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Random Generalized Set-Valued Strongly Nonlinear Implicit Quasi-Variational Inequalities

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The purpose of this paper is to introduce a class of new random generalized set-valued strongly nonlinear implicit quasi-variational inequalities, to construct new random iterative algorithms, and to give some existence theorems of random solutions for this class of random generalized set-valued strongly nonlinear implicit quasi-variational inequalities. We also prove the convergence of random iterative sequences generated by the algorithms. Our results extend and improve the earlier and recent results.

Keywords: Random generalized nonlinear implicit quasi-variational inequality; Random set-valued mapping; Random algorithm; Convergence

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1. INTRODUCTION

Variational inequality theory has become a rich source of inspiration in pure and applied mathematics. Variational inequalities not only have stimulated new results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming,

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economics and transportation equilibrium and engineering sciences, etc. Quasi-variational inequalities are the extended form of variational inequalities in which the constrained set depends upon the solutions. These were introduced and studied by Bensoussan *et al.* [3]. For more details, we may refer to [1,2,4,5,12,18–20,26], and the references therein.

Recently, the random variational inequality and random quasi-variational inequality problems have been introduced and studied by Chang [5], Chang and Huang [7,8], Chang and Zhu [9], Ganguly and Wadhwa [10], Huang [13,14], Huang and Cho [15,16], Huang *et al.* [17], Noor and Elsanousi [22], Tan [24] and Yuan [25].

In this paper, we introduce and study a class of new random generalized set-valued strongly nonlinear implicit quasi-variational inequalities and construct some new random iterative algorithms. We prove the existence of random solutions for this class or random generalized set-valued strongly nonlinear implicit quasi-variational inequalities without compactness and the convergence of the random iterative sequences generated by the random algorithms. Our results extend and improve the earlier and recent results including the corresponding results of Chang [5], Chang and Huang [8], Huang [13,14], Huang and Cho [16], Noor [21], Noor and Elsanousi [22], Siddiqi and Ansari [23].

2. PRELIMINARIES AND FORMULATIONS

Throughout this paper, let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and H be a separable real Hilbert space. We denote by $\mathcal{B}(H)$, $(\cdot,\cdot)$ and $\|\cdot\|$ the class of Borel σ-fields in H, the inner product and the norm on H, respectively. In the sequel, we denote 2^H, $CB(H)$ and h by

$$2^H = \{A\colon A \in H\},$$

$$CB(H) = \{A \subset H\colon A \text{ is nonempty, bounded and closed}\},$$

the Hausdorff metric on $CB(H)$, respectively.

DEFINITION 2.1 *A mapping $x:\Omega \to H$ is said to be measurable if for any $B \in \mathcal{B}(H)$, $\{\omega \in \Omega\colon x(\omega) \in B\} \in \mathcal{A}$.*

DEFINITION 2.2 *A mapping $T:\Omega \times H \to H$ is called a random operator if for any $x \in H$, $T(\omega, x) = x(\omega)$ is measurable. A random operator T is said to be continuous (resp., linear, bounded) if for any $\omega \in \Omega$, the mapping $T(\omega, \cdot): H \to H$ is continuous (resp., linear, bounded).*

DEFINITION 2.3 *A set-valued mapping $V:\Omega \to 2^H$ is said to be measurable if for any $B \in \mathcal{B}$, $V^{-1}(B) = \{\omega \in \Omega\colon V(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$.*

DEFINITION 2.4 *A mapping $u:\Omega \to H$ is called a measurable selection of a set-valued measurable mapping $V:\Omega \to 2^H$ if u is measurable and for any $\omega \in \Omega$, $u(\omega) \in V(\omega)$.*

LEMMA 2.1 ([16]) *Suppose that a random function $a:\Omega \times H \times H \to \mathbb{R} = (-\infty, +\infty)$ satisfies the following conditions:*

(1) *For any $\omega \in \Omega$, $a(\omega, \cdot, \cdot): H \times H \to \mathbb{R}$ is a bounded bilinear function,*
(2) *For any $u, v, \in H$, $a(\cdot, u, v): \Omega \to \mathbb{R}$ is a measurable function.*

Then there exists a unique random bounded linear operator $A:\Omega \times H \to H$ such that

$$(A(\omega, u), v) = a(\omega, u, v) \quad \text{and} \quad \|A(\omega, \cdot)\| = \|a(\omega, \cdot, \cdot)\|$$

for all $u, v \in H$ and $\omega \in \Omega$, where

$$\|A(\omega, \cdot)\| = \sup\{\|A(\omega, u)\|\colon \|u\| \leq 1\},$$
$$\|a(\omega, \cdot, \cdot)\| = \sup\{|a(\omega, u, v)|\colon \|u\| \leq 1, \|v\| \leq 1\}.$$

DEFINITION 2.5 *A mapping $V:\Omega \times H \to 2^H$ is called a random set-valued mapping if for any $x \in H$, $V(\cdot, x)$ is measurable. A random set-valued mapping $V:\Omega \times H \to CB(H)$ is said to be h-continuous if for any $\omega \in \Omega$, $V(\omega, \cdot)$ is continuous in the Hausdorff metric h.*

