. Let $t_0 \geq 0$ be such that $y(t) > 0$ and $y'(t) > 0$ for $t \geq t_0$. Then

$$y'(t) \geq y'(t_0) \quad \text{and} \quad y(t) \geq y(t_0) + y'(t_0)(t - t_0), \qquad t \geq t_0. \tag{5}$$

Integrating (A) on $[t_0, t]$ and letting $t \to \infty$, we obtain

$$\int_{t_0}^{\infty} q(t)(y(t))^{\beta}\,\mathrm{d}t < \infty.$$

This combined with the second inequality of (5) shows that

$$(y'(t_0))^{\beta} \int_{t_0}^{\infty} (t - t_0)^{\beta} q(t)\,\mathrm{d}t \leq \int_{t_0}^{\infty} q(t)(y(t))^{\beta}\,\mathrm{d}t < \infty,$$

which implies (4)

(The "if" part) Assume that (4) holds. Let $y_0 \geq 0$ be an arbitrarily fixed constant. We show that (A) has both increasing and decreasing proper solutions emanating from the initial point $(0, y_0)$.

Choose first a constant $\eta_1 > 0$ sufficiently small so that

$$\int_0^\infty (y_0 + \eta_1 t)^\beta q(t)\,dt \le (2^\alpha - 1)\eta_1^{-\alpha}, \tag{6}$$

and define the set $Y \subset C[0,\infty)$ and the integral operator $\mathcal{F} : Y \to C[0,\infty)$ by

$$Y = \left\{y \in C[0,\infty) \colon y_0 + \frac{\eta_1}{2} t \le y(t) \le y_0 + \eta_1 t,\ t \ge 0\right\} \tag{7}$$

and

$$(\mathcal{F}y)(t) = y_0 + \int_0^t \left[\eta_1^{-\alpha} + \int_s^\infty q(r)(y(r))^\beta\,dr\right]^{-1/\alpha} ds, \quad t \ge 0, \tag{8}$$

respectively. That $\mathcal{F}$ maps Y into itself is guaranteed by (6). It is easy to show that $\mathcal{F}$ is a continuous mapping and $\mathcal{F}(Y)$ is a relatively compact subset of $C[0,\infty)$. The Schauder–Tychonoff fixed point theorem, therefore, ensures the existence of a fixed element $y \in Y$ of $\mathcal{F}$, which, in view of (8), satisfies

$$y(t) = y_0 + \int_0^t \left[\eta_1^{-\alpha} + \int_s^\infty q(r)(y(r))^\beta\,dr\right]^{-1/\alpha} ds, \quad t \ge 0. \tag{9}$$

From (9) we conclude that $y(t)$ is a positive increasing solution of (A) defined on $[0,\infty)$ and satisfying $\lim_{t\to\infty} y'(t) = \eta_1 > 0$.

Let $\zeta_1 > 0$ be a constant such that

$$\int_0^\infty (y_0 + 2\zeta_1 t)^\beta q(t)\,dt \le (1 - 2^{-\alpha})\zeta_1^{-\alpha} \tag{10}$$

and consider the set $Z \subset C[0,\infty)$ and the mapping $\mathcal{G} : Z \to C[0,\infty)$ defined by

$$Z = \{z \in C[0,\infty) \colon y_0 - 2\zeta_1 t \le z(t) \le y_0 - \zeta_1 t,\ t \ge 0\} \tag{11}$$

and

$$(\mathcal{G}z)(t) = y_0 - \int_0^t \left[\zeta_1^{-\alpha} - \int_s^\infty q(r)|z(r)|^\beta\,dr\right]^{-1/\alpha} ds, \quad t \ge 0. \tag{12}$$

Then, as is easily verified, the Schauder–Tychonoff fixed point theorem applies, and there exists an element $z \in Z$ such that $z = \mathcal{G}z$, that is,

$$z(t) = y_0 - \int_0^t \left[\zeta_1^{-\alpha} - \int_s^\infty q(r)|z(r)|^\beta \, \mathrm{d}r \right]^{-1/\alpha} \mathrm{d}s, \quad t \geq 0. \tag{13}$$

It follows that $z(t)$ is a decreasing proper solution of (A) on $[0, \infty)$ which is eventually negative and satisfies $\lim_{t\to\infty} z'(t) = -\zeta_1 < 0$. The case where $y_0 < 0$ can be treated similarly. This completes the proof.

