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Singular and Nonsingular Boundary Value Problems with Sign Changing Nonlinearities

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This paper presents existence results for singular boundary value problems where the nonlinearity is allowed to change sign. Our theory is then applied to an example which arises naturally in the theory of shallow membrane caps.

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1. INTRODUCTION

This paper discusses problems of the form

$$\begin{aligned} &(py')' + p(t)q(t)f(t,y) = 0, \quad 0 < t < 1, \\ &\lim_{t\to 0^+} p(t)y'(t) = 0, \\ &y(1) = A \ge 0, \end{aligned} \tag{1.1}$$

where f is allowed to change sign and $1/p$ is *not* necessarily in $L^1[0, 1]$. In addition our nonlinear term $f(t, y)$ may not be Carathéodory function due to the singular behaviour of its y variable. When $A = 0$ we will refer to the problem as singular whereas if $A > 0$ we will say the problem is

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nonsingular. The theory presented in this paper was motivated by a nonsingular problem arising in the theory of shallow membrane caps [3,4,6], namely

$$\begin{aligned}
&(t^3y')' + \left(\frac{t^3}{8y^2} - a_0\frac{t^3}{y} - b_0t^{2\gamma-1}\right) = 0, \quad 0 < t < 1 \\
&\lim_{t\to 0^+} t^3y'(t) = 0, \\
&y(1) = A > 0, \quad a_0 \geq 0,\ b_0 > 0 \text{ and } \gamma > 1.
\end{aligned} \tag{1.2}$$

Our paper will be divided into two main sections. In Section 2 we present a slight variation of the classical theory of upper and lower solutions (see [3]) so that (1.1) can be discussed in both the singular and nonsingular situation. Section 3 discusses in more detail the singular problem. The theory presented here extends and generalizes some ideas introduced in [1,8]. In particular the results in [1] only hold if $f(t,y)=f(y)$ for $t\in[0,1]\backslash(\frac{1}{3},\frac{2}{3})$. In our paper we replace this with the less restrictive assumption $f(\cdot,y)$ is nondecreasing on $(0,\frac{1}{3})$ for each fixed $y\in(0,\infty)$ (see Remark 3.4 for a more general situation).

2. UPPER AND LOWER SOLUTION APPROACH

In this section we discuss the singular and nonsingular problem

$$\begin{aligned}
&(py')' + p(t)q(t)f(t,y) = 0, \quad 0 < t < 1, \\
&\lim_{t\to 0^+} p(t)y'(t) = 0, \\
&y(1) = A \geq 0.
\end{aligned} \tag{2.1}$$

Suppose the following conditions are satisfied:

$$p \in C[0,1] \cap C^1(0,1) \quad \text{with } p > 0 \text{ on } (0,1), \tag{2.2}$$

$$q \in C(0,1) \quad \text{with } q > 0 \text{ on } (0,1), \tag{2.3}$$

$$\int_0^1 p(s)q(s)\,ds < \infty \quad \text{and} \quad \int_0^1 \frac{1}{p(t)}\int_0^t p(s)q(s)\,ds\,dt < \infty, \tag{2.4}$$

$$\begin{aligned}&\text{there exists } \beta \in C[0,1] \cap C^2(0,1) \text{ with } p\beta' \in AC[0,1],\ \ \beta(1) \geq A,\\ &\lim_{t\to 0^+} p(t)\beta'(t) \leq 0 \text{ and } p(t)q(t)f(t,\beta(t)) + (p\beta')'(t) \leq 0\\ &\text{for } t \in (0,1),\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\text{there exists } \alpha \in C[0,1] \cap C^2(0,1) \text{ with } p\alpha' \in AC[0,1],\\ &\alpha(t) \leq \beta(t) \text{ on } [0,1],\ \alpha(1) \leq A,\ \lim_{t\to 0^+} p(t)\alpha'(t) \geq 0 \text{ and}\\ &p(t)q(t)f(t,\alpha(t)) + (p\alpha')'(t) \geq 0 \text{ for } t \in (0,1)\end{aligned} \tag{2.6}$$

and

$$\text{for each } t \in [0,1],\ f(t,u) \in \mathbf{R} \quad \text{for } u \in [\alpha(t),\beta(t)]. \tag{2.7}$$

Let

$$f^*(t,y) = \begin{cases} f(t,\beta(t)) + r(\beta(t)-y), & y \geq \beta(t),\\ f(t,y), & \alpha(t) < y < \beta(t),\\ f(t,\alpha(t)) + r(\alpha(t)-y), & y \leq \alpha(t)\end{cases}$$

and $r : \mathbf{R} \to [-1,1]$ is the radial retraction defined by

$$r(x) = \begin{cases} x, & |x| \leq 1,\\ \dfrac{x}{|x|}, & |x| > 1.\end{cases}$$

Finally we assume

$$f^* : [0,1] \times \mathbf{R} \to \mathbf{R} \text{ is continuous.} \tag{2.8}$$

THEOREM 2.1 *Suppose* (2.2)–(2.8) *hold. Then* (2.1) *has a solution* y (*here* $y \in C[0,1] \cap C^2(0,1)$ *with* $py' \in AC[0,1]$) *with* $\alpha(t) \leq y(t) \leq \beta(t)$ *for* $t \in [0,1]$.

