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On Lyapunov Inequality in Stability Theory for Hill's Equation on Time Scales

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In this paper we obtain sufficient conditions for instability and stability to hold for second order linear Δ-differential equations on time scales with periodic coefficients.

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1. INTRODUCTION; FORMULATION OF THE MAIN RESULTS

Let $\mathbb{T}$ be a closed subset (so called a time scale or a measure chain) of the real number $\mathbb{R}$.

DEFINITION *If there exists a positive number* $\Lambda \in \mathbb{R}$ *such that* $t + n\Lambda \in \mathbb{T}$ *for all* $t \in \mathbb{T}$ *and* $n \in \mathbb{Z}$, *then we call* $\mathbb{T}$ *a* periodic time scale *with period* Λ.

Suppose $\mathbb{T}$ is a Λ-periodic time scale. For the sake of simplicity we will assume that $0 \in \mathbb{T}$. Consider the second order linear Δ-differential

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equation given by

$$[p(t)y^{\Delta}(t)]^{\Delta} + q(t)y(\sigma(t)) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where σ is the forward jump operator and the coefficients $p(t)$ and $q(t)$ are real valued Λ-periodic functions defined on $\mathbb{T}$,

$$p(t+\Lambda) = p(t), \quad q(t+\Lambda) = q(t), \qquad t \in \mathbb{T}. \tag{1.2}$$

Besides we assume that

$$p(t) > 0, \quad p(t) \in C^1_{rd}[0,\Lambda], \quad q(t) \in C_{rd}[0,\Lambda], \tag{1.3}$$

where $[0,\Lambda] = \{t \in \mathbb{T} : 0 \leq t \leq \Lambda\}$.

For the definition of the Δ-derivative and other concepts related with time scales we refer to [5,10,15,17,18]. Further results include [2,3,7,11,12].

Equation (1.1) is said to be *unstable* if all nontrivial solutions are unbounded on $\mathbb{T}$, *conditionally stable* if there exist a nontrivial solution which is bounded on $\mathbb{T}$, and *stable* if all solutions are bounded on $\mathbb{T}$.

Main results of this paper are the following two theorems.

THEOREM 1.1 *If $q(t) \leq 0$ and $q(t) \not\equiv 0$, then Eq.* (1.1) *is unstable.*

By definition, put

$$p_0 = \max_{t\in[0,\rho(\Lambda)]} \frac{\sigma(t)-t}{p(t)}, \qquad q_+(t) = \max\{q(t), 0\}, \tag{1.4}$$

where ρ is the backward jump operator.

THEOREM 1.2 *If*

$$\text{(i)} \qquad \int_0^{\Lambda} q(t)\Delta t \geq 0, \quad q(t) \not\equiv 0; \tag{1.5}$$

$$\text{(ii)} \qquad \left[p_0 + \int_0^{\Lambda} \frac{\Delta t}{p(t)}\right] \cdot \int_0^{\Lambda} q_+(t)\Delta t \leq 4, \tag{1.6}$$

then Eq. (1.1) *is stable.*

We dwell on the three special cases as follows:

1. If $\mathbb{T}=\mathbb{R}$ we can take as Λ any $\omega\in\mathbb{R}$, $\omega>0$. Equation (1.1) takes the form

$$[p(x)y'(x)]' + q(x)y(x) = 0, \quad x \in \mathbb{R}, \tag{1.7}$$

and the periodicity condition (1.2) becomes

$$p(x+\omega) = p(x), \quad q(x+\omega) = q(x), \qquad x \in \mathbb{R}.$$

The conditions (i) and (ii) of Theorem 1.2 transform into

(i)$_1$ $\displaystyle\int_0^\omega q(x)\,\mathrm{d}x \geq 0, \quad q(x) \not\equiv 0;$

(ii)$_1$ $\displaystyle\left[\int_0^\omega \frac{\mathrm{d}x}{p(x)}\right] \cdot \int_0^\omega q_+(x)\,\mathrm{d}x \leq 4.$

We note here that the last conditions lead to a well-known result. In [19] Lyapunov has proved that, if a real, continuous and periodic function $q(x)$ of period $\omega>0$ satisfies the conditions

(1) $q(x) \geq 0, \quad q(x) \not\equiv 0;$

(2) $\displaystyle\omega\int_0^\omega q(x)\,\mathrm{d}x \leq 4,$

then the equation

$$y''(x) + q(x)y(x) = 0, \quad x \in \mathbb{R} \tag{1.8}$$

is stable. In [6] Borg extended Lyapunov's result to functions $q(x)$ of variable sign, showing that if

(1′) $\displaystyle\int_0^\omega q(x)\,\mathrm{d}x \geq 0, \quad q(x) \not\equiv 0;$

(2′) $\displaystyle\omega\int_0^\omega |q(x)|\,\mathrm{d}x \leq 4,$

hold, then Eq. (1.8) is stable. Then in [16] Krein improved Borg's result replacing in condition (2′) $|q(x)|$ by $q_+(x)=\max\{q(x),0\}$.

