On Simpson's Inequality and Applications

S.S. DRAGOMIR*, R.P. AGARWAL[†] and P. CERONE[‡]

School of Communications Informatics, Victoria University of Technology, P.O. Box 14428, Mc Melbourne City, 8001 Victoria, Australia

New inequalities of Simpson type and their application to quadrature formulae in Numerical Analysis are given.

Keywords: Simpson's inequality; Quadrature formulae

1. INTRODUCTION

The following inequality is well known in the literature as *Simpson's inequality*:

$$\left| \int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right|$$

$$\leq \frac{1}{2880}\left\| f^{(4)} \right\|_\infty (b-a)^5, \tag{1.1}$$

where the mapping $f:[a,b] \to \mathbb{R}$ is assumed to be four times continuously differentiable on the interval (a,b) and for the fourth derivative

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* Corresponding author. E-mail: sever@matilda.vu.edu.au.
URL: http://matilda.vu.edu.au/~rgmia/dragomirweb.html.
[†] Current address: Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260. E-mail: matravip@leonis.nus.sg.
[‡] *URL:* http://matilda.vu.edu.au/~rgmia.

to be bounded on (a, b), that is

$$\|f^{(4)}\|_\infty := \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty.$$

Now, if we assume that $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition of the interval $[a, b]$ and f is as above, then we have the classical *Simpson's quadrature formula*:

$$\int_a^b f(x)\, dx = A_S(f, I_n) + R_S(f, I_n), \qquad (1.2)$$

where $A_S(f, I_n)$ is the *Simpson rule*

$$A_S(f, I_n) =: \frac{1}{6} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i + \frac{2}{3} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \qquad (1.3)$$

and the *remainder term* $R_S(f, I_n)$ satisfies the estimate

$$|R_S(f, I_n)| \le \frac{1}{2880} \|f^{(4)}\|_\infty \sum_{i=0}^{n-1} h_i^5, \qquad (1.4)$$

where $h_i := x_{i+1} - x_i$ for $i = 0, \ldots, n-1$.

When we have an equidistant partitioning of $[a, b]$ given by

$$I_n : x_i := a + \frac{b-a}{n} \cdot i, \quad i = 0, \ldots, n; \qquad (1.5)$$

then we have the formula:

$$\int_a^b f(x)\, dx = A_{S,n}(f) + R_{S,n}(f), \qquad (1.6)$$

where

$$A_{S,n}(f) := \frac{b-a}{6n} \sum_{i=0}^{n-1} \left[f\left(a + \frac{b-a}{n} \cdot i\right) + f\left(a + \frac{b-a}{n} \cdot (i+1)\right) \right]$$

$$+ \frac{2(b-a)}{3n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot \frac{2i+1}{2}\right) \qquad (1.7)$$

and the remainder satisfies the estimation

$$|R_{S,n}(f)| \leq \frac{1}{2880} \cdot \frac{(b-a)^5}{n^4} \|f^{(4)}\|_\infty. \qquad (1.8)$$

For some other integral inequalities see the recent book [1] and the papers [2–4] and [5–37].

The main purpose of this survey paper is to point out some very recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

It is well known that if the mapping f is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on (a, b), then we cannot apply the classical Simpson quadrature formula, which, actually, is one of the most used quadrature formulae in practical applications.

The first section of our paper deals with an upper bound for the remainder in Simpson's inequality for the class of functions of bounded variation.

The second section provides some estimates for the remainder when f is a Lipschitzian mapping while the third section is concerned with the same problem for absolutely continuous mappings whose derivatives are in the Lebesgue spaces $L_p[a, b]$.

The fourth section is devoted to the application of a celebrated result due to Grüss to estimate the remainder in the Simpson quadrature rule in terms of the supremum and infimum of the first derivative. The fifth section deals with a general convex combination of trapezoid and interior point quadrature formula from which, in particular, we can obtain the classical Simpson rule.

The last section contains some results related to Simpson, trapezoid and midpoint formulae for monotonic mappings and some applications for probability distribution functions.

Last, but not least, we would like to mention that every section contains a special subsection in which the theoretical results are applied for the special means of two positive numbers: identric mean, logarithmic mean, p-logarithmic mean etc. and provides improvements and related results to the classical sequence of inequalities

$$H \leq G \leq L \leq I \leq A,$$

where H, G, L, I and A are defined in the sequel.