DEFINITION 2.6 *A mapping $a:\Omega \times H \times H \to \mathbb{R}$ is called a random coercive bounded bilinear function if the following conditions are satisfied:*

(1) *For any $\omega \in \Omega$, $a(\omega, \cdot, \cdot)$ is bilinear and there exist measurable functions $\alpha, \beta:\Omega \to (0, \infty)$ such that*

$$a(\omega, u, u) \geq \alpha(\omega)\|u\|^2 \quad \text{and} \quad |a(\omega, u, v)| \leq \beta(\omega)\|u\|\|v\|$$

for all $u, v \in H$ and $\omega \in \Omega$,
(2) *For any $u, v \in H$, $a(\cdot, u, v)$ is a measurable function.*

The measurable functions $\alpha(\omega)$ and $\beta(\omega)$ are called the coercive coefficients.

From Definition 2.6, it is easy to see that $\alpha(\omega) \leq \beta(\omega)$ for all $\omega \in \Omega$.

Let $U, V, K : \Omega \times H \to 2^H$ be three random set-valued mappings such that for each $\omega \in \Omega$ and $x \in H$, $K(\omega, x)$ is a nonempty closed convex subset of H. Let $g : \Omega \times H \to H$ be a random operator and $a : \Omega \times H \times H \to \mathbb{R}$ be a random function. Now, we consider the following problem.

Find measurable mappings $u, x, y : \Omega \to H$ such that

$$\begin{aligned} &g(\omega, u(\omega)) \in K(\omega, x(\omega)), \quad x(\omega) \in U(\omega, u(\omega)), \ y(\omega) \in V(\omega, u(\omega)), \\ &a(\omega, u(\omega), v - g(\omega, u(\omega))) \geq (y(\omega), v - g(\omega, u(\omega))) \end{aligned} \tag{2.1}$$

for all $\omega \in \Omega$ and $v \in K(\omega, x(\omega))$.

The problem (2.1) is called a random generalized set-valued strongly nonlinear implicit quasi-variational inequality.

If U is an identity mapping, then the problem (2.1) is equivalent to the problem finding measurable mappings $u, y : \Omega \to H$ such that

$$\begin{aligned} &g(\omega, u(\omega)) \in K(\omega, u(\omega)), \quad y(\omega) \in V(\omega, u(\omega)), \\ &a(\omega, u(\omega), v - g(\omega, u(\omega))) \geq (y(\omega), v - g(\omega, u(\omega))) \end{aligned} \tag{2.2}$$

for all $\omega \in \Omega$ and $v \in K(\omega, u(\omega))$.

The problem (2.2) is called a random generalized set-valued nonlinear implicit quasi-variational inequality and appears to be a new one.

If g is an identity mapping, then the problem (2.1) is equivalent to the problem finding measurable mappings $u, x, y : \Omega \to H$ such that

$$\begin{aligned} &u(\omega) \in K(\omega, x(\omega)), \quad x(\omega) \in U(\omega, u(\omega)), \quad y(\omega) \in V(\omega, u(\omega)), \\ &a(\omega, u(\omega), v - u(\omega)) \geq (y(\omega), v - u(\omega)) \end{aligned} \tag{2.3}$$

for all $\omega \in \Omega$ and $v \in K(\omega, x(\omega))$, and the problem (2.2) is equivalent to the problem finding measurable mappings $u, y : \Omega \to H$ such that

$$\begin{aligned} &u(\omega) \in K(\omega, u(\omega)), \quad y(\omega) \in V(\omega, u(\omega)), \\ &a(\omega, u(\omega), v - u(\omega)) \geq (y(\omega), v - u(\omega)) \end{aligned} \tag{2.4}$$

for all $\omega \in \Omega$ and $v \in K(\omega, u(\omega))$, respectively.

The problems (2.3) and (2.4) are called the random generalized set-valued strongly nonlinear quasi-variational inequality and random generalized set-valued nonlinear quasi-variational inequality, respectively.

If $K(\omega, u) = m(\omega, u) + K$ for all $\omega \in \Omega$ and $u \in H$, where $m : \Omega \times H \to H$ is a random operator, K is a nonempty closed convex subset of H, then the problem (2.1) is equivalent to the problem finding measurable mappings $u, x, y : \Omega \to H$ such that

$$\begin{aligned} & g(\omega, u(\omega)) - m(\omega, x(\omega)) \in K, \quad x(\omega) \in U(\omega, u(\omega)), \; y(\omega) \in V(\omega, u(\omega)), \\ & a(\omega, u(\omega), v - g(\omega, u(\omega))) \geq (y(\omega), v - g(\omega, u(\omega))) \end{aligned} \tag{2.5}$$

for all $\omega \in \Omega$ and $v \in m(\omega, x(\omega)) + K$.

The problem (2.5) is called the random completely generalized set-valued implicit quasi-variational inequality, considered by Huang and Cho [16].