Notice that Eq. (1.7) can be transformed into an equation of the type (1.8) by the change of variable. For direct investigation of Eq. (1.7) we refer to [14].

2. If $\mathbb{T}=\mathbb{Z}$ we can take as Λ any integer $N>0$. Equation (1.1) takes the form

$$\Delta[p(n)\Delta y(n)] + q(n)y(n+1) = 0, \quad n \in \mathbb{Z},$$

where Δ is the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n)$. The periodicity condition (1.2) can be written as

$$p(n+N) = p(n), \quad q(n+N) = q(n), \qquad n \in \mathbb{Z}.$$

The conditions (i) and (ii) of Theorem 1.2 become

$$(\mathrm{i})_2 \qquad \sum_{n=0}^{N-1} q(n) \geq 0, \quad q(n) \not\equiv 0;$$

$$(\mathrm{ii})_2 \qquad \left[\frac{1}{p} + \sum_{n=0}^{N-1} \frac{1}{p(n)}\right] \cdot \sum_{n=0}^{N-1} q_+(n) \leq 4,$$

where $p = \min\{p(0), p(1), \ldots, p(N-1)\}$, $q_+(n) = \max\{q(n), 0\}$. This result was established in [4].

3. Let ω be a positive real number and N be a positive integer. Setting $\Lambda = \omega + N$ consider the time scale $\mathbb{T}$ defined by

$$\mathbb{T} = \bigcup_{k\in\mathbb{Z}} [\{x \in \mathbb{R}\colon k\Lambda \leq x \leq k\Lambda + \omega\} \cup \{k\Lambda + \omega + n\colon n = 1, 2, \ldots, N-1\}].$$

Evidently the set $\mathbb{T}$ defined in such a way is a Λ-periodic time scale. Equation (1.1) takes the form

$$[p(x)y'(x)]' + q(x)y(x) = 0, \quad x \in \bigcup_{k\in\mathbb{Z}} \{x \in \mathbb{R}\colon k\Lambda \leq x \leq k\Lambda + \omega\},$$

$$\Delta[p(t)\Delta y(t)] + q(t)y(t+1) = 0,$$

$$t \in \bigcup_{k\in\mathbb{Z}} \{k\Lambda + \omega + n\colon n = 0, 1, \ldots, N-2\},$$

which incorporates both differential and difference equations. In this case, the conditions (i) and (ii) of Theorem 1.2 can be written as

$$\text{(i)}_3 \qquad \int_0^{\omega} q(x)\,\mathrm{d}x + \sum_{n=0}^{N-1} q(\omega+n) \geq 0, \quad q(t) \not\equiv 0;$$

$$\text{(ii)}_3 \qquad \left[\int_0^{\omega} \frac{\mathrm{d}x}{p(x)} + \frac{1}{p} + \sum_{n=0}^{N-1} \frac{1}{p(\omega+n)}\right] \cdot \left[\int_0^{\omega} q_+(x)\mathrm{d}x + \sum_{n=0}^{N-1} q_+(\omega+n)\right] \leq 4,$$

where $p=\min\{p(\omega), p(\omega+1), \ldots, p(\omega+N-1)\}$, $q_+(t)=\max\{q(t), 0\}$.

As a conclusive remark we note that the assumption $0 \in \mathbb{T}$ is not necessary. Instead we can take any fixed point $t_0 \in \mathbb{T}$ in place of zero and in that case, the integrals in the conditions (i) and (ii) of Theorem 1.2 will be considered from t_0 to $t_0+\Lambda$.

2. AUXILIARY PROPOSITIONS

Consider Eq. (1.1), where $\mathbb{T}$ is a Λ-periodic time scale containing zero, with the coefficients $p(t)$ and $q(t)$ being real valued and satisfying the conditions (1.2) and (1.3). Floquet theory applies for Eq. (1.1). For the details of Floquet theory we may refer to, for example, [8,20] for differential equations, and [13, pp. 113–115, 4, 9, pp. 144–149] for difference equations.

Let us denote by $y^{[\Delta]}(t)=p(t)y^{\Delta}(t)$, the quasi-$\Delta$-derivative of $y(t)$. For arbitrary complex numbers c_0 and c_1, Eq. (1.1) has a unique solution $y(t)$ satisfying the initial conditions

$$y(0)=c_0, \qquad y^{[\Delta]}(0)=c_1.$$

Denote by $\theta(t)$ and $\varphi(t)$ the solutions of Eq. (1.1) under the initial conditions

$$\theta(0)=1, \quad \theta^{[\Delta]}(0)=0; \qquad \varphi(0)=0, \quad \varphi^{[\Delta]}(0)=1. \qquad (2.1)$$

There exist a nonzero complex number β and a nontrivial solution $\psi(t)$ of Eq. (1.1) such that

$$\psi(t+\Lambda) = \beta\psi(t), \quad t \in \mathbb{T}. \tag{2.2}$$

The number β is a root of the quadratic equation

$$\beta^2 - D\beta + 1 = 0, \tag{2.3}$$

where

$$D = \theta(\Lambda) + \varphi^{[\Delta]}(\Lambda). \tag{2.4}$$

The roots of (2.3) are defined by

$$\beta_{1,2} = \tfrac{1}{2}(D \pm \sqrt{D^2 - 4}).$$

Since the coefficients of Eq. (1.1) and the initial conditions (2.1) are real, the solutions $\theta(t)$, $\varphi(t)$ and hence the number D defined by (2.4) will be real.