2. SIMPSON'S INEQUALITY FOR MAPPINGS OF BOUNDED VARIATION

2.1. Simpson's Inequality

The following result holds [2].

THEOREM 1   *Let $f: [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then we have the inequality:*

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{3} \cdot \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \le \frac{1}{3}(b-a) \bigvee_a^b (f),$$

$$(2.1)$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{3}$ is the best possible.

Proof   Using the integration by parts formula for Reimann–Stieltjes integral we have

$$\int_a^b s(x)\, df(x) = \frac{b-a}{3} \cdot \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(x)\, dx,$$

$$(2.2)$$

where

$$s(x) := \begin{cases} x - \dfrac{5a+b}{6}, & x \in \left[a, \dfrac{a+b}{2}\right), \\ x - \dfrac{a+5b}{6}, & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Indeed,

$$\int_a^b s(x)\, df(x)$$

$$= \int_a^{(a+b)/2} \left(x - \frac{5a+b}{6} \right) df(x) + \int_{(a+b)/2}^b \left(x - \frac{a+5b}{6} \right) df(x)$$

$$= \left[\left(x - \frac{5a+b}{6} \right) f(x) \right]_a^{(a+b)/2} + \left[\left(x - \frac{a+5b}{6} \right) f(x) \right]_{(a+b)/2}^b$$

$$- \int_a^b f(x)\, dx$$

$$= \frac{b-a}{3} \cdot \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(x)\, dx$$

and the identity is proved.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) :=$ $\max_{i \in \{0,\dots,n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)} \right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)} \right]$. If $p : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then

$$\left| \int_a^b p(x)\, dv(x) \right| = \left| \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)} \right) \left[v\left(x_{i+1}^{(n)} \right) - v\left(x_i^{(n)} \right) \right] \right|$$

$$\leq \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} \left| p\left(\xi_i^{(n)} \right) \right| \left| v\left(x_{i+1}^{(n)} \right) - v\left(x_i^{(n)} \right) \right|$$

$$\leq \max_{x \in [a,b]} |p(x)| \sup_{\Delta_n} \sum_{i=0}^{n-1} \left| v\left(x_{i+1}^{(n)} \right) - v\left(x_i^{(n)} \right) \right|$$

$$= \max_{x \in [a,b]} |p(x)| \bigvee_a^b (v). \tag{2.3}$$

Applying the inequality (2.3) for $p(x) = s(x)$ and $v(x) = f(x)$ we get

$$\left| \int_a^b s(x)\, df(x) \right| \leq \max_{x \in [a,b]} |s(x)| \bigvee_a^b (f). \tag{2.4}$$

Taking into account the fact that the mapping s is monotonic nondecreasing on the intervals $[a, (a+b)/2)$ and $[(a+b)/2, b]$ and

$$s(a) = -\frac{b-a}{6},$$

$$s\left(\frac{a+b}{2} - 0 \right) = \frac{1}{3}(b-a),$$

$$s\left(\frac{a+b}{2} \right) = -\frac{1}{3}(b-a)$$

and

$$s(b) = \frac{b-a}{6},$$

we deduce that

$$\max_{x \in [a,b]} |s(x)| = \frac{1}{3}(b-a).$$

Now, using the inequality (2.4) and the identity (2.2) we deduce the desired result (2.1).

Now, for the best constant. Assume that the following inequality holds

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \le C(b-a) \bigvee_a^b (f)$$

with a constant $C > 0$.

Let us choose the mapping $f: [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[a, \dfrac{a+b}{2} \right) \cup \left(\dfrac{a+b}{2}, b \right], \\ -1 & \text{if } x = \dfrac{a+b}{2}. \end{cases}$$

Then we have

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| = \frac{4}{3}(b-a)$$

and

$$(b-a) \bigvee_a^b (f) = 4(b-a).$$

Now, using the above inequality, we get $4C(b-a) \ge \frac{4}{3}(b-a)$ which implies that $C \ge \frac{1}{3}$ and then $\frac{1}{3}$ is the best possible constant in (2.1).

It is natural to consider the following corollary which follows from identity (2.2).