Proof To show (2.1) has a solution we consider the problem

$$\begin{aligned}&(py')' + p(t)q(t)f^*(t,y) = 0, \quad 0 < t < 1,\\ &\lim_{t\to 0^+} p(t)y'(t) = 0,\\ &y(1) = A \geq 0.\end{aligned} \tag{2.9}$$

Solving (2.9) is equivalent to finding a $y \in C[0,1]$ to

$$y(t) = A + \int_t^1 \frac{1}{p(x)} \int_0^x p(s)q(s)f^*(s, y(s))\,\mathrm{d}s\,\mathrm{d}x.$$

Define the operator $N: C[0,1] \to C[0,1]$ by

$$Ny(t) = A + \int_t^1 \frac{1}{p(x)} \int_0^x p(s)q(s)f^*(s, y(s))\,\mathrm{d}s\,\mathrm{d}x.$$

A standard argument [3,9] implied $N: C[0,1] \to C[0,1]$ is continuous and compact. Now Schauder's fixed point theorem guarantees that N has a fixed point i.e. (2.9) has a solution $y \in C[0,1] \cap C^2(0,1)$ with $py' \in AC[0,1]$. The result will follow once we show

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for } t \in [0,1]. \tag{2.10}$$

We now show

$$y(t) \le \beta(t) \quad \text{for } t \in [0,1]. \tag{2.11}$$

Suppose (2.11) is not true. Then $y - \beta$ has a positive absolute maximum at $t_1 \in [0,1)$ (note $y(1) = A \le \beta(1)$). First let us take $t_1 \in (0,1)$. Then $(y-\beta)'(t_1) = 0$ and $(p(y-\beta)')'(t_1) \le 0$. However since $y(t_1) > \beta(t_1)$ we have

$$\begin{aligned}
&(p(y - \beta')'(t_1) \\
&\quad = -p(t_1)q(t_1)[f(t_1, \beta(t_1)) + r(\beta(t_1) - y(t_1))] - (p\beta')'(t_1) \\
&\quad \ge -p(t_1)q(t_1)r(\beta(t_1) - y(t_1)) > 0,
\end{aligned}$$

a contradiction. It remains to consider the case $t_1 = 0$. Notice

$$\lim_{t \to 0^+} p(t)[y - \beta]'(t) = -\lim_{t \to 0^+} p(t)\beta'(t) \ge 0,$$

which is a contradiction unless $\lim_{t\to 0^+} p(t)\beta'(t) = 0$. So assume $\lim_{t\to 0^+} p(t)\beta'(t) = 0$. Now there exists $\mu > 0$ with $y(s) - \beta(s) > 0$ for $s \in [0, \mu]$. Thus for $t \in (0, \mu)$ we have

$$\begin{aligned}
&p(y - \beta)'(t) \\
&\quad = -\int_0^t \{p(s)q(s)[f(s, \beta(s)) + r(\beta(s) - y(s))] + (p\beta')'(s)\}\,\mathrm{d}s > 0,
\end{aligned}$$

and this contradicts the fact that $y-\beta$ has a positive absolute maximum at $t_1=0$. Thus (2.11) holds. Similarly we can show

$$\alpha(t) \leq y(t) \quad \text{for } t \in [0,1]. \tag{2.12}$$

Our result follows.

The following examples arise in the theory of shallow membrane caps, see [2,4,6] and their references.

Example 2.1 Consider

$$\begin{aligned} &(t^3y')' + \left(\frac{t^3}{8y^2} - a_0\frac{t^3}{y} - b_0t^{2\gamma-1}\right) = 0, \quad 0 < t < 1, \\ &\lim_{t\to 0^+} t^3y'(t) = 0, \\ &y(1) = A > 0, \end{aligned} \tag{2.13}$$

where $a_0 > 0$, $b_0 > 0$ and $\gamma > 1$. Let

$$\beta(t) = \max\left\{A, \frac{1}{8a_0}\right\} \equiv \beta_0.$$

If $\gamma \geq 2$ let

$$\alpha(t) = \alpha_0 \quad \text{where } \alpha_0 = \min\left\{A, \frac{1}{2}\left(-\frac{a_0}{b_0} + \sqrt{\frac{a_0^2}{b_0^2} + \frac{1}{2b_0}}\right)\right\}, \tag{2.14}$$

whereas if $1 < \gamma < 2$ let

$$\alpha(t) = \alpha_0 t^{2-\gamma}.$$

Then (2.13) has a solution $y \in C[0,1] \cap C^2(0,1)$ with $t^3y' \in AC[0,1]$ and $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [0,1]$.