PROPOSITION 2.1 *Equation* (1.1) *is unstable if* $|D| > 2$, *and stable if* $|D| < 2$.

Proof If the discriminant $D^2 - 4$ is nonzero, then (2.3) has two distinct roots β_1 and β_2, and hence there exist two nontrivial solutions $\psi_1(t)$ and $\psi_2(t)$ of Eq. (1.1) such that

$$\psi_1(t+\Lambda) = \beta_1\psi_1(t), \quad \psi_2(t+\Lambda) = \beta_2\psi_2(t), \qquad t \in \mathbb{T}. \tag{2.5}$$

It is easy to see that $\psi_1(t)$ and $\psi_2(t)$ are linearly independent. From (2.5) it follows that, for all $k \in \mathbb{Z}$,

$$\psi_1(t+k\Lambda) = \beta_1^k\psi_1(t), \quad \psi_2(t+k\Lambda) = \beta_2^k\psi_2(t), \qquad t \in \mathbb{T}. \tag{2.6}$$

If $|D| > 2$, then the numbers β_1 and β_2 will be distinct and real. Therefore from the equality $\beta_1\beta_2 = 1$ it follows that $|\beta_1| \neq 1$ and $|\beta_2| \neq 1$, since if this were false, we would get $\beta_1 = \beta_2 = \pm 1$. Obviously, if $|\beta_1| > 1$, then

$|\beta_2|<1$, and if $|\beta_1|<1$, we have $|\beta_2|>1$. Consequently from (2.6) as $k \to \pm\infty$ it follows that every nontrivial linear combination of $\psi_1(t)$ and $\psi_2(t)$ will be unbounded on $\mathbb{T}$, that is Eq. (1.1) is unstable.

If $|D|<2$, then the numbers β_1 and β_2 will be distinct, nonreal and such that $|\beta_1| = |\beta_2| = 1$. Therefore, from (2.5) we have

$$|\psi_1(t+\Lambda)| = |\psi_1(t)|, \quad |\psi_2(t+\Lambda)| = |\psi_2(t)|, \qquad t \in \mathbb{T}.$$

Consequently, $\psi_1(t)$ and $\psi_2(t)$ and hence every solution of Eq. (1.1) which is a linear combination of $\psi_1(t)$ and $\psi_2(t)$ will be bounded on $\mathbb{T}$, that is Eq. (1.1) will be stable. This completes the proof of the proposition.

Remark 2.1 If $|D|=2$, then Eq. (1.1) will be stable in the case $\theta^{[\Delta]}(\Lambda) = \varphi(\Lambda) = 0$; but conditionally stable and not stable otherwise.

The following mean value result (for the case $\mathbb{T} = \mathbb{Z}$ see [1, p. 24, 4]) will play a significant role in Section 4.

PROPOSITION 2.2 *Let $a<b$ be any two points in the time scale $\mathbb{T}$, and let $f(t)$ and $g(t)$ be two real functions continuous on the segment $[a,b] = \{t \in \mathbb{T}: a \le t \le b\}$ and Δ-differentiable on $(a,b) = \{t \in \mathbb{T}: a<t<b\}$. Suppose the function $g(t)$ is increasing on $[a,b]$. Then there exist $\xi, \tau \in \langle a,b)$ such that*

$$\frac{f^{\Delta}(\tau)}{g^{\Delta}(\tau)} \le \frac{f(b)-f(a)}{g(b)-g(a)} \le \frac{f^{\Delta}(\xi)}{g^{\Delta}(\xi)}, \tag{2.7}$$

where $\langle a,b) = (a,b)$ if $\sigma(a) = a$, and $\langle a,b) = [a,b)$ if $\sigma(a) > a$.

Proof We prove the right-hand side of (2.7), with the proof of the left-hand side being similar. Assume, on the contrary, that

$$\frac{f(b)-f(a)}{g(b)-g(a)} > \frac{f^{\Delta}(t)}{g^{\Delta}(t)}$$

for all t in $\langle a,b)$. Since $g(t)$ is increasing on $[a,b]$, $g^{\Delta}(t)>0$ on $\langle a,b)$. So we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} g^{\Delta}(t) > f^{\Delta}(t), \quad \forall t \in \langle a,b).$$

Integrating (in the sense of Δ-integral) both sides from a to b, we arrive at the contradiction

$$f(b) - f(a) > f(b) - f(a),$$

thereby establishing the result.

We will also make use of the following formulae which can easily be verified.