Obviously, the problem (2.1) includes many kinds of variational inequalities and quasi-variational inequalities in [5,8,13,14,16,21–23] as special cases.

If $a(\omega, u, v)$ is a random coercive bounded bilinear function, then, from Lemma 2.1, there exists a unique random bounded linear operator $A : \Omega \times H \to H$ such that $(A(\omega, u), v) = a(\omega, u, v)$ for all $u, v \in H$ and $\omega \in \Omega$.

In this case, it follows that:

(1) The problem (2.1) is equivalent to the problem finding measurable mappings $u, x, y : \Omega \to H$ such that

$$\begin{aligned} & g(\omega, u(\omega)) \in K(\omega, x(\omega)), \quad x(\omega) \in U(\omega, u(\omega)), \; y(\omega) \in V(\omega, u(\omega)), \\ & (A(\omega, u(\omega)), v - g(\omega, u(\omega))) \geq (y(\omega), v - g(\omega, u(\omega))) \end{aligned} \tag{2.1$'$}$$

for all $\omega \in \Omega$ and $v \in K(\omega, x(\omega))$.

(2) The problem (2.2) is equivalent to the problem finding measurable mappings $u, y : \Omega \to H$ such that

$$\begin{aligned} & g(\omega, u(\omega)) \in K(\omega, u(\omega)), \quad y(\omega) \in V(\omega, u(\omega)), \\ & (A(\omega, u(\omega)), v - g(\omega, u(\omega))) \geq (y(\omega), v - g(\omega, u(\omega))) \end{aligned} \tag{2.2$'$}$$

for all $\omega \in \Omega$ and $v \in K(\omega, u(\omega))$.

(3) The problem (2.3) is equivalent to the problem finding measurable mappings $u, x, y : \Omega \to H$ such that

$$u(\omega) \in K(\omega, x(\omega)), \quad x(\omega) \in U(\omega, u(\omega)), \quad y(\omega) \in V(\omega, u(\omega)), \\ (A(\omega, u(\omega)), v - u(\omega)) \geq (y(\omega), v - u(\omega)) \tag{2.3$'$}$$

for all $\omega \in \Omega$ and $v \in K(\omega, u(\omega))$.

(4) The problem (2.4) is equivalent to the problem finding measurable mappings $u, y : \Omega \to H$ such that

$$u(\omega) \in K(\omega, u(\omega)), \quad y(\omega) \in V(\omega, u(\omega)), \\ (A(\omega, u(\omega)), v - u(\omega)) \geq (y(\omega), v - u(\omega)) \tag{2.4$'$}$$

for all $\omega \in \Omega$ and $v \in K(\omega, u(\omega))$.

3. RANDOM ALGORITHMS

We first give the following lemmas and definition for our main results.

LEMMA 3.1 ([6]) *Let $V : \Omega \times H \to CB(H)$ be a h-continuous random set-valued mapping. Then for any measurable mapping $u : \Omega \to H$, the set-valued mapping $V(\cdot, u(\cdot)) : \Omega \to CB(H)$ is measurable.*

LEMMA 3.2 ([6]) *Let $V, W : \Omega \to CB(H)$ be two measurable set-valued mappings, $\epsilon > 0$ be a constant and $u : \Omega \to H$ be a measurable selection of V. Then there exists a measurable selection $v : \Omega \to H$ of W such that for all $\omega \in \Omega$,*

$$\|u(\omega) - v(\omega)\| \leq (1 + \epsilon) h(V(\omega), W(\omega)).$$

LEMMA 3.3 ([5]) *If K is a closed convex subset of H and $z \in H$, then $u \in K$ satisfies the inequality $(u - z, v - u) \geq 0$ for all $v \in K$ if and only if*

$$u = P_K(z), \tag{3.1}$$

where P_K is the projection of H onto K.

Note that the mapping P_K defined by (3.1) *is nonexpensive, i.e., for all $u, v \in H$,*

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|.$$

LEMMA 3.4 ([7]) *Let K be a closed convex subset of H and $m:\Omega\times H\to H$ be a random operator. If $K(\omega,u)=m(\omega,u)+K$ for all $\omega\in\Omega$ and $u\in H$, then for any $v\in H$, $P_{K(\omega,u)}(v)=m(\omega,u)+P_K(v-m(\omega,u))$ for all $\omega\in\Omega$ and $u\in H$.*

From (2.1)′ and Lemma 3.3, we have the following lemma.

LEMMA 3.5 *Let $U,V,K:\Omega\times H\to 2^H$ be three random set-valued mappings such that for each $\omega\in\Omega$ and $x\in H$, $K(\omega,x)$ is a nonempty closed convex subset of H. Let $g:\Omega\times H\to H$ be a random operator and $a:\Omega\times H\times H\to\mathbb{R}$ be a random coercive bounded bilinear function. Then the measurable mappings $u,x,y:\Omega\to H$ are the solutions of (2.1) if and only if for any $\omega\in\Omega$,*

$$\begin{aligned}&x(\omega)\in U(\omega,u(\omega)),\quad y(\omega)\in V(\omega,u(\omega)),\\&g(\omega,u(\omega))=P_{K(\omega,x(\omega))}(g(\omega,u(\omega))+\rho(\omega)(y(\omega)-A(\omega,u(\omega)))),\end{aligned}\tag{3.2}$$

where $\rho:\Omega\to(0,\infty)$ is a measurable function and $A:\Omega\times H\to H$ is a random bounded linear operator defined by $(A(\omega,u),v)=a(\omega,u,v)$ for all $u,v\in H$ and $\omega\in\Omega$.