COROLLARY 1 *Suppose that $f: [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and*

$$\|f'\|_1 := \int_a^b |f'(x)|\, dx < \infty.$$

Then we have the inequality:

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{3}\|f'\|_1 (b-a)^2.$$

(2.5)

The following corollary for Simpson's composite formula holds:

COROLLARY 2   *Let $f:[a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and I_h a partition of $[a,b]$. Then we have the Simpson's quadrature formula (1.2) and the remainder term $R_S(f, I_h)$ satisfies the estimate:*

$$|R_S(f, I_h)| \leq \frac{1}{3}\gamma(h) \bigvee_a^b (f),$$

(2.6)

where $\gamma(h) := \max\{h_i \,|\, i = 0, \ldots, n-1\}$.

The case of equidistant partitioning is embodied in the following corollary:

COROLLARY 3   *Let I_n be an equidistant partitioning of $[a,b]$ and f be as in Theorem 1. Then we have the formula (1.6) and the remainder satisfies the estimate:*

$$|R_{S,n}(f)| \leq \frac{1}{3n}(b-a) \bigvee_a^b (f).$$

(2.7)

Remark 1   If we want to approximate the integral $\int_a^b f(x)\,dx$ by Simpson's formula $A_{S,n}(f)$ with an accuracy less than $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := \left[\frac{1}{3\varepsilon} \cdot (b-a) \bigvee_a^b (f)\right] + 1$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

Comments   If the mapping $f:[a,b] \to \mathbb{R}$ is neither four times differentiable nor the fourth derivative is bounded on (a, b), then we cannot apply the classical estimation in Simpson's formula using the fourth derivative.

But if we assume that f is of bounded variation, then we can use instead the formula (2.6).

We give here a class of mappings which are of bounded variation but which have the fourth derivative unbounded on the given interval.

Let $f_p : [a, b] \to \mathbb{R}, f_p(x) := (x - a)^p$ where $p \in (3, 4)$. Then obviously

$$f_p'(x) := p(x - a)^{p-1}, \quad x \in (a, b)$$

and

$$f_p^{(4)}(x) = \frac{p(p - 1)(p - 2)(p - 3)}{(x - a)^{4-p}}, \quad x \in (a, b).$$

It is clear that f_p is of bounded variation and

$$\bigvee_a^b (f) = (b - a)^p < \infty,$$

but $\lim_{x \to a+} f_p^{(4)}(x) = +\infty$.

2.2. Applications for Special Means

Let us recall the following means:

(1) *The arithmetic mean*

$$A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0;$$

(2) *The geometric mean*

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

(3) *The harmonic mean*

$$H = H(a, b) := \frac{2}{1/a + 1/b}, \quad a, b > 0;$$

(4) *The logarithmic mean*

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a,b > 0, \ a \neq b;$$

(5) *The identric mean*

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \quad a,b > 0, \ a \neq b;$$

(6) *The p-logarithmic mean*

$$L_p = L_p(a,b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p},$$

$$p \in \mathbb{R}\setminus\{-1,0\}, \quad a,b > 0, \ a \neq b.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

Using Theorem 1, some new inequalities are derived for the above means.

1. Let $f : [a,b] \to \mathbb{R}$ $(0 < a < b), f(x) = x^p, p \in \mathbb{R}\setminus\{-1,0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = L_p(a,b),$$

$$\frac{f(a) + f(b)}{2} = A(a^p, b^p),$$

$$f\left(\frac{a+b}{2}\right) = A^p(a,b)$$

and

$$\|f'\|_1 = |p|(b-a)L_{p-1}^{p-1}, \quad p \in \mathbb{R}\setminus\{-1,0,1\}.$$

Using the inequality (2.5) we get

$$\left|L_p^p(a,b) - \frac{1}{3}A(a^p,b^p) - \frac{2}{3}A^p(a,b)\right| \leq \frac{|p|}{3}L_{p-1}^{p-1}(b-a)^2.$$