Remark 1 Suppose that

$$\int_0^\infty t^\beta q(t) \, \mathrm{d}t = \infty. \tag{14}$$

It then follows from Theorem 1 that (A) has no proper solution, and consequently all of its solutions are singular.

Remark 2 Let $y(t)$ be a proper solution of (A). Since $|y'(t)|$ is increasing by (A), $|y(t)|$ grows to infinity at least as fast as a constant multiple of t as $t \to \infty$. Such a solution $y(t)$ is said to be *subdominant* if $|y(t)|$ grows exactly like a constant multiple of t as $t \to \infty$, that is, $\lim_{t\to\infty} y(t)/t = \lim_{t\to\infty} y'(t) = \text{const.} \neq 0$. The coexistence of both subdominant and non-subdominant solutions can take place as the following example shows.

Example 1 Consider the equation

$$(|y'|^{-\alpha})' + \alpha \mathrm{e}^{-(\alpha+\beta)t} |y|^\beta = 0, \quad t \geq 0. \tag{15}$$

Since (4) is satisfied by $q(t) = \alpha \mathrm{e}^{-(\alpha+\beta)t}$, this equation possesses subdominant proper solutions by the proof of Theorem 1. Note that $y(t) = \mathrm{e}^t$ is a non-subdominant proper solution of (15).

2. SINGULAR SOLUTIONS

Let us turn our attention to singular solutions of the Eq. (A). Surprisingly it turns out that the behavior of singular solutions of (A) in the case $\alpha > 1$ is distinct from that of (A) in the case $\alpha < 1$.

THEOREM 2 *The Eq.* (A) *with* $\alpha < 1$ *always possesses blowing-up singular solutions.*

To prove this theorem we need the following lemma comparing the behavior of solutions of the two singular differential equations

$$(|u'|^{-\alpha})' + a(t)|u|^{\beta} = 0, \tag{16}$$

$$(|v'|^{-\alpha})' + A(t)|v|^{\beta} = 0, \tag{17}$$

where $a(t)$ and $A(t)$ are positive continuous functions on $[0, T)$.

LEMMA 1 *Suppose that* $a(t) \leq A(t)$ *on* $[0, T)$. *Let* $u(t)$ *and* $v(t)$ *be positive increasing solutions of* (16) *and* (17), *respectively, existing on* $[0, T)$. *If* $u(0) \leq v(0)$ *and* $u'(0) < v'(0)$, *then* $u(t) < v(t)$ *and* $u'(t) < v'(t)$ *on* $(0, T)$.

Proof of Lemma 1 We integrate (16) and (17) from 0 to t to obtain for $t \in [0, T)$

$$u'(t) = \left[(u'(0))^{-\alpha} - \int_0^t a(s)(u(s))^{\beta}\,\mathrm{d}s\right]^{-1/\alpha}, \tag{18}$$

$$u(t) = u(0) + \int_0^t \left[(u'(0))^{-\alpha} - \int_0^s a(r)(u(r))^{\beta}\,\mathrm{d}r\right]^{-1/\alpha} \mathrm{d}s, \tag{19}$$

$$v'(t) = \left[(v'(0))^{-\alpha} - \int_0^t A(s)(v(s))^{\beta}\,\mathrm{d}s\right]^{-1/\alpha}, \tag{20}$$

$$v(t) = v(0) + \int_0^t \left[(v'(0))^{-\alpha} - \int_0^s A(r)(v(r))^{\beta}\,\mathrm{d}r\right]^{-1/\alpha} \mathrm{d}s. \tag{21}$$

Since $u(0) \leq v(0)$ and $u'(0) < v'(0)$, there exists $\tau \in (0, T)$ such that $u(t) < v(t)$ for $t \in (0, \tau)$. Suppose now that the conclusion of the lemma is false. Then there exists $t_0 \in (0, T)$ such that $u(t) < v(t)$ for $t \in (0, t_0)$ and $u(t_0) = v(t_0)$. We then easily see that

$$\left[(u'(0))^{-\alpha} - \int_0^t a(s)(u(s))^{\beta}\,\mathrm{d}s\right]^{-1/\alpha} < \left[(v'(0))^{-\alpha} - \int_0^t A(s)(v(s))^{\beta}\,\mathrm{d}s\right]^{-1/\alpha} \tag{22}$$

for $t \in (0, t_0]$, which implies that the right hand side of (21) is greater than that of (19) for all $t \in (0, t_0]$. This, however, is a contradiction, because the left hand sides of (19) and (21) coincide at $t = t_0$. Therefore, we must have $u(t) < v(t)$ for $(0, T)$. That $u'(t) < v'(t)$ on $(0, T)$ is an immediate consequence of (18), (20) and (22). The proof of Lemma 1 is thus complete.