PROPOSITION 2.3 *Let $a, b \in \mathbb{T}$, $a < b$ and let $f(t)$ be a continuous function on $[a, \rho(b)] \subset \mathbb{T}$. Then*

$$\int_a^b f(t)\Delta t = \int_a^{\rho(b)} f(t)\Delta t + [b - \rho(b)] \cdot f(\rho(b)),$$

$$\int_a^b f(t)\Delta t = [\sigma(a) - a] \cdot f(a) + \int_{\sigma(a)}^b f(t)\Delta t,$$

where σ and ρ are the "forward" and "backward" jump operators, respectively.

3. PROOF OF THEOREM 1.1

Let $\theta(t)$ and $\varphi(t)$ be solutions of Eq. (1.1) satisfying the initial conditions (2.1). Our aim is to show that, under the hypotheses of the theorem, the inequalities

$$\theta(\Lambda) \geq 1, \qquad \varphi^{[\Delta]}(\Lambda) > 1$$

hold. Then $D = \theta(\Lambda) + \varphi^{[\Lambda]}(\Lambda) > 2$ will be obtained and therefore by Proposition 2.1, Eq. (1.1) will be unstable.

Firstly we show that

$$\theta(t) \geq 1, \quad \theta^{[\Delta]}(t) \geq 0, \quad \varphi(t) \geq 0, \quad \varphi^{[\Delta]}(t) \geq 1, \qquad \forall t \in \mathbb{T}, \ t \geq 0. \tag{3.1}$$

For this purpose we apply the induction principle developed for Time-Scales (see, for example, [5,18]), to the statement,

$$A(t): \theta(t) \geq 1 \quad \text{and} \quad \theta^{[\Delta]}(t) \geq 0 \qquad \text{for all } t \in \mathbb{T} \text{ and } t \geq 0.$$

(I) The statement $A(0)$ is true, since $\theta(0) = 1$ and $\theta^{[\Delta]}(0) = 0$.

(II) Let t be *right-scattered* and $A(t)$ be true, i.e., $\theta(t) \geq 1$ and $\theta^{[\Delta]}(t) \geq 0$. We need to show that $\theta(\sigma(t)) \geq 1$ and $\theta^{[\Delta]}(\sigma(t)) \geq 0$. But, in view of the induction assumptions, the required result are immediate from the relations

$$\theta(\sigma(t)) = \theta(t) + \mu^*(t)\theta^{\Delta}(t),$$

$$\theta^{[\Delta]}(\sigma(t)) = \theta^{[\Delta]}(t) - \mu^*(t)q(t)\theta(\sigma(t)),$$

the first one following from the definition of Δ-derivative and the second from Eq. (1.1) for $\theta(t)$. Here $\mu^*(t) = \sigma(t) - t$.

(III) Let t_0 be *right-dense*, $A(t_0)$ be true and $t \in [t_0, t_1] = \{t \in \mathbb{T}: t_0 \leq t \leq t_1\}$, where $t_1 \in \mathbb{T}$ is such that $t_1 > t_0$ and is sufficiently closed to t_0. We need to prove that $A(t)$ is true $t \in [t_0, t_1]$.

From Eq. (1.1) with $y(t) = \theta(t)$, the equations

$$\theta^{[\Delta]}(t) = \theta^{[\Delta]}(t_0) - \int_{t_0}^{t} q(s)\theta(\sigma(s))\Delta s, \tag{3.2}$$

$$\theta(t) = \theta(t_0) + \theta^{[\Delta]}(t_0) \cdot \int_{t_0}^{t} \frac{\Delta\tau}{p(\tau)} - \int_{t_0}^{t} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} q(s)\theta(\sigma(s))\Delta s\right] \Delta\tau \tag{3.3}$$

follow. To investigate the term $\theta(t)$ appearing in (3.3), we consider the equation

$$y(t) = \theta(t_0) + \theta^{[\Delta]}(t_0) \cdot \int_{t_0}^{t} \frac{\Delta\tau}{p(\tau)} - \int_{t_0}^{t} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} q(s)y(\sigma(s))\Delta s\right] \Delta\tau, \tag{3.4}$$

where $y(t)$ is a desired solution. Our aim is to show that for t_1 sufficiently close to t_0, Eq. (3.4) has a unique, continuous (in the topology of $\mathbb{T}$) solution $y(t)$ satisfying the inequality

$$y(t) \geq \theta(t_0) + \theta^{[\Delta]}(t_0) \cdot \int_{t_0}^{t} \frac{\Delta\tau}{p(t)}, \quad t_0 \leq t \leq t_1. \tag{3.5}$$

We solve Eq. (3.4) by the method of successive approximations, setting

$$
\begin{aligned}
y_0(t) &= \theta(t_0) + \theta^{[\Delta]}(t_0) \cdot \int_{t_0}^{t} \frac{\Delta\tau}{p(\tau)}, \\
y_j(t) &= -\int_{t_0}^{t} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} q(s) y_{j-1}(\sigma(s)) \Delta s \right] \Delta\tau, \quad j = 1, 2, 3, \ldots
\end{aligned}
\tag{3.6}
$$

If the series $\sum_{j=0}^{\infty} y_j(t)$ converges uniformly with respect to $t \in [t_0, t_1]$, then its sum will be, obviously, a continuous solution of Eq. (3.4). To prove the uniform convergence of this series we let

$$
c_0 = \theta(t_0) + \theta^{[\Delta]}(t_0) \cdot \int_{t_0}^{t_1} \frac{\Delta\tau}{p(\tau)}, \qquad c_1 = \int_{t_0}^{t_1} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} |q(s)| \Delta s \right] \Delta\tau.
$$

Then the estimate

$$
0 \le y_j(t) \le c_0 c_1^j \quad (t_0 \le t \le t_1), \qquad j = 0, 1, 2, \ldots \tag{3.7}
$$

can easily be obtained.