To see this we will apply Theorem 2.1. Choose ϵ, $0 \leq \epsilon < 2$ so that $\gamma \geq 2 - \epsilon/2$. Take $p(t) = t^3$, $q(t) = t^{-\epsilon}$ and

$$f(t,y) = \frac{t^\epsilon}{8y^2} - a_0\frac{t^\epsilon}{y} - b_0t^{2\gamma-4+\epsilon}.$$

Notice (2.2)–(2.4) and (2.7) are satisfied. If $\gamma \geq 2$ then

$$f^*(t,y) = \begin{cases} \dfrac{t^\epsilon}{8\beta_0^2} - a_0\dfrac{t^\epsilon}{\beta_0} - b_0 t^{2\gamma-4+\epsilon} + r(\beta_0 - y), & y \geq \beta_0, \\ \dfrac{t^\epsilon}{8y^2} - a_0\dfrac{t^\epsilon}{y} - b_0 t^{2\gamma-4+\epsilon}, & \alpha_0 < y < \beta_0, \\ \dfrac{t^\epsilon}{8\alpha_0^2} - a_0\dfrac{t^\epsilon}{\alpha_0} - b_0 t^{2\gamma-4+\epsilon} + r(\alpha_0 - y), & y \leq \alpha_0, \end{cases}$$

whereas if $1 < \gamma < 2$ then

$$f^*(t,y) = \begin{cases} \dfrac{t^\epsilon}{8\beta_0^2} - a_0\dfrac{t^\epsilon}{\beta_0} - b_0 t^{2\gamma-4+\epsilon} + r(\beta_0 - y), & y \geq \beta_0, \\ \dfrac{t^\epsilon}{8y^2} - a_0\dfrac{t^\epsilon}{y} - b_0 t^{2\gamma-4+\epsilon}, & \alpha_0 t^{2-\gamma} < y < \beta_0, \\ \dfrac{t^{2\gamma-4+\epsilon}}{8\alpha_0^2} - a_0\dfrac{t^{\gamma-2+\epsilon}}{\alpha_0} - b_0 t^{2\gamma-4+\epsilon} & \\ \quad + r(\alpha_0 t^{2-\gamma} - y), & y \leq \alpha_0 t^{2-\gamma}. \end{cases}$$

Clearly (2.8) is satisfied since $\epsilon \geq 4 - 2\gamma$. To see that $\beta(t) = \beta_0$ satisfied (2.5) notice $\beta(1) \geq A$, $\lim_{t\to 0^+} t^3\beta'(t) = 0$ and

$$\begin{aligned} &p(t)q(t)f(t,\beta(t)) + (p\beta')'(t) \\ &\quad = \frac{t^3}{\beta_0}\left(\frac{1}{8\beta_0} - a_0\right) - b_0 t^{2\gamma-1} \leq -b_0 t^{2\gamma-1} \leq 0 \end{aligned}$$

for $t \in (0,1)$ since $\beta(t) = \beta_0 \geq 1/(8a_0)$. It remains to show (2.6). We consider the cases $\gamma \geq 2$ and $1 < \gamma < 2$ seperately.

Case (i) $\gamma \geq 2$ Now $\alpha(t) \leq \beta(t)$, $\alpha(1) \leq A$, $\lim_{t\to 0^+} t^3\alpha'(t) = 0$ and

$$\begin{aligned} &p(t)q(t)f(t,\alpha(t)) + (p\alpha')'(t) \\ &\quad = \frac{t^3}{8\alpha_0^2} - \frac{t^3 a_0}{\alpha_0} - b_0 t^{2\gamma-1} = t^3\left(\frac{1}{8\alpha_0^2} - \frac{a_0}{\alpha_0} - b_0 t^{2\gamma-4}\right) \\ &\quad \geq \frac{t^3}{8\alpha_0^2}\left(1 - 8a_0\alpha_0 - 8b_0\alpha_0^2\right) = -\frac{t^3 b_0}{\alpha_0^2}\left(\alpha_0^2 + \frac{a_0}{b_0}\alpha_0 - \frac{1}{8b_0}\right) \\ &\quad = -\frac{t^3 b_0}{\alpha_0^2}(\alpha_0 - x_0)(\alpha_0 + x_0) \geq 0 \end{aligned}$$

for $t \in (0, 1)$ since $\alpha_0 \le x_0$; here

$$x_0 = \frac{1}{2}\left(-\frac{a_0}{b_0} + \sqrt{\frac{a_0^2}{b_0^2} + \frac{1}{2b_0}}\right).$$

Hence (2.6) is true in this case.

Case (ii) $1 < \gamma < 2$ Now $\alpha(1) \le A$, $\lim_{t \to 0^+} t^3 \alpha'(t) = \alpha_0(2-\gamma)$ $\lim_{t \to 0^+} t^{4-\gamma} = 0$ and

$$\begin{aligned}
&p(t)q(t)f(t,\alpha(t)) + (p\alpha')'(t) \\
&= \alpha_0(2-\gamma)(4-\gamma)t^{3-\gamma} + t^{2\gamma-1}\left(\frac{1}{8\alpha_0^2} - \frac{a_0 t^{2-\gamma}}{\alpha_0} - b_0\right) \\
&\ge \alpha_0(2-\gamma)(4-\gamma)t^{3-\gamma} + t^{2\gamma-1}\left(\frac{1}{8\alpha_0^2} - \frac{a_0}{\alpha_0} - b_0\right) \\
&= \alpha_0(2-\gamma)(4-\gamma)t^{3-\gamma} - \frac{t^{2\gamma-1}b_0}{\alpha_0^2}(\alpha_0 - x_0)(\alpha_0 + x_0) \ge 0
\end{aligned}$$

for $t \in (0, 1)$ since $\alpha_0 \le x_0$; here

$$x_0 = \frac{1}{2}\left(-\frac{a_0}{b_0} + \sqrt{\frac{a_0^2}{b_0^2} + \frac{1}{2b_0}}\right).$$

Hence (2.6) is true in this case.