Proof of Theorem 2 Let $y(t)$ be a positive increasing solution of (A) satisfying (1). Let a subinterval $[0, t_0]$ of $[0, \infty)$ be fixed. Put $q_0 = \min_{t\in[0,t_0]} q(t) > 0$ and consider the singular differential equation

$$(|z'|^{-\alpha})' + q_0|z|^{\beta} = 0. \tag{23}$$

Let $z(t)$ be the solution of (23) subject to the same initial condition as $y(t)$, i.e., $z(0) = y_0$ and $z'(0) = y_1$, and let J denote its maximal interval of existence. Multiplying (23) by $z'(t)$, we have

$$\frac{\alpha}{1-\alpha}\left[(z'(t))^{1-\alpha}\right]' = \frac{q_0}{\beta+1}\left[(z(t))^{\beta+1}\right]', \quad t \in J, \tag{24}$$

which, upon integration, gives

$$z'(t) = \left[\gamma\left((z(t))^{\beta+1} - y_0^{\beta+1}\right) + y_1^{1-\alpha}\right]^{1/(1-\alpha)}, \quad t \in J, \tag{25}$$

where $\gamma = q_0(1-\alpha)/\alpha(\beta+1)$. From (25) we see that $z(t)$ is determined implicitly by the equation

$$t = \int_{y_0}^{z(t)} \frac{d\zeta}{\left[\gamma\left(\zeta^{\beta+1} - y_0^{\beta+1}\right) + y_1^{1-\alpha}\right]^{1/(1-\alpha)}}, \quad t \in J. \tag{26}$$

Since

$$\tau_0 := \int_{y_0}^{\infty} \frac{d\zeta}{\left[\gamma\left(\zeta^{\beta+1} - y_0^{\beta+1}\right) + y_1^{1-\alpha}\right]^{1/(1-\alpha)}} < \infty, \tag{27}$$

the relation (26) shows that $z(t)$ exists on $J = [0, \tau_0)$ and blows up at $t = \tau_0$. Choose $y_1 > 0$ so large that $\tau_0 < t_0$, which is possible since $\tau_0 \to 0$ as

$y_1 \to \infty$, and consider $y(t)$ and $z(t)$ determined by such initial values $\{y_0, y_1\}$ at $t = 0$. Comparison of (A) with (23) then shows by Lemma 1 that $y(t) \geq z(t)$ on a common interval of their existence, from which it follows that $y(t)$ must blow up at some finite point less than τ_0. This completes the proof of Theorem 2.

THEOREM 3 *Suppose that* $q(t)$ *is continuously differentiable on* $[0, \infty)$. *The Eq.* (A) *with* $\alpha > 1$ *has no blowing-up singular solutions.*

Proof Let $y(t)$ be a blowing-up solution of (A). Without loss of generality we may suppose that $y(t)$ and $y'(t)$ are positive on $[0, T)$, $0 < T < \infty$, and $\lim_{t \to T-0} y(t) = \lim_{t \to T-0} y'(t) = \infty$. Define the function $V[y](t)$ by

$$V[y](t) = \frac{\alpha}{\alpha - 1}(y'(t))^{1-\alpha} + \frac{q(t)}{\beta + 1}(y(t))^{\beta+1}, \quad t \in [0, T). \tag{28}$$

As is easily verified, $V[y](t)$ is positive and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V[y](t) = \frac{q'(t)}{\beta + 1}(y(t))^{\beta+1}, \tag{29}$$

whence it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} V[y](t) \leq \frac{q'_+(t)}{q(t)} V[y](t), \quad t \in [0, T), \tag{30}$$

where $q'_+(t) = \max\{q'(t), 0\}$. Integrating (30), we have

$$V[y](t) \leq V[y](0) \exp\left(\int_0^t \frac{q'_+(s)}{q(s)} \mathrm{d}s\right), \quad t \in [0, T). \tag{31}$$

Letting $t \to T - 0$ in (31), we find that the right hand side remains bounded, while the left hand side tends to ∞ since $(y'(t))^{1-\alpha} \to 0$ and $q(t)(y(t))^{\beta+1} \to \infty$. This contradiction proves the truth of the conclusion of the theorem.