2. Let $f: [a, b] \to \mathbb{R}$ $(0 < a < b), f(x) = 1/x$. Then

$$\frac{1}{b-a} \int_a^b f(x)\,dx = L^{-1}(a, b),$$

$$\frac{f(a)+f(b)}{2} = H^{-1}(a, b),$$

$$f\left(\frac{a+b}{2}\right) = A^{-1}(a, b)$$

and

$$\|f'\|_1 = \frac{b-a}{G^2(a, b)}.$$

Using the inequality (2.5) we get

$$|3AH - AL - 2HL| \leq \frac{(b-a)^2}{G^2} LHA.$$

3. Let $f: [a, b] \to \mathbb{R}$ $(0 < a < b), f(x) = \ln x$. Then

$$\frac{1}{b-a} \int_a^b f(x)\,dx = \ln I(a, b),$$

$$\frac{f(a)+f(b)}{2} = \ln G(a, b),$$

$$f\left(\frac{a+b}{2}\right) = \ln A(a, b)$$

and

$$\|f'\|_1 = \frac{b-a}{L(a, b)}.$$

Using the inequality (2.5) we obtain

$$\left| \ln\left[\frac{I}{G^{1/3} A^{2/3}}\right] \right| \leq \frac{(b-a)^2}{3L}.$$

3. SIMPSON'S INEQUALITY FOR LIPSCHITZIAN MAPPINGS

3.1. Simpson's Inequality

The following result holds [3]:

THEOREM 2   *Let $f:[a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$. Then we have the inquality:*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \le \frac{5}{36} L(b-a)^2.$$

(3.1)

Proof   Using the integration by parts formula for Riemann–Stieltjes integral we have (see also the proof of Theorem 1) that

$$\int_a^b s(x)\,df(x) = \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(x)\,dx,$$

(3.2)

where

$$s(x) := \begin{cases} x - \dfrac{5a+b}{6}, & x \in \left[a, \dfrac{a+b}{2}\right), \\[2mm] x - \dfrac{a+5b}{6}, & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i \in \{0,\dots,n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)} \right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)} \right]$. If $p:[a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $v:[a,b] \to \mathbb{R}$ is L-Lipschitzian on $[a,b]$, then

$$\left| \int_a^b p(x)\,dv(x) \right| = \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right) \right] \right|$$

$$\le \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| p\left(\xi_i^{(n)}\right) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) \left| \frac{v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)}{x_{i+1}^{(n)} - x_i^{(n)}} \right|$$

$$\le L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| p(\xi_i^{(n)}) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right)$$

$$= L \int_a^b |p(x)| \, dx. \tag{3.3}$$

Applying the inequality (3.3) for $p(x) = s(x)$ and $v(x) = f(x)$ we get

$$\left| \int_a^b s(x) \, df(x) \right| \le L \int_a^b |s(x)| \, dx. \tag{3.4}$$

Let us compute

$$\int_a^b |s(x)| \, dx = \int_a^{(a+b)/2} \left| x - \frac{5a+b}{6} \right| dx + \int_{(a+b)/2}^b \left| x - \frac{a+5b}{6} \right| dx$$

$$= \int_a^{(5a+b)/6} \left(\frac{5a+b}{6} - x \right) dx + \int_{(5a+b)/6}^{(a+b)/2} \left(x - \frac{5a+b}{6} \right) dx$$

$$+ \int_{(a+b)/2}^{(a+5b)/6} \left(\frac{a+5b}{6} - x \right) dx + \int_{(a+5b)/6}^b \left(x - \frac{a+5b}{6} \right) dx$$

$$= \frac{5}{36} (b-a)^2.$$

Now, using the inequality (3.4) and the identity (3.2) we deduce the desired result (3.1).

COROLLARY 4 *Suppose that $f:[a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b). Then we have the inequality:*

$$\left| \int_a^b f(x) \, dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] \right| \le \frac{5}{36} \|f'\|_\infty (b-a)^2. \tag{3.5}$$

The following corollary for Simpson's composite formula holds:

COROLLARY 5 *Let $f:[a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$ and I_h a partition of $[a,b]$. Then we have the Simpson's quadrature*

formula (1.2) *and the remainder term* $R_S(f, I_h)$ *satisfies the estimation:*

$$|R_S(f, I_h)| \leq \frac{5}{36} L \sum_{i=0}^{n-1} h_i^2. \tag{3.6}$$

The case of equidistant partitioning is embodied in the following corollary:

COROLLARY 6   *Let I_n be an equidistant partitioning of $[a, b]$ and f be as in Theorem 2. Then we have the formula (1.6) and the remainder satisfies the estimation:*

$$|R_{S,n}(f)| \leq \frac{5}{36} \cdot \frac{L}{n} (b - a)^2. \tag{3.7}$$

Remark 2   If we want to approximate the integral $\int_a^b f(x) \, dx$ by Simpson's formula $A_{S,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := \left[\frac{5}{36} \cdot \frac{L}{\varepsilon} (b - a)^2 \right] + 1$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

Comments   If the mapping $f : [a, b] \rightarrow \mathbb{R}$ is neither four time differentiable nor the fourth derivative is bounded on (a, b), then we cannot apply the classical estimation in Simpson's formula using the fourth derivative. But if we assume that f is Lipschitzian, then we can use instead the formula (3.6).