Now Theorem 2.1 guarantees that (2.13) has the desired solution.

Example 2.2 Consider

$$\begin{aligned}
&(t^3y')' + \left(\frac{t^3}{8y^2} - b_0 t^{2\gamma-1}\right) = 0, \quad 0 < t < 1, \\
&\lim_{t \to 0^+} t^3 y'(t) = 0, \\
&y(1) = A > 0,
\end{aligned} \tag{2.15}$$

where $b_0 > 0$ and $\gamma > 1$ (note (2.15) is (2.13) with $a_0 = 0$). If $\gamma \ge 2$ let

$$\alpha(t) = \min\left\{A, \frac{1}{\sqrt{8b_0}}\right\} \equiv \alpha_1 \quad \text{and} \quad \beta(t) = \frac{1}{24A^2}(1-t) + A,$$

whereas if $1 < \gamma < 2$ let

$$\alpha(t) = \alpha_1 t^{2-\gamma} \quad \text{and} \quad \beta(t) = A + \frac{1}{\sqrt{8b_0}}.$$

Then (2.15) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $t^3 y' \in AC[0, 1]$ and $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0, 1]$.

To see this we will apply Theorem 2.1. Choose ϵ, $0 \le \epsilon < 2$ so that $\gamma \ge 2 - \epsilon/2$. Take $p(t) = t^3$, $q(t) = t^{-\epsilon}$ and

$$f(t, y) = \frac{t^\epsilon}{8y^2} - b_0 t^{2\gamma-4+\epsilon}.$$

Notice (2.2)–(2.4), (2.7) and (2.8) are satisfied. To show (2.5) and (2.6) hold we consider the cases $\gamma \ge 2$ and $1 < \gamma < 2$ seperately.

Case (i) $\gamma \ge 2$ Notice $\beta(t) = (1/(24A^2))(1 - t) + A$ satisfies (2.5) since $\beta(1) = A$, $\lim_{t \to 0^+} t^3 \beta'(t) = 0$ and

$$\begin{aligned} &p(t)q(t)f(t, \beta(t)) + (p\beta')'(t) \\ &= -\frac{t^2}{8A^2} + \frac{t^3}{8[(1/(24A^2))(1 - t) + A]^2} - b_0 t^{2\gamma-1} \\ &\le t^2 \left[-\frac{1}{8A^2} + \frac{t}{8A^2} - b_0 t^{2\gamma-3} \right] \le 0 \end{aligned}$$

for $t \in (0, 1)$. Also $\alpha(t) = \alpha_1$ satisfies (2.6) since $\alpha(1) \le A$, $\lim_{t \to 0^+} t^3 \alpha'(t) = 0$ and

$$\begin{aligned} p(t)q(t)f(t, \alpha(t)) + (p\alpha')'(t) &= t^3 \left(\frac{1}{8\alpha_1^2} - b_0 t^{2\gamma-4} \right) \ge t^3 (b_0 - b_0 t^{2\gamma-4}) \\ &= b_0 t^3 (1 - t^{2\gamma-4}) \ge 0 \end{aligned}$$

for $t \in (0, 1)$.

Case (ii) $1 < \gamma < 2$ Notice $\beta(t) = A + 1/\sqrt{8b_0}$ satisfies (2.5) since

$$\begin{aligned} p(t)q(t)f(t, \beta(t)) + (p\beta')'(t) &\le t^3 b_0 - b_0 t^{2\gamma-1} \\ &= b_0 t^{2\gamma-1} (t^{4-2\gamma} - 1) \le 0 \end{aligned}$$

for $t \in (0,1)$. Also $\alpha(t) = \alpha_1 t^{2-\gamma}$ satisfies (2.6) since

$$p(t)q(t)f(t,\alpha(t)) + (p\alpha')'(t)$$
$$= \alpha_1(2-\gamma)(4-\gamma)\,t^{3-\gamma} + t^{2\gamma-1}\left(\frac{1}{8\alpha_1^2} - b_0\right) \geq 0$$

for $t \in (0,1)$.

Now Theorem 2.1 guarantees that (2.15) has the desired solution.