DEFINITION 3.1 *Let $K:\Omega\times H\to 2^H$ be a random set-valued mapping such that for each $\omega\in\Omega$ and $x\in H$, (ω,x) is a nonempty closed convex subset of H. The projection $P_{K(\omega,x)}$ is said to be a Lipschitz continuous random operator if*

(1) *for any given $x,z\in H$, $P_{K(\cdot,x)}z$ is measurable;*
(2) *there exists a measurable function $\eta:\Omega\to(0,\infty)$ such that for all $x,y,z\in H$ and $\omega\in\Omega$,*

$$\|P_{K(\omega,x)}z-P_{K(\omega,y)}z\|\le\eta(\omega)\|x-y\|.$$

PROPOSITION 3.1 *If $K(\omega,x)$ is defined as in Lemma 3.4, and $m:\Omega\times H\to H$ is a Lipschitz continuous random operator, then $P_{K(\omega,x)}$ is a Lipschitz continuous random operator.*

Proof It is easy to see that for any given $x,z\in H$, $P_{K(\cdot,x)}z$ is measurable. Furthermore, for any $x,y,z\in H$ and $\omega\in\Omega$, it follows from Lemmas 3.3

and 3.4 that

$$\begin{aligned}\|P_{K(\omega,x)}z - P_{K(\omega,y)}z\| &= \|m(\omega,x) + P_K(z - m(\omega,x)) \\ &\quad - m(\omega,y) - P_K(z - m(\omega,y))\| \\ &\le \|m(\omega,x) - m(\omega,y)\| \\ &\quad + \|P_K(z - m(\omega,x)) - P_K(z - m(\omega,y))\| \\ &\le 2\|m(\omega,x) - m(\omega,y)\|.\end{aligned}$$

Since m is a Lipschitz continuous random operator, we know that $P_{K(\omega,x)}$ is also a Lipschitz continuous random operator.

Now, by using Lemma 3.5, we construct the random algorithm for the random generalized set-valued strongly nonlinear implicit quasi-variational inequality (2.1).

ALGORITHM 3.1 *Let $g:\Omega\times H\to H$ be a continuous random operator, $K:\Omega\times H\to 2^H$ a random set-valued mapping such that for each $\omega\in\Omega$ and $x\in H$, $K(\omega,x)$ is a nonempty closed convex subset of H and the projection $P_{K(\omega,x)}$ is a Lipschitz continuous random operator. Let $U,V:\Omega\times H\to CB(H)$ be two h-continuous random set-valued mappings and $a:\Omega\times H\times H\to\mathbb{R}$ be a random coercive bounded bilinear function. For any measurable mapping $u_0:\Omega\to H$, the set-valued mappings $U(\cdot,u_0(\cdot))$, $V(\cdot,u_0(\cdot)):\Omega\to CB(H)$ are measurable by Lemma* 3.1. *Hence there exist measurable selections $x_0(\cdot):\Omega\to H$ of $U(\cdot,u_0(\cdot))$ and $y_0(\cdot):\Omega\to H$ of $V(\cdot,u_0(\cdot))$ by Himmelberg* [11]. *Letting*

$$\begin{aligned}u_1(\omega) = u_0(\omega) - g(\omega,u_0(\omega)) + P_{K(\omega,x_0(\omega))}(g(\omega,u_0(\omega)) \\ + \rho(\omega)(y_0(\omega) - A(\omega,u_0(\omega)))),\end{aligned}$$

where ρ and A are the same as in Lemma 3.5, *then it is easy to see that $u_1:\Omega\to H$ is measurable. Since $x_0(\omega)\in U(\omega,u_0(\omega))\in CB(H)$ and $y_0(\omega)\in V(\omega,u_0(\omega))\in CB(H)$, by Lemma* 3.2, *there exist measurable selections $x_1:\Omega\to H$ of $U(\omega,u_1(\omega))$ and $y_1:\Omega\to H$ of $V(\omega,u_1(\omega))$ such that for all $\omega\in\Omega$,*

$$\begin{aligned}\|x_0(\omega) - x_1(\omega)\| &\le \left(1+\tfrac{1}{1}\right)h(U(\omega,u_0(\omega)),U(\omega,u_1(\omega))), \\ \|y_0(\omega) - y_1(\omega)\| &\le \left(1+\tfrac{1}{1}\right)h(V(\omega,u_0(\omega)),V(\omega,u_1(\omega))).\end{aligned}$$