We give here a class of mappings which are Lipschitzian but having the fourth derivative unbounded on the given interval.

Let $f_p : [a, b] \rightarrow \mathbb{R}, f_p(x) := (x - a)^p$ where $p \in (3, 4)$. Then obviously

$$f_p'(x) := p(x - a)^{p-1}, \quad x \in (a, b)$$

and

$$f_p^{(4)}(x) = \frac{p(p - 1)(p - 2)(p - 3)}{(x - a)^{4-p}}, \quad x \in (a, b).$$

It is clear that f_p is Lipschitzian with the constant

$$L = p(b-a)^{p-1} < \infty,$$

but $\lim_{x \to a+} f_p^{(4)}(x) = +\infty$.

3.2. Applications for Special Means

Using Theorem 2, we now point out some new inequalities for the special means defined in the previous section.

1. Let $f:[a,b] \to \mathbb{R}$ $(0 < a < b), f(x) = x^p, p \in \mathbb{R} \setminus \{-1, 0\}$. Then

$$\|f'\|_\infty = \delta_p(a,b) := \begin{cases} pb^{p-1} & \text{if } p \geq 1, \\ |p|a^{p-1} & \text{if } p \in (-\infty, 1) \setminus \{-1, 0\}. \end{cases}$$

Using the inequality (3.5) we get

$$|L_p^p(a,b) - \tfrac{1}{3}A(a^p, b^p) - \tfrac{2}{3}A^p(a,b)| \leq \tfrac{5}{35}\delta_p(a,b)(b-a).$$

2. Let $f:[a,b] \to \mathbb{R}$ $(0 < a < b), f(x) = 1/x$. Then

$$\|f'\|_\infty = \frac{1}{a^2}.$$

Using the inequality (3.5) we get

$$|3HA - LA - 2LH| \leq \frac{5}{12} \cdot \frac{b-a}{a^2} LAH.$$

3. Let $f:[a,b] \to \mathbb{R}$ $(0 < a < b), f(x) = \ln x$. Then

$$\|f'\|_\infty = \frac{1}{a}.$$

Using the inequality (3.5) we get

$$\left| \ln \left[\frac{I}{G^{1/3} A^{2/3}} \right] \right| \leq \frac{5}{36} \left(\frac{b-a}{a} \right).$$

4. SIMPSON'S INEQUALITY IN TERMS OF THE p-NORM

4.1. Simpson's Inequality

The following result holds [4]:

THEOREM 3   *Let $f:[a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right|$$
$$\leq \frac{1}{6} \left[\frac{2^{q+1}+1}{3(q+1)} \right]^{1/q} (b-a)^{1+1/q} \|f'\|_p, \tag{4.1}$$

where $(1/p) + 1/q = 1, p > 1$.

Proof Using the integration by parts formula for absolutely continuous mappings, we have

$$\int_a^b s(x)f'(x)\,dx = \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(x)\,dx,$$
$$\tag{4.2}$$

where

$$s(x) := \begin{cases} x - \dfrac{5a+b}{6}, & x \in \left[a, \dfrac{a+b}{2} \right), \\[2mm] x - \dfrac{a+5b}{6}, & x \in \left[\dfrac{a+b}{2}, b \right]. \end{cases}$$

Indeed

$$\int_a^b s(x)f'(x)\,dx$$

$$= \int_a^{(a+b)/2} \left(x - \frac{5a+b}{6} \right) f'(x)\,dx + \int_{(a+b)/2}^b \left(x - \frac{a+5b}{6} \right) f'(x)\,dx$$

$$= \left[\left(x - \frac{5a+b}{6}\right)f(x)\right]_a^{(a+b)/2} + \left[\left(x - \frac{a+5b}{6}\right)f(x)\right]_{(a+b)/2}^b$$

$$- \int_a^b f(x)\,dx$$

$$= \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(x)\,dx,$$

and the identity is proved.