Indeed, (3.7) evidently holds for $j = 0$. Let it also hold for $j = n$. Then from (3.6), applying Proposition 2.3, we get

$$
\begin{aligned}
0 \le y_{n+1}(t) &\le \int_{t_0}^{t} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} |q(s)|\, y_n(\sigma(s)) \Delta s \right] \Delta\tau \\
&= \int_{t_0}^{\rho(t_1)} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} |q(s)|\, y_n(\sigma(s)) \Delta s \right] \Delta\tau \\
&\quad + [t_1 - \rho(t_1)] \cdot \frac{1}{p(\rho(t_1))} \cdot \int_{t_0}^{\rho(t_1)} |q(s)|\, y_n(\sigma(s)) \Delta s \\
&\le c_0 c_1^n \left\{ \int_{t_0}^{\rho(t_1)} \frac{1}{p(\tau)} \left[\int_{t_0}^{\tau} |q(s)| \Delta s \right] \Delta\tau \right. \\
&\quad \left. + [t_1 - \rho(t_1)] \cdot \frac{1}{p(\rho(t_1))} \cdot \int_{t_0}^{\rho(t_1)} |q(s)| \Delta s \right\} \\
&= c_0 c_1^n \int_{t_0}^{t_1} \frac{1}{p(\tau)} \cdot \left[\int_{t_0}^{\tau} |q(s)| \Delta s \right] \Delta\tau = c_0 c_1^{n+1}.
\end{aligned}
$$

Therefore by the usual mathematical induction principle, (3.7) holds for all $j = 0, 1, 2, \ldots$

Now choosing t_1 appropriately we obtain $c_1 < 1$. Then Eq. (3.4) will have a continuous solution

$$y(t) = \sum_{j=0}^{\infty} y_j(t) \quad \text{for } t \in [t_0, t_1].$$

Since $y_j(t) \geq 0$, it follows that $y(t) \geq y_0(t)$ thereby proving the validity of inequality (3.5).

Uniqueness of the solution of Eq. (3.4) can be proven in the usual way.

From (3.3) and (3.4) in view of the uniqueness of the solution we get that $\theta(t) \equiv y(t)$, $t_0 \leq t \leq t_1$. Therefore

$$\theta(t) \geq \theta(t_0) + \theta^{[\Delta]}(t_0) \cdot \int_{t_0}^{t} \frac{\Delta\tau}{p(\tau)}, \quad t_0 \leq t \leq t_1.$$

Hence by making use of the induction hypothesis $A(t_0)$ being true, we obtain from the above inequality

$$\theta(t) \geq 1 \quad \text{for } t \in [t_0, t_1].$$

Taking this into account, from (3.2) we also get

$$\theta^{[\Delta]}(t) \geq 0 \quad \text{for } t \in [t_0, t_1].$$

Thus, $A(t)$ is true for all $t \in [t_0, t_1]$.

(IV) Let $t \in \mathbb{T}$, $t > 0$ be *left-dense* and such that $A(s)$ is true for all $s < t$, i.e., $\theta(s) \geq 1$, $\theta^{[\Delta]}(s) \geq 0$, $\forall s < t$. Passing here to the limit as $s \to t$ we get by the continuity of $\theta(s)$ and $\theta^{[\Delta]}(s)$ that $\theta(t) \geq 1$ and $\theta^{[\Delta]}(t) \geq 0$, thereby verifying the validity of $A(t)$.

Consequently, by the continuous type induction principle on a time-scale, (3.1) holds for $\theta(t)$ and $\theta^{[\Delta]}(t)$, $\forall t \in \mathbb{T}$, $t \geq 0$.

Proof for $\varphi(t) \geq 0$ and $\varphi^{[\Delta]}(t) \geq 1$ is similar.