Letting

$$u_2(\omega) = u_1(\omega) - g(\omega, u_1(\omega)) + P_{K(\omega,x_1(\omega))}(g(\omega, u_1(\omega)) + \rho(\omega)(y_1(\omega) - A(\omega, u_1(\omega)))),$$

then $u_2 : \Omega \to H$ *is measurable.*

Inductively, we can define three sequences $\{u_n(\omega)\}$, $\{x_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *of measurable mappings such that*

$$\begin{aligned} &x_n(\omega) \in U(\omega, u_n(\omega)),\ \ y_n(\omega) \in V(\omega, u_n(\omega)),\\ &\|x_n(\omega) - x_{n+1}(\omega)\| \le \left(1 + \frac{1}{n+1}\right) h(U(\omega, u_n(\omega)), U(\omega, u_{n+1}(\omega))),\\ &\|y_n(\omega) - y_{n+1}(\omega)\| \le \left(1 + \frac{1}{n+1}\right) h(V(\omega, u_n(\omega)), V(\omega, u_{n+1}(\omega))),\\ &u_{n+1}(\omega) = u_n(\omega) - g(\omega, u_n(\omega)) + P_{K(\omega,x_n(\omega))}(g(\omega, u_n(\omega))\\ &\qquad + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \end{aligned} \tag{3.3}$$

for all $\omega \in \Omega$ *and* $n = 0, 1, 2, \ldots,$ *where* ρ *and* A *are the same as in Lemma* 3.5.

Similarly, we have the following algorithms.

ALGORITHM 3.2 *Let* K, V, g *and* a *be the same as in Algorithm* 3.1. *Then for any measurable mapping* $u_0 : \Omega \to H$, *we can define two sequences* $\{u_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *of measurable mappings by*

$$\begin{aligned} &y_n(\omega) \in V(\omega, u_n(\omega)),\\ &\|y_n(\omega) - y_{n+1}(\omega)\| \le \left(1 + \frac{1}{n+1}\right) h(V(\omega), u_n(\omega)), V(\omega, u_{n+1}(\omega))),\\ &u_{n+1}(\omega) = u_n(\omega) - g(\omega, u_n(\omega)) + P_{K(\omega,u_n(\omega))}(g(\omega, u_n(\omega))\\ &\qquad + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \end{aligned} \tag{3.4}$$

for all $\omega \in \Omega$ *and* $n = 0, 1, 2, \ldots,$ *where* ρ *and* A *are the same as in Lemma* 3.5.

ALGORITHM 3.3 *Let* K, U, V *and* a *be the same as in Algorithm* 3.1. *Then for any measurable mapping* $u_0 : \Omega \to H$, *we can define three sequences* $\{u_n(\omega)\}$, $\{x_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *of measurable mappings such that*

$$
\begin{aligned}
&x_n(\omega) \in U(\omega, u_n(\omega)), \ y_n(\omega) \in V(\omega, u_n(\omega)), \\
&\|x_n(\omega) - x_{n+1}(\omega)\| \leq \left(1 + \frac{1}{n+1}\right) h(U(\omega, u_n(\omega)), U(\omega, u_{n+1}(\omega))), \\
&\|y_n(\omega) - y_{n+1}(\omega)\| \leq \left(1 + \frac{1}{n+1}\right) h(V(\omega, u_n(\omega)), V(\omega, u_{n+1}(\omega))), \\
&u_{n+1}(\omega) = P_{K(\omega, x_n(\omega))}(u_n(\omega) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega))))
\end{aligned} \tag{3.5}
$$

for all $\omega \in \Omega$ *and* $n = 0, 1, 2, \ldots,$ *where* ρ *and* A *are the same as in Lemma* 3.5.

ALGORITHM 3.4 *Let* K, V *and* a *be the same as in Algorithm* 3.1. *Then for any measurable mapping* $u_0 : \Omega \to H$, *we can define two sequences* $\{u_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *of measurable mappings by*

$$
\begin{aligned}
&y_n(\omega) \in V(\omega, u_n(\omega)), \\
&\|y_n(\omega) - y_{n+1}(\omega)\| \leq \left(1 + \frac{1}{n+1}\right) h(V(\omega), u_n(\omega)), V(\omega, u_{n+1}(\omega))), \\
&u_{n+1}(\omega) = P_{K(\omega, u_n(\omega))}(u_n(\omega) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega))))
\end{aligned} \tag{3.6}
$$

for all $\omega \in \Omega$ *and* $n = 0, 1, 2, \ldots,$ *where* ρ *and* A *are the same as in Lemma* 3.5.

4. EXISTENCE AND CONVERGENCE

In this section, we discuss the existence of random solutions for the random generalized set-valued strongly nonlinear implicit quasi-variational inequality (2.1) and the convergence of the random iterative sequences generated by the algorithm.

DEFINITION 4.1 *A random operator $f:\Omega \times H \to H$ is said*

(1) *to be strongly monotone if there exists a measurable function $\delta:\Omega \to (0,\infty)$ such that*

$$(f(\omega,u) - f(\omega,v), u - v) \geq \delta(\omega)\|u - v\|^2$$

for all $\omega \in \Omega$ and $u, v \in H$,

(2) *to be Lipschitz continuous if there exists a measurable function $\gamma:\Omega \to (0,\infty)$ such that*

$$\|f(\omega,u) - f(\omega,v)\| \leq \gamma(\omega)\|u - v\|$$

for all $\omega \in \Omega$ and $u, v \in H$.