3. SINGULAR PROBLEM USING A GROWTH APPROACH

In this section we discuss the singular problem

$$\begin{aligned} &(py')' + p(t)q(t)f(t,y) = 0, \quad 0 < t < 1, \\ &\lim_{t\to 0^+} p(t)y'(t) = 0, \\ &y(1) = 0. \end{aligned} \tag{3.1}$$

We are interested in nonnegative solutions (in fact solutions y with $y > 0$ on $[0,1)$). One can observe that if we use the upper and lower solution technique of Section 2 (we use (3.7) and (3.8) to construct the upper solution and (3.5) and (3.6) to construct the lower solution) we have to assume (2.8). As a result we present a different approach for singular problems in this section. We remark here that a similar theory could be obtained for nonsingular problems (the proofs are a lot easier in this case). In particular the results in this section improve those in [1] since in [1] we had to assume $f(t,y) = f(y)$ for $t \in [0,1]\backslash(1/n, 1-(1/n))$ for some $n \in \{3,4,\ldots\}$.

Throughout this section we will assume (2.1)–(2.4) hold. For our first result we will suppose the following conditions are satisfied:

$$f\colon [0,1]\times(0,\infty) \to \mathbf{R} \text{ is continuous.} \tag{3.2}$$

$$f(\cdot,y) \text{ is nondecreasing on } \left(0,\tfrac{1}{3}\right) \quad \text{for each fixed } y \in (0,\infty). \tag{3.3}$$

$$\begin{aligned} &|f(t,y)| \leq g(y) + h(y) \text{ on } [0,1]\times(0,\infty) \\ &\text{with } g > 0 \text{ continuous and nonincreasing on } (0,\infty), \\ &h \geq 0 \text{ continuous on } [0,\infty) \text{ and} \\ &h/g \text{ nondecreasing on } (0,\infty). \end{aligned} \tag{3.4}$$

let $n \in \{3,4,\ldots\}$ and associated with each n we have a constant ρ_n such that $\{\rho_n\}$ is a nonincreasing sequence with $\lim_{t\to\infty} \rho_n = 0$ and such that for $1/n \le t \le 1$ we have $p(t)q(t)f(t,\rho_n) \ge 0$ (3.5)

there exists a function $\alpha \in C[0,1] \cap C^2(0,1)$ with $p\alpha' \in AC[0,1]$, $\lim_{t\to 0^+} p(t)\alpha'(t) = \alpha(1) = 0$, $\alpha > 0$ on $[0,1)$ such that $p(t)q(t)f(t,y) + (p(t)\alpha'(t))' > 0$ for $(t,y) \in (0,1) \times \{y \in (0,\infty) : y < \alpha(t)\}$ (3.6)

for any $R > 0$, $1/g$ is differentiable on $(0,R]$ with $g' < 0$ a.e. on $(0,R]$ and $g'/g^2 \in L^1[0,R]$ (3.7)

and

$$\sup_{c\in(0,\infty)} \left(\frac{1}{\{1+h(c)/g(c)\}} \int_0^c \frac{\mathrm{d}u}{g(u)} \right) > \int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.8}$$

THEOREM 3.1 *Suppose* (2.2)–(2.4) *and* (3.2)–(3.8) *hold. In addition assume*

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)g(\alpha(x))\,\mathrm{d}x\,\mathrm{d}s < \infty \tag{3.9}$$

is satisfied. Then (3.1) *has a solution* y (*here* $y \in C[0,1] \cap C^2(0,1)$ *with* $py' \in AC[0,1]$) *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,1]$.

Proof Choose $M > 0$ and $\epsilon > 0$ ($\epsilon < M$) with

$$\frac{1}{\{1+h(M)/g(M)\}} \int_\epsilon^M \frac{\mathrm{d}u}{g(u)} > \int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.10}$$

Let $m_0 \in \{3,4,\ldots\}$ be chosen so that $\rho_{m_0} < \epsilon$ and let $N^+ = \{m_0, m_0+1,\ldots\}$. We begin by showing

$$\begin{aligned} &(py')' + p(t)q(t)f^{**}(t,y) = 0, \quad 0 < t < 1,\\ &\lim_{t\to 0^+} p(t)y'(t) = 0,\\ &y(1) = \rho_n, \end{aligned} \tag*{$(3.11)^n$}$$

has a solution for each $n \in N^+$; here

$$f^{**}(t,y) = \begin{cases} f\left(\frac{1}{n}, y\right), & y \geq \rho_n \text{ and } 0 \leq t \leq \frac{1}{n} \\ f(t,y), & y \geq \rho_n \text{ and } \frac{1}{n} \leq t \leq 1 \\ f(t,\rho_n) + \rho_n - y, & y < \rho_n \text{ and } \frac{1}{n} \leq t \leq 1 \\ f\left(\frac{1}{n}, \rho_n\right) + \rho_n - y, & y < \rho_n \text{ and } 0 \leq t \leq \frac{1}{n}. \end{cases}$$

Remark 3.1 Notice (3.4) implies $|f^{**}(t,y)| \leq g(y) + h(y)$ if $y \geq \rho_n$ and $t \in [0,1]$. Also (3.3) implies $f^{**}(t,y) \geq f(t,y)$, $t \in (0,1)$ for each fixed $y \geq \rho_n$.