Combining Theorem 1 with Theorem 3 enables us to indicate a situation in which the equation (A) really possesses black hole solutions.

THEOREM 4 *Suppose that $q(t)$ is continuously differentiable on* $[0, \infty)$. *All solutions of* (A) *are black hole solutions if* $\alpha > 1$ *and the condition* (14) *holds.*

Example 2 Consider the equation

$$(|y'|^{-\alpha})' + \gamma|y|^{\beta} = 0, \quad t \geq 0, \tag{32}$$

where γ is a positive constant. All solutions of (32) are singular because $q(t) = \gamma$ satisfies (14) (Theorem 1). There exist blowing-up solutions of (32) if $\alpha < 1$ (Theorem 2) and all solutions of (32) are black hole solutions if $\alpha > 1$ (Theorem 4). The same property is enjoyed by the Eq. (A) with $q(t) \in C^1[0, \infty)$ satisfying $\liminf_{t\to\infty} t^{\beta+1} q(t) > 0$.

3. EXTENSION

The above-mentioned results for (A) can be extended to the Eq. (B) provided the function $p(t)$ satifies the condition

$$\int_0^{\infty} (p(t))^{1/\alpha}\,\mathrm{d}t = \infty. \tag{33}$$

This is a direct consequence of the fact that the change of variables $(t, y) \to (\tau, Y)$ defined by

$$\tau = P(t), \qquad Y(\tau) = y(t), \tag{34}$$

where

$$P(t) = \int_0^t (p(s))^{1/\alpha}\,\mathrm{d}s, \tag{35}$$

transforms (B) into the equation

$$(|\dot{Y}|^{-\alpha})^{\cdot} + Q(\tau)|Y|^{\beta} = 0, \quad \tau \geq 0, \tag{36}$$

where $Q(\tau) = (p(t))^{-1/\alpha} q(t)$ and a dot denotes differentiation with respect to τ. Application of Theorems 1–4 to (36) yields the statements for

(B) listed below.

(i) The Eq. (B) has a positive increasing proper solution if and only if

$$\int_0^\infty (P(t))^\beta q(t)\,\mathrm{d}t < \infty. \tag{37}$$

(ii) The Eq. (B) with $\alpha < 1$ always possesses blowing-up singular solutions.
(iii) Suppose that $p(t)$ and $q(t)$ are continuously differentiable on $[0, \infty)$. The Eq. (B) with $\alpha > 1$ has no blowing-up singular solutions.
(iv) Suppose that $p(t)$ and $q(t)$ are continuously differentiable on $[0, \infty)$. All positive increasing solutions of (A) are black hole solutions if $\alpha > 1$ and

$$\int_0^\infty (P(t))^\beta q(t)\,\mathrm{d}t = \infty. \tag{38}$$

Example 3 From the statement (i) above it follows that the equation

$$(\mathrm{e}^{\alpha t}|y'|^{-\alpha})' + \mathrm{e}^{-\beta t}|y|^\beta = 0, \quad t \geq 0, \tag{39}$$

has no positive increasing proper solution. By (iv) all positive increasing solutions of (39) are black hole solutions if $\alpha > 1$. In case $\alpha < 1$ the statement (ii) implies that (39) has blowing-up singular solutions. It is not known if there exists a black hole solution in this case.

References

[1] M. Kurokiba, T. Kusano and J. Wang, Positive solutions of second order quasilinear differential equations with singular nonlinearities, *Differentsial'nye Uravneniya* **32** (1996), 1630–1637 (in Russian).
[2] T. Kusano and T. Tanigawa, Positive solutions to a class of second order differential equations with singular nonlinearities, *Appl. Anal.* **69** (1998), 315–331.
[3] T. Tanigawa, Positive solutions to second order singular differential equations involving the one-dimensional m-Laplace operator, *Georgian Math. J.* **6** (1999), 347–362.
[4] T. Tanigawa, Asymptotic behavior of positive solutions to nonlinear singular differential equations of second order, *Studia Sci. Math. Hungar.* (to appear).