The measurable functions $\delta(\omega)$ and $\gamma(\omega)$ are called the strongly monotone coefficient and Lipschitz coefficient, respectively.

DEFINITION 4.2 *A random set-valued mapping $V:\Omega \times H \to CB(H)$ is said to be h-Lipschitz continuous if there exists a measurable function $\eta:\Omega \to (0,\infty)$ such that*

$$h(V(\omega,u), V(\omega,v)) \leq \eta(\omega)\|u - v\|$$

for all $\omega \in \Omega$ and $u, v \in H$. The measurable function $\eta(\omega)$ is called the h-Lipschitz coefficient.

THEOREM 4.1 *Suppose that $K:\Omega \times H \to 2^H$ is a random set-valued mapping such that for each $\omega \in \Omega$ and $x \in H$, $K(\omega,x)$ is a nonempty closed convex subset of H and the projection $P_{K(\omega,x)}$ is a Lipschitz continuous random operator with the Lipschitz coefficient $\gamma(\omega)$. Let $g:\Omega \times H \to H$ be a strongly monotone Lipschitz continuous random operator with the strongly monotone coefficient $\delta(\omega)$ and Lipschitz coefficient $\sigma(\omega)$, respectively. Suppose that $U, V:\Omega \times H \to CB(H)$ are h-Lipschitz continuous random set-valued mappings with the h-Lipschitz coefficients $\zeta(\omega)$ and $\eta(\omega)$, respectively, and $a:\Omega \times H \times H \to \mathbb{R}$ is a random coercive* *bounded bilinear function with the coercive coefficients* $\alpha(\omega)$ *and* $\beta(\omega)$, *respectively. If, for any* $\omega \in \Omega$,

$$\left|\rho(\omega) - \frac{\alpha(\omega) - (1 - k(\omega))\eta(\omega)}{\beta^2(\omega) - \eta^2(\omega)}\right| < \sqrt{\frac{[\alpha(\omega) - (1 - k(\omega))\eta(\omega)]^2 - (\beta^2(\omega) - \eta^2(\omega))k(\omega)(2 - k(\omega))}{\beta^2(\omega) - \eta^2(\omega)}}, \tag{4.1}$$

$$\alpha(\omega) > (1 - k(\omega))\eta(\omega) + \sqrt{(\beta^2(\omega) - \eta^2(\omega))k(\omega)(2 - k(\omega))}, \tag{4.2}$$

$$\rho(\omega)\eta(\omega) < 1 - k(\omega), \quad \eta(\omega) < \alpha(\omega), \tag{4.3}$$

$$k(\omega) = \gamma(\omega)\zeta(\omega) + 2\sqrt{1 - 2\delta(\omega) + \sigma^2(\omega)} < 1, \tag{4.4}$$

then there exists measurable mappings $u, x, y : \Omega \to H$ *which are the solutions of the random generalized set-valued strongly nonlinear implicit quasi-variational inequality* (2.1) *and for any* $\omega \in \Omega$,

$$u_n(\omega) \to u(\omega), \ x_n(\omega) \to x(\omega), \ y_n(\omega) \to y(\omega) \quad \text{as } n \to \infty,$$

where $\{u_n(\omega)\}$, $\{x_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *are three sequence of measurable mappings generated by Algorithm* 3.1.

Proof By Algorithm 3.1, Definition 3.1 and Lemma 3.3, we have

$$\begin{aligned}
&\|u_{n+1}(\omega) - u_n(\omega)\| \\
&\quad = \|u_n(\omega) - g(\omega, u_n(\omega)) - u_{n-1}(\omega) + g(\omega, u(\omega)) \\
&\qquad + P_{K(\omega,x_n(\omega))}(g(\omega, u_n(\omega)) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \\
&\qquad - P_{K(\omega,x_{n-1}(\omega))}(g(\omega, u_{n-1}(\omega)) + \rho(\omega)(y_{n-1}(\omega) - A(\omega, u_{n-1}(\omega))))\| \\
&\quad \le \|u_n(\omega) - u_{n-1}(\omega) - (g(\omega, u_n(\omega)) - g(\omega, u_{n-1}(\omega)))\| \\
&\qquad + \|P_{K(\omega,x_n(\omega))}(g(\omega, u_n(\omega)) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \\
&\qquad - P_{K(\omega,x_n(\omega))}(g(\omega, u_{n-1}(\omega)) + \rho(\omega)(y_{n-1}(\omega) - A(\omega, u_{n-1}(\omega))))\| \\
&\qquad + \|P_{K(\omega,x_n(\omega))}(g(\omega, u_{n-1}(\omega)) + \rho(\omega)(y_{n-1}(\omega) - A(\omega, u_{n-1}(\omega)))) \\
&\qquad - P_{K(\omega,x_{n-1}(\omega))}(g(\omega, u_{n-1}(\omega)) + \rho(\omega)(y_{n-1}(\omega) - A(\omega, u_{n-1}(\omega))))\|
\end{aligned}$$