To show $(3.11)^n$ has a solution for each $n \in N^+$ we apply [1, Theorem 2.9]. Fix $n \in N^+$ and consider the family of problems

$$\begin{aligned} &(py')' + \lambda p(t)q(t)f^{**}(t,y) = 0, \quad 0 < t < 1, \ 0 < \lambda < 1, \\ &\lim_{t \to 0^+} p(t)y'(t) = 0, \\ &y(1) = \rho_n. \end{aligned} \tag{$3.12)^n_\lambda$}$$

First we show

$$y(t) \geq \rho_n \quad \text{for } t \in [0,1] \tag{3.13}$$

for any solution y to $(3.12)^n_\lambda$. Suppose (3.13) is not true. Then $y - \rho_n$ has a negative absolute minimum at $t_0 \in [0,1)$ (note $y(1) - \rho_n = 0$). First let us take the case $t_0 \in (0,1)$. Then $y'(t_0) = 0$ and $(py')'(t_0) \geq 0$ (note $y(t_0) - \rho_n < 0$). However

$$\begin{aligned} (py')'(t_0) &= -\lambda p(t_0)q(t_0)f^{**}(t_0, y(t_0)) \\ &= \begin{cases} -\lambda p(t_0)q(t_0)[f(t_0,\rho_n) + \rho_n - y(t_0)] \\ \quad \text{if } \frac{1}{n} \leq t_0 < 1 \\ -\lambda p(t_0)q(t_0)\left[f\left(\frac{1}{n}, \rho_n\right) + \rho_n - y(t_0)\right] \\ \quad \text{if } 0 < t_0 \leq \frac{1}{n} \end{cases} \\ &< 0, \end{aligned}$$

a contradiction. It remains to consider the case $t_0 = 0$. Notice $\lim_{t \to 0^+} p(t)[y - \rho_n]'(t) = 0$. Also since $y(0) - \rho_n < 0$ there exists $\delta > 0$

with $y(s) - \rho_n < 0$ for $s \in [0, \delta]$. Thus for $t \in (0, \delta)$,

$$p(t)(y - \rho_n)'(t) = -\int_0^t \lambda p(s)q(s)f^{**}(s, y(s))\, ds < 0,$$

and this contradicts the fact that $y - \rho_n$ has a negative absolute maximum at $t_0 = 0$. Thus (3.13) holds.

Now since $y(1) = \rho_n$ and $y(t) \geq \rho_n$ on [0, 1] we may assume the absolute maximum of y occurs at say $t_n \in [0, 1)$, so $\lim_{t \to t_n} p(t)y'(t) = 0$. Without loss of generality assume $y(t_n) > \epsilon$. For $x \in (0, 1)$ we have from Remark 3.1 that

$$\frac{-(p(x)y'(x))'}{g(y(x))} \leq p(x)q(x)\left\{1 + \frac{h(y(x))}{g(y(x))}\right\}.$$

Integrate from t_n to $t(t > t_n)$ to obtain

$$\frac{-p(t)y'(t)}{g(y(t))} + \int_{t_n}^t \left\{\frac{-g'(y(x))}{g^2(y(x))}\right\} p(x)[y'(x)]^2\, dx \leq \left\{1 + \frac{h(y(t_n))}{g(y(t_n))}\right\} \int_{t_n}^t p(x)q(x)\, dx$$

and so (see (3.7))

$$\frac{-y'(t)}{g(y(t))} \leq \left\{1 + \frac{h(y(t_n))}{g(y(t_n))}\right\} \frac{1}{p(t)} \int_0^t p(s)q(s)\, ds \quad \text{for } t \in (t_n, 1).$$

Integrate from t_n to 1 to obtain

$$\int_{\rho_n}^{y(t_n)} \frac{du}{g(u)} \leq \left\{1 + \frac{h(y(t_n))}{g(y(t_n))}\right\} \int_{t_n}^1 \frac{1}{p(t)} \int_0^t p(s)q(s)\, ds\, dt$$

and so

$$\int_{\epsilon}^{y(t_n)} \frac{du}{g(u)} \leq \left\{1 + \frac{h(y(t_n))}{g(y(t_n))}\right\} \int_0^1 \frac{1}{p(t)} \int_0^t p(s)q(s)\, ds\, dt. \tag{3.14}$$

Now (3.10) and (3.14) imply $|y|_0 = \sup_{t \in [0,1]} |y(t)| \neq M$. Thus [1, Theorem 2.9] implies $(3.11)^n$ has a solution y_n with $|y_n|_0 \leq M$.