From (3.1), in particular, we have $\theta(\Lambda) \geq 1$, $\varphi^{[\Delta]}(\Lambda) \geq 1$. Actually $\varphi^{[\Delta]}(\Lambda) > 1$. Indeed, consider the equations

$$\varphi^{[\Delta]}(t) = 1 - \int_0^t q(s)\varphi(\sigma(s))\Delta s, \tag{3.8}$$

$$\varphi(t) = \int_0^t \frac{\Delta\tau}{p(\tau)} - \int_0^t \frac{1}{p(\tau)} \left[\int_0^\tau q(s)\varphi(\sigma(s))\Delta s\right] \Delta\tau, \tag{3.9}$$

which follow from Eq. (1.1) with $y=\varphi$ using initial conditions (2.1). If $\varphi^{[\Delta]}(\Lambda)=1$, (3.8) gives rise to

$$\int_0^{\Lambda} q(s)\varphi(\sigma(s))\Delta s = 0. \tag{3.10}$$

On the other hand, since $\varphi(t)\geq 0$ by (3.1), from (3.9) it follows that $\varphi(t)>0$, $\forall t>0$. Therefore, (3.10) yields $q(s)=0$, $\forall s\in[0,\rho(\Lambda)]$. Hence employing the Λ-periodicity of $q(t)$ it follows that $q(t)\equiv 0$, $t\in\mathbb{T}$, contradicting the hypothesis of the theorem. Hence $\varphi^{[\Delta]}(\Lambda)>1$ and Theorem 1.1 is proven.

4. PROOF OF THEOREM 1.2

To prove Theorem 1.2 it is sufficient by Proposition 2.1 to show that $D^2<4$. Assuming on the contrary that $D^2\geq 4$ will lead to contradiction.

DEFINITION *We say that a function* $f:\mathbb{T}\to\mathbb{R}$ *has a generalized zero (a node) at* $t_0\in\mathbb{T}$ *if either* $f(t_0)=0$ *or* $f(\rho(t_0))\cdot f(t_0)<0$, *where* ρ *is the backward jump operator.*

Next we develop the proof of Theorem 1.2 in the form of two lemmas.

LEMMA 4.1 *If* $D^2\geq 4$, *then Eq.* (1.1) *has a real, nontrivial solution* $\psi(t)$ *possessing the following properties: there exist two points a and b in* $\mathbb{T}$ *such that* $0\leq a\leq\rho(\Lambda)$, $b>a$, $b-a\leq\Lambda$, $\psi(t)$ *has generalized zeros at a and b, and* $\psi(t)>0$ *for* $a<t<b$.

Proof If $D^2\geq 4$, it then follows from Section 2 that Eq. (1.1) has a nontrivial solution $y(t)$ having the property $y(t+\Lambda)=\beta y(t)$ $(t\in\mathbb{T})$, where β is a real nonzero number. Since $\operatorname{Re} y(t)$ and $\operatorname{Im} y(t)$ are also solutions of Eq. (1.1) with the same property, we may assume that Eq. (1.1) has a real, nontrivial solution $\psi(t)$ satisfying

$$\psi(t+\Lambda)=\beta\psi(t),\quad t\in\mathbb{T}, \tag{4.1}$$

where β is a real nonzero number.

Firstly we show that $\psi(t)$ must have at least one generalized zero a in the segment $[0,\rho(\Lambda)]=\{t\in\mathbb{T}\colon 0\leq t\leq\rho(\Lambda)\}$. If not, then by (4.1), $\psi(t)$

does not have any generalized zero in $\mathbb{T}$, so $\psi(t) \neq 0$ and $\psi(\rho(t)) \cdot \psi(t) > 0$ for all t in $\mathbb{T}$. Hence we also have $\psi(t) \cdot \psi(\sigma(t)) > 0$, $\forall t \in \mathbb{T}$.

From the equation

$$[p(t)\psi^{\Delta}(t)]^{\Delta} + q(t)\psi(\sigma(t)) = 0, \quad t \in \mathbb{T} \tag{4.2}$$

we have

$$\int_0^{\Lambda} \frac{[p(t)\psi^{\Delta}(t)]^{\Delta}}{\psi(\sigma(t))} \Delta t + \int_0^{\Lambda} q(t)\Delta t = 0.$$

Therefore using the integration by parts formula

$$\int_0^{\Lambda} U^{\Delta}(t) \cdot V(\sigma(t))\Delta t = U(t)V(t)\Big|_0^{\Lambda} - \int_0^{\Lambda} U(t) \cdot V^{\Delta}(t)\Delta t$$

and noting that, by the periodicity of $p(t)$ and the Eq. (4.1),

$$\frac{p(t)\psi^{\Delta}(t)}{\psi(t)}\Big|_0^{\Lambda} = \frac{p(\Lambda)\psi^{\Delta}(\Lambda)}{\psi(\Lambda)} - \frac{p(0)\psi^{\Delta}(0)}{\psi(0)} = [p(\Lambda) - p(0)]\frac{\psi^{\Delta}(0)}{\psi(0)} = 0,$$

we get that

$$\int_0^{\Delta} \frac{p(t)[\psi^{\Delta}(t)]^2}{\psi(t)\psi(\sigma(t))} \Delta t + \int_0^{\Lambda} q(t)\Delta t = 0.$$