$$\leq 2\|u_n(\omega) - u_{n-1}(\omega) - (g(\omega, u_n(\omega)) - g(\omega, u_{n-1}(\omega))\|$$
$$+ \gamma(\omega)\|x_n(\omega) - x_{n-1}(\omega)\| + \rho(\omega)\|y_n(\omega) - y_{n-1}(\omega)\|$$
$$+ \|u_n(\omega) - u_{n-1}(\omega) - \rho(\omega)(A(\omega, u_n(\omega)) - A(\omega, u_{n-1}(\omega)))\|. \quad (4.5)$$

Since $a(\omega, u, v)$ is a random coercive bounded bilinear function and g is a strongly monotone Lipschitz continuous random operator, we have

$$\|u_n(\omega) - u_{n-1}(\omega) - \rho(\omega)(A(\omega, u_n(\omega)) - A(\omega, u_{n-1}(\omega)))\|^2$$
$$\leq (1 - 2\rho(\omega)\alpha(\omega) + \rho^2(\omega)\beta^2(\omega))\|u_n(\omega) - u_{n-1}(\omega)\|^2 \quad (4.6)$$

and

$$\|u_n(\omega) - u_{n-1}(\omega) - (g(\omega, u_n(\omega)) - g(\omega, u_{n-1}(\omega)))\|^2$$
$$\leq (1 - 2\delta(\omega) + \sigma^2(\omega))\|u_n(\omega) - u_{n-1}(\omega)\|^2. \quad (4.7)$$

Further, since U, V are h-Lipschitz continuous, from (3.3), it follows that

$$\|x_n(\omega) - x_{n-1}(\omega)\| \leq \left(1 + \frac{1}{n}\right) h(U(\omega, u_n(\omega)), U(\omega, u_{n-1}(\omega)))$$
$$\leq \left(1 + \frac{1}{n}\right) \zeta(\omega)\|u_n(\omega) - u_{n-1}(\omega)\| \quad (4.8)$$

and

$$\|y_n(\omega) - y_{n-1}(\omega)\| \leq \left(1 + \frac{1}{n}\right) h(V(\omega, u_n(\omega)), V(\omega, u_{n-1}(\omega)))$$
$$\leq \left(1 + \frac{1}{n}\right) \eta(\omega)\|u_n(\omega) - u_{n-1}(\omega)\|. \quad (4.9)$$

It follows from the above inequalities, (4.5)–(4.9), that

$$\|u_{n+1}(\omega) - u_n(\omega)\| \leq \theta_n(\omega)\|u_n(\omega) - u_{n-1}(\omega)\|, \quad (4.10)$$

where

$$\theta_n(\omega) = k_n(\omega) + \sqrt{1 - 2\rho(\omega)\alpha(\omega) + \rho^2(\omega)\beta^2(\omega)} + \rho(\omega)\eta(\omega)\left(1 + \frac{1}{n}\right)$$

and

$$k_n(\omega) = \gamma(\omega)\zeta(\omega)\left(1 + \frac{1}{n}\right) + 2\sqrt{1 - 2\delta(\omega) + \sigma^2(\omega)}.$$

In view of (4.1)–(4.4), we know that $0 < \theta_n(\omega) < 1$ for sufficiently large n. By (4.10), $\{u_n(\omega)\}$ is a Cauchy sequence and so $u_n(\omega) \to u(\omega)$ as $n \to \infty$. By virtue of (4.8) and (4.9), it is easy to see that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are both Cauchy sequences in H. Let $x_n(\omega) \to x(\omega)$ and $y_n(\omega) \to y(\omega)$ as $n \to \infty$. Since $\{u_n(\omega)\}$, $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are all measurable sequences of mappings, we know that $u, x, y : \Omega \to H$ are measurable.

Now we prove that

$$g(\omega, u(\omega)) = P_{K(\omega,x(\omega))}(g(\omega, u(\omega)) + \rho(\omega)(y(\omega) - A(\omega, u(\omega)))). \quad (4.11)$$

In fact, from (3.3), we know that it is enough to prove that

$$\begin{aligned}
&\lim_{n\to\infty} P_{K(\omega,x_n(\omega))}(g(\omega, u_n(\omega)) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \\
&\quad = P_{K(\omega,x(\omega))}(g(\omega, u(\omega)) + \rho(\omega)(y(\omega) - A(\omega, u(\omega)))). \qquad (4.12)
\end{aligned}$$