Also (as above),

$$\rho_n \le y_n(t) < M \quad \text{for } t \in [0,1]. \tag{3.15}$$

Next we obtain a sharper lower bound on y_n, namely we will show

$$y_n(t) \ge \alpha(t) \quad \text{for } t \in [0,1]. \tag{3.16}$$

Suppose (3.16) is not true. Then $y_n - \alpha$ has a negative absolute minimum at $t_1 \in [0,1)$ (note $y_n(1) - \alpha(1) = \rho_n > 0$). First let us take $t_1 \in (0,1)$. Then $(y_n - \alpha)'(t_1) = 0$ and $(p(y_n - \alpha)')'(t_1) \ge 0$. However since $0 < y_n(t_1) < \alpha(t_1)$ and $y_n(t_1) \ge \rho_n$ we have from (3.6) and Remark 3.1 that

$$\begin{aligned}(p(y_n - \alpha)')'(t_1) &= -[p(t_1)q(t_1)f^{**}(t_1, y_n(t_1)) + (p\alpha')'(t_1)] \\ &\le -[p(t_1)q(t_1)f(t_1, y_n(t_1)) + (p\alpha')'(t_1)] \\ &< 0,\end{aligned}$$

a contradiction. It remains to consider the case $t_1 = 0$. Notice $\lim_{t\to 0^+} p(t)[y_n - \alpha]'(t) = 0$. Now there exists $\mu > 0$ with $0 < y_n(s) < \alpha(s)$ for $t \in [0,\mu]$ (also note $y_n(s) \ge \rho_n$ for $s \in [0,\mu]$). Thus for $t \in (0,\mu)$ we have

$$p(y_n - \alpha)'(t) = -\int_0^t [p(s)q(s)f^{**}(s, y_n(s)) + (p\alpha')'(s)]\, ds < 0,$$

and this contradicts the fact that $y_n - \alpha$ has a negative absolute minimum at $t_1 = 0$. Thus (3.16) is true.

Remark 3.2 It is easy to check directly, using (3.6) and the ideas used to prove (3.14), that $\alpha(t) \le M$ for all $t \in [0,1]$.

We shall now obtain a solution of (3.1) by means of the Arzela–Ascoli theorem, as a limit of solutions of $(3.11)^n$. To this end we will show

$$\{y_n\}_{n\in N^+} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{3.17}$$

To show equicontinuity notice

$$\begin{aligned}|f^{**}(t, y_n(t))| &\le g(y_n(t))\left\{1 + \frac{h(y_n(t))}{g(y_n(t))}\right\} \\ &\le g(\alpha(t))\left\{1 + \frac{h(M)}{g(M)}\right\} \quad \text{for } t \in (0,1).\end{aligned}$$

This together with the differential equation gives

$$|y_n'(t)| \le \frac{1}{p(t)}\left\{1+\frac{h(M)}{g(M)}\right\}\int_0^t p(s)q(s)g(\alpha(s))\,\mathrm{d}s \quad \text{for } t\in(0,1)$$

and this together with (3.9) establishes (3.17).

The Arzela–Ascoli theorem guarantees the existence of a subsequence N_0 of N^+ and a function $y\in C[0,1]$ with y_n converging uniformly on $[0,1]$ to y as $n\to\infty$ through N_0. Also $y(1)=0$ and $y(t)\ge\alpha(t)$ for $t\in[0,1]$. Fix $t\in(0,1)$ and let $n_1\in N_0$ be such that $1/n_1<t<1$. Let $N_1=\{n\in N_0\colon n\ge n_1\}$. Now y_n, $n\in N_1$, satisfies the integral equation

$$\begin{aligned} y_n(t) &= y_n(0) - \int_0^{1/n}\frac{1}{p(x)}\int_0^x p(s)q(s)f\left(\frac{1}{n},y_n(s)\right)\mathrm{d}s\,\mathrm{d}x \\ &\quad - \int_0^t \frac{1}{p(x)}\chi_{[1/n,t]}(x)\left[\int_0^{1/n} p(s)q(s)f\left(\frac{1}{n},y_n(s)\right)\mathrm{d}s\right. \\ &\quad \left. + \int_0^x p(s)q(s)f(s,y_n(s))\chi_{[1/n,x]}(s)\,\mathrm{d}s\right]\mathrm{d}x. \end{aligned}$$

For $s\in[0,t]$ we have $f(s,y_n(s))\to f(s,y(s))$ uniformly on compact subsets of $[0,t]\times(0,M]$, so letting $n\to\infty$ through N_1 gives

$$y(t) = y(0) - \int_0^t\frac{1}{p(x)}\int_0^x p(s)q(s)f(s,y(s))\,\mathrm{d}s\,\mathrm{d}x. \tag{3.18}$$

We can do this argument for each $t\in(0,1)$.

Remark 3.3 Notice to apply this step we need only $\int_0^a 1/p(x)\int_0^x p(s)\times q(s)g(\alpha(s))\,\mathrm{d}s\,\mathrm{d}x<\infty$ for any $a\in(0,1)$. This is automatically satisfied since (2.4) holds and $\alpha(s)>0$ for $s\in[0,a]$. As a result (3.9) is *not* needed in this step.

Therefore from the integral equation (3.18) we see that $(py')'(t)+p(t)q(t)f(t,y(t))=0$, $0<t<1$ and $\lim_{t\to0^+}p(t)y'(t)=0$.