Hence taking into account the condition (1.5) of the theorem along with the facts that $\psi(t)\psi(\sigma(t)) > 0$ and $p(t) > 0$, we obtain $\psi^{\Delta}(t) = 0$ for all $t \in [0, \rho(\Lambda)]$, i.e. $\psi(t) = C = \text{const.}$ on $[0, \Lambda]$. Note that $C \neq 0$, since $\psi(t)$ is a nontrivial solution of Eq. (1.1). Therefore, setting $t = 0$ in (4.1) we get $\beta = 1$. Hence $\psi(t) = C$ for all $t \in \mathbb{T}$. Consequently, from Eq. (4.2) we have $q(t) \cdot C = 0$ on $\mathbb{T}$ and hence $q(t) \equiv 0$ on $\mathbb{T}$, which contradicts the condition (1.5) of the theorem. Thus $\psi(t)$ has at least one generalized zero a in $[0, \rho(\Lambda)]$. From (4.1) we get that $\psi(t)$ will have also a generalized zero at $a + \Lambda$. It is not difficult to show that on the segment $[a, a + \Lambda]$ the solution $\psi(t)$ may have only finitely many generalized zeros. Denote by b the smallest generalized zero of $\psi(t)$ lying to right of a and different from a. Then $b \leq a + \Lambda$, therefore $b > a$, $b - a \leq \Lambda$, and $\psi(t)$ does not have generalized zero at t for $a<t<b$. Since $\psi(t)$ must keep a constant sign on (a,b), and together with $\psi(t)$ the function $-\psi(t)$ also is a solution of Eq. (1.1) with the same generalized zeros at a and b, we may assume that $\psi(t)>0$ for $a<t<b$. The lemma is therefore proven.

LEMMA 4.2 *Under the hypothesis of the preceding lemma the inequality*

$$\left[p_0+\int_0^{\Lambda}\frac{\Delta t}{p(t)}\right]\cdot\int_0^{\Lambda}q_+(t)\Delta t>4 \tag{4.3}$$

holds, where p_0 and $q_+(t)$ are defined by (1.4).

Proof Let us set

$$g(t)=\int_0^t\frac{\Delta s}{p(s)},\qquad G(t)=[p(t)\psi^{\Delta}(t)]^{\Delta}$$
$$G_+(t)=\max\{G(t),0\},\qquad G_-(t)=-\min\{G(t),0\}.$$

Evidently

$$G_+(t)+G_-(t)=|G(t)|,\qquad G_+(t)-G_-(t)=G(t).$$

Let $\psi(t)$ be a solution of Eq. (1.1) possesing the properties indicated in Lemma 4.1. There are four possibilities with respect to a and b.

Case 1: $\psi(\rho(a))\cdot\psi(a)<0$ and $\psi(\rho(b))\cdot\psi(b)<0$.
Case 2: $\psi(\rho(a))\cdot\psi(a)<0$ and $\psi(b)=0$.
Case 3: $\psi(a)=0$ and $\psi(b)=0$.
Case 4: $\psi(a)=0$ and $\psi(\rho(b))\cdot\psi(b)<0$.

Let one of the Cases 1 and 2 hold. Then $\rho(a)<a$ and $\rho(b)\le b$. Choose $c\in[a,\rho(b)]$ such that

$$\psi(c)=\max_{a\le t\le\rho(b)}\psi(t).$$

It is easy to see that, in the cases considered,

$$\psi(c)-\psi(\rho(a))>\psi(c),\qquad \psi(c)-\psi(b)\ge\psi(c),$$

with the strict inequality in the latter one occurring in Case 1.

Consequently,

$$
\begin{aligned}
&\frac{1}{g(c)-g(\rho(a))}+\frac{1}{g(b)-g(c)} \\
&\quad<\frac{1}{g(c)-g(\rho(a))}\cdot\frac{\psi(c)-\psi(\rho(a))}{\psi(c)}+\frac{1}{g(b)-g(c)}\cdot\frac{\psi(c)-\psi(b)}{\psi(c)} \\
&\quad=\left[\frac{\psi(c)-\psi(\rho(a))}{g(c)-g(\rho(a))}-\frac{\psi(b)-\psi(c)}{g(b)-g(c)}\right]\frac{1}{\psi(c)}.
\end{aligned}
$$

Furthermore, by Proposition 2.2 there exist $\xi\in[\rho(a),c)$ and $\tau\in\langle c,b)$ such that

$$
\frac{\psi(c)-\psi(\rho(a))}{g(c)-g(\rho(a))}\le\frac{\psi^{\Delta}(\xi)}{g^{\Delta}(\xi)},\qquad \frac{\psi(b)-\psi(c)}{g(b)-g(c)}\ge\frac{\psi^{\Delta}(\tau)}{g^{\Delta}(\tau)}.
$$

Therefore

$$
\begin{aligned}
\frac{1}{g(c)-g(\rho(a))}+\frac{1}{g(b)-g(c)}&<\left[\frac{\psi^{\Delta}(\xi)}{g^{\Delta}(\xi)}-\frac{\psi^{\Delta}(\tau)}{g^{\Delta}(\tau)}\right]\frac{1}{\psi(c)} \\
&=-\frac{1}{\psi(c)}\int_{\xi}^{\tau}\left[\frac{\psi^{\Delta}(t)}{g^{\Delta}(t)}\right]^{\Delta}\Delta t \\
&=-\frac{1}{\psi(c)}\int_{\xi}^{\tau}[p(t)\psi^{\Delta}(t)]^{\Delta}\Delta t \\
&=-\frac{1}{\psi(c)}\int_{\xi}^{\tau}G(t)\Delta t \\
&\le\frac{1}{\psi(c)}\int_{\xi}^{\tau}G_{-}(t)\Delta t \\
&\le\int_{\xi}^{\tau}\frac{G_{-}(t)}{\psi(\sigma(t))}\Delta t, \qquad (4.4)
\end{aligned}
$$

since $\xi\le t<\tau\Rightarrow\rho(a)\le t<\rho(b)\Rightarrow a\le\sigma(t)\le\rho(b)$.