It follows from Definition 3.1 and Lemma 3.3, we have

$$\begin{aligned}
&\|P_{K(\omega,x_n(\omega))}(g(\omega, u_n(\omega)) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \\
&\quad - P_{K(\omega,x(\omega))}(g(\omega, u(\omega)) + \rho(\omega)(y(\omega) - A(\omega, u(\omega))))\| \\
&\qquad \le \|P_{K(\omega,x_n(\omega))}(g(\omega, u_n(\omega)) + \rho(\omega)(y_n(\omega) - A(\omega, u_n(\omega)))) \\
&\qquad\quad - P_{K(\omega,x_n(\omega))}(g(\omega, u(\omega)) + \rho(\omega)(y(\omega) - A(\omega, u(\omega))))\| \\
&\qquad\quad + \|P_{K(\omega,x_n(\omega))}(g(\omega, u(\omega)) + \rho(\omega)(y(\omega) - A(\omega, u(\omega)))) \\
&\qquad\quad - P_{K(\omega,x(\omega))}(g(\omega, u(\omega)) + \rho(\omega)(y(\omega) - A(\omega, u(\omega))))\| \\
&\qquad \le \|g(\omega, u_n(\omega)) - (g(\omega, u(\omega))\| + \rho(\omega)\|y_n(\omega) - y(\omega)\| \\
&\qquad\quad + \rho(\omega)\|A(\omega, u_n(\omega)) - A(\omega, u(\omega))\| + \gamma(\omega)\|x_n(\omega) - x(\omega)\|.
\end{aligned}$$

This implies that the equality (4.12) holds and so the equality (4.11) is true.

Next, we prove that $x(\omega) \in U(\omega, u(\omega))$. In fact, we have

$$\begin{aligned} d(x(\omega), U(\omega, u(\omega))) &= \inf\{\|x(\omega) - z\|: z \in U(\omega, u(\omega))\} \\ &\leq \|x(\omega) - x_n(\omega)\| + d(x_n(\omega), U(\omega, u(\omega))) \\ &\leq \|x(\omega) - x_n(\omega)\| + h(U(\omega, u_n(\omega)), U(\omega, u(\omega))) \\ &\leq \|x(\omega) - x_n(\omega)\| + \zeta(\omega)\|u_n(\omega) - u(\omega)\|. \end{aligned} \tag{4.13}$$

From the above inequality (4.13), it is easy to see that $d(x(\omega), U(\omega, v(\omega))) = 0$. This implies that $x(\omega) \in U(\omega, u(\omega))$. Similarly, we have $y(\omega) \in V(\omega, u(\omega))$. Therefore, by (4.11) and Lemma 3.5, we know that u, x, y are the random solutions of random generalized set-valued strongly nonlinear implicit quasi-variational inequality (2.1) and $x_n(\omega) \to x(\omega)$, $y_n(\omega) \to y(\omega)$, $u_n(\omega) \to u(\omega)$ as $n \to \infty$. This completes the proof.

From Theorem 4.1, the following results can be obtained immediately.

THEOREM 4.2 *Let g, K, P_K, V and a be the same as in Theorem* 4.1. *If, for any $\omega \in \Omega$, the conditions* (4.1)–(4.4) *in Theorem* 4.1 *are satisfied for*

$$k(\omega) = \gamma(\omega) + 2\sqrt{1 - 2\delta(\omega) + \sigma^2(\omega)},$$

then there exist measurable mappings $u, y : \Omega \to H$ which are the random solutions of the random generalized set-valued nonlinear implicit quasi-variational inequality (2.2) *and for any $\omega \in \Omega$, $u_n(\omega) \to u(\omega)$, $y_n(\omega) \to y(\omega)$ as $n \to \infty$, where $\{u_n(\omega)\}$ and $\{y_n(\omega)\}$ are two sequences of measurable mappings generated by Algorithm* 3.2.

THEOREM 4.3 *Let K, P_K, U, V and a be the same as in Theorem* 4.1. *If for any $\omega \in \Omega$, the conditions* (4.1)–(4.4) *in Theorem* 4.1 *are satisfied for $k(\omega) = \gamma(\omega)\zeta(\omega)$, then there exist measurable mappings $u, x, y : \Omega \to H$ which are the random solution of the random generalized set-valued strongly nonlinear quasi-variational inequality* (2.3) *and for any $\omega \in \Omega$, $u_n(\omega) \to u(\omega)$, $x_n(\omega) \to x(\omega)$, $y_n(\omega) \to y(\omega)$ as $n \to \infty$, where $\{u_n(\omega)\}$, $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are three sequences of measurable mappings generated by Algorithm* 3.3.

THEOREM 4.4 *Let* K, P_K, V *and* a *be the same as in Theorem* 4.1. *If for any* $\omega \in \Omega$ *the conditions* (4.1)–(4.4) *in Theorem* 4.1 *are satisfied for* $k(\omega) = \gamma(\omega)$, *then there exist measurable mappings* $u, y : \Omega \to H$ *which are the random solution of the random generalized set-valued nonlinear quasi-variational inequality* (2.4) *and for any* $\omega \in \Omega$, $u_n(\omega) \to u(\omega)$ *and* $y_n(\omega) \to y(\omega)$ *as* $n \to \infty$, *where* $\{u_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *are two sequences of measurable mappings generated by Algorithm* 3.4.

Remark 4.1 From Theorems 4.1–4.4, we can obtain several known results of [5,8,13,14,16,21–23] as special cases.

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