Remark 3.4 If in (3.5) we replace $1/n\le t\le 1$ with $0\le t\le 1-1/n$ then one would replace (3.3) with: $f(t,y)$ is nonincreasing on $(\frac{2}{3},1)$ for each fixed $y \in (0, \infty)$. More generally if in (3.5) we replace $1/n \leq t \leq 1$ with $1/n \leq t \leq 1 - 1/n$ then one would replace (3.3) with the following: for any fixed $y \in (0, \infty)$ there exists $\epsilon, 0 < \epsilon < \frac{1}{2}$ with $f(t, y)$ nondecreasing on $(0, \epsilon)$ and $f(t, y)$ nonincreasing on $(1 - \epsilon, 1)$. Finally if in (3.5) we replace $1/n \leq t \leq 1$ with $0 \leq t \leq 1$ then assumption (3.3) is not needed.

It is worth remarking that the only place we needed assumption (3.9) was in proving (3.17). It is possible to put other conditions on p, q and f to guarantee that (3.17) holds.

THEOREM 3.2 *Suppose* (2.2)–(2.4) *and* (3.2)–(3.8) *hold. In addition assume*

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty \tag{3.19}$$

and

$$\int_0^\infty \frac{|g'(t)|^{1/2}}{g(t)} \, \mathrm{d}t = \infty \tag{3.20}$$

are satisfied. Then (3.1) *has a solution* y (*here* $y \in C[0, 1] \cap C^2(0, 1)$ *with* $py' \in AC[0, 1]$) *with* $y(t) \geq \alpha(t)$ *for* $t \in [0, 1]$.

Proof The proof is essentially the same as in Theorem 3.1 except to prove (3.17) we use the argument in [7, p. 74].

Remark 3.5 One can usually "construct" α from the differential equation. For a more detailed discussion we refer the reader to [1,5,8].

Example 3.1 The boundary value problem

$$\begin{aligned} &(t^3 y')' + t^2\left(\frac{1}{\sqrt{y}} - \mu\right) = 0, \quad 0 < t < 1, \\ &\lim_{t \to 0^+} t^3 y'(t) = 0, \\ &y(1) = 0, \quad \mu > 0 \end{aligned} \tag{3.21}$$

has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $py' \in AC[0, 1]$ and $y(t) > 0$ for $t \in [0, 1)$.

We will apply Theorem 3.1. Take $p(t) = t^3$, $q(t) = 1/t$, $f(t, y) = 1/\sqrt{y} - \mu$, $g(y) = 1/\sqrt{y}$ and $h(y) = \mu$. Notice (2.2)–(2.4) and

(3.2)–(3.4) are satisfied. Choose $n_0 \in \{1,2,\dots\}$ so that $n_0 \geq \mu^2$ and let

$$\rho_n = \frac{1}{n+n_0}.$$

Now (3.5) is true since

$$\begin{aligned} p(t)q(t)f(t,\rho_n) &= t^2[(n+n_0)^{1/2} - \mu] \\ &\geq t^2(n_0^{1/2} - \mu) \geq 0 \quad \text{for } t \in (0,1). \end{aligned}$$

Next let

$$a(t) = a_0(1-t) \quad \text{where } a_0 \geq 0 \text{ is chosen so that } 3a_0^{3/2} + \mu a_0^{1/2} < 1.$$

Now $\alpha(1) = 0$, $\lim_{t\to 0^+} t^3\alpha'(t) = 0$ and for $(t,y) \in (0,1) \times \{y \in (0,\infty)\colon y < \alpha(t)\}$,

$$\begin{aligned} &p(t)q(t)f(t,y) + (p\alpha')'(t) \\ &= t^2\left(\frac{1}{\sqrt{y}} - \mu\right) - 3a_0t^2 \geq t^2\left(\frac{1}{\sqrt{a_0(1-t)}} - \mu\right) - 3a_0t^2 \\ &= t^2\left(\frac{1}{\sqrt{a_0}\sqrt{(1-t)}} - \mu - 3a_0\right) \geq t^2\left(\frac{1}{\sqrt{a_0}} - \mu - 3a_0\right) \\ &= \frac{t^2}{\sqrt{a_0}}(1 - \mu\sqrt{a_0} - 3a_0^{3/2}) > 0. \end{aligned}$$

Thus (3.6) holds. In addition (3.7) and (3.8) are satisfied since

$$\sup_{c\in(0,\infty)}\left(\frac{1}{\{1+h(c)/g(c)\}}\int_0^c \frac{\mathrm{d}u}{g(u)}\right) = \sup_{c\in(0,\infty)}\left(\frac{2}{1+\mu\sqrt{c}}\frac{c^{3/2}}{3}\right) = \infty.$$

Finally note (3.9) hold since

$$\begin{aligned} \int_0^1 \frac{1}{t^3}\int_0^t s^2 g(\alpha(s))\,\mathrm{d}s\,\mathrm{d}t &= \frac{1}{\sqrt{a_0}}\int_0^1 \frac{1}{t^3}\int_0^t \frac{s^2}{\sqrt{1-s}}\,\mathrm{d}s\,\mathrm{d}t \\ &\leq \frac{1}{\sqrt{a_0}}\int_0^1 \frac{1}{t^3}\frac{1}{\sqrt{1-t}}\int_0^t s^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{3\sqrt{a_0}}\int_0^1 \frac{1}{\sqrt{1-t}}\,\mathrm{d}t < \infty. \end{aligned}$$

Now Theorem 3.1 guarantees the result.

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