On the other hand for arbitrary real numbers x, y, z satisfying $x<y<z$ the inequality

$$
\frac{1}{y-x}+\frac{1}{z-y}\ge\frac{4}{z-x}
$$

holds. Consequently, from the inequality (4.4) we get

$$[g(b) - g(\rho(a))] \cdot \int_{\xi}^{\tau} \frac{G_{-}(t)}{\psi(\sigma(t))} \Delta t > 4. \tag{4.5}$$

Since

$$g(b) - g(\rho(a)) = \int_{\rho(a)}^{b} \frac{\Delta t}{p(t)}, \quad \frac{G_{-}(t)}{\psi(\sigma(t))} = q_{+}(t), \qquad t \in [\rho(a), \rho(b)),$$

it follows from (4.5) that

$$\left[\int_{\rho(a)}^{b} \frac{\Delta t}{p(t)}\right] \cdot \int_{\xi}^{\tau} q_{+}(t)\Delta t > 4. \tag{4.6}$$

Next, taking into account that $\rho(a) \leq \xi < \tau \leq \rho(b)$, $b \leq a + \Lambda$, and that for any periodic function $f(t)$ on $\mathbb{T}$ with period Λ the equality

$$\int_{t_0}^{t_0+\Lambda} f(t)\Delta t = \int_{0}^{\Lambda} f(t)\Delta t$$

holds for all $t_0 \in \mathbb{T}$, we have

$$\int_{\rho(a)}^{b} \frac{\Delta t}{p(t)} \leq \int_{\rho(a)}^{a+\Lambda} \frac{\Delta t}{p(t)} = \frac{a - \rho(a)}{p(\rho(a))} + \int_{a}^{a+\Lambda} \frac{\Delta t}{p(t)} \leq p_0 + \int_{0}^{\Lambda} \frac{\Delta t}{p(t)},$$

$$\int_{\xi}^{\tau} q_{+}(t)\Delta t \leq \int_{\rho(a)}^{\rho(b)} q_{+}(t)\Delta t \leq \int_{\rho(a)}^{\rho(a)+\Lambda} q_{+}(t)\Delta t = \int_{0}^{\Lambda} q_{+}(t)\Delta t,$$

since by the Λ-periodicity of $\mathbb{T}$ and $p(t)$

$$\max_{a \in [0, \rho(\Lambda)]} \frac{a - \rho(a)}{p(\rho(a))} = \max_{t \in [0, \rho(\Lambda)]} \frac{\sigma(t) - t}{p(t)} = p_0,$$

and from the inequality $b \leq a + \Lambda$ it follows that $\rho(b) \leq \rho(a) + \Lambda$. Consequently, (4.3) follows from (4.6).

Now we assume that one of the Cases 3 or 4 holds. In these cases choosing $c \in (a, b)$ such that

$$\psi(c) = \max_{\sigma(a) \leq t \leq \rho(b)} \psi(t)$$

yields

$$\psi(c) - \psi(a) = \psi(c), \qquad \psi(c) - \psi(b) \geq \psi(c),$$

with the strict inequality in the second one holding for the Case 4. Consequently,

$$\begin{aligned} &\frac{1}{g(c) - g(a)} + \frac{1}{g(b) - g(c)} \\ &\quad \leq \frac{1}{g(c) - g(a)} \cdot \frac{\psi(c) - \psi(a)}{\psi(c)} + \frac{1}{g(b) - g(c)} \cdot \frac{\psi(c) - \psi(b)}{\psi(c)} \\ &\quad = \left[\frac{\psi(c) - \psi(a)}{g(c) - g(a)} - \frac{\psi(b) - \psi(c)}{g(b) - g(c)}\right] \frac{1}{\psi(c)}. \end{aligned}$$

Finally, reasoning as in the previous case we obtain the inequality,

$$\left[\int_0^{\Lambda} \frac{\Delta t}{p(t)}\right] \cdot \int_0^{\Lambda} q_+(t) \Delta t > 4.$$

Therefore the inequality (4.3) is true in these cases as well. The Lemma is thus proven.

Since the inequality (4.3) contradicts the condition (1.6) of the theorem, the inequality $D^2 \geq 4$ cannot be true. Thus $D^2 < 4$ and Eq. (1.1) is stable and the Theorem 1.2 is proven.

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