Applying Hölder's integral inequality we obtain

$$\left|\int_a^b s(x)f'(x)\,dx\right| \le \left(\int_a^b |s(x)|^q\,dx\right)^{1/q} \|f'\|_p. \qquad (4.3)$$

Let us compute

$$\int_a^b |s(x)|^q\,dx$$

$$= \int_a^{(a+b)/2} \left|x - \frac{5a+b}{6}\right|^q dx + \int_{(a+b)/2}^b \left|x - \frac{a+5b}{6}\right|^q dx$$

$$= \int_a^{(5a+b)/6} \left(\frac{5a+b}{6} - x\right)^q dx + \int_{(5a+b)/6}^{(a+b)/2} \left(x - \frac{5a+b}{6}\right)^q dx$$

$$+ \int_{(a+b)/2}^{(a+5b)/2} \left(\frac{a+5b}{6} - x\right)^q dx + \int_{(a+5b)/6}^b \left(x - \frac{a+5b}{6}\right)^q dx$$

$$= \frac{1}{q+1}\left[-\left(\frac{5a+b}{6} - x\right)^{q+1}\bigg|_a^{(5a+b)/6} + \left(x - \frac{5a+b}{6}\right)^{q+1}\bigg|_{(5a+b)/6}^{(a+b)/2}\right.$$

$$\left. -\left(\frac{a+5b}{6} - x\right)^{q+1}\bigg|_{(a+b)/2}^{(a+5b)/6} + \left(x - \frac{a+5b}{6}\right)^{q+1}\bigg|_{(a+5b)/6}^b\right]$$

$$= \frac{1}{q+1}\left[\left(\frac{5a+b}{6} - a\right)^{q+1} + \left(\frac{a+b}{2} - \frac{5a+b}{6}\right)^{q+1}\right.$$

$$\left. + \left(\frac{a+5b}{6} - \frac{a+b}{2}\right)^{q+1} + \left(b - \frac{a+5b}{6}\right)^{q+1}\right]$$

$$= \frac{(2^{q+1} + 1)(b-a)^{q+1}}{3(q+1)6^q}.$$

Now, using the inequaltiy (4.3) and the identity (4.2) we deduce the desired result (4.1).

The following corollary for Simpson's composite formula holds:

COROLLARY 7   *Let f and I_h be as above. The we have Simpson's rule* (1.2) *and the remainder $R_S(f, I_h)$ satisfies the estimate:*

$$|R_S(f, I_h)| \le \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)}\right]^{1/q} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{1+q}\right)^{1/q}. \qquad (4.4)$$

Proof Apply Theorem 3 on the interval $[x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ to obtain

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h_i}{3}\left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(\frac{x_i + x_{i+1}}{2}\right)\right]\right|$$

$$\le \frac{1}{6}\left[\frac{2^{q+1} + 1}{3(q+1)}\right]^{1/q} \|f'\|_p\, h_i^{1+1/q} \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p\,dt\right)^{1/p}.$$

Summing the above inequalities over i from 0 to $n-1$, using the generalized triangle inequality and Hölder's discrete inequality, we get

$$|R_S(f, I_h)|$$

$$\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h_i}{3}\left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(\frac{x_i + x_{i+1}}{2}\right)\right]\right|$$

$$\le \frac{1}{6}\left[\frac{2^{q+1} + 1}{3(q+1)}\right]^{1/q} \sum_{i=0}^{n-1} h_i^{1+1/q}\left(\int_{x_i}^{x_{i+1}} |f'(t)|^p\,dt\right)^{1/p}$$

$$\le \frac{1}{6}\left[\frac{2^{q+1} + 1}{3(q+1)}\right]^{1/q}\left[\sum_{i=0}^{n-1}\left(h_i^{1+1/q}\right)^q\right]^{1/q}$$

$$\times \left(\sum_{i=0}^{n-1}\left[\left(\int_{x_i}^{x_{i+1}} |f'(t)|^p\,dt\right)^{1/p}\right]^p\right)^{1/p}$$

$$= \frac{1}{6}\left[\frac{2^{q+1} + 1}{3(q+1)}\right]^{1/q} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{1+q}\right)^{1/q},$$

and the corollary is proved.

The case of equidistant partitioning is embodied in the following corollary:

COROLLARY 8   *Let f be as above and if I_n is an equidistant partitioning of $[a, b]$, then we have the estimate:*

$$|R_{S,n}(f)| \le \frac{1}{6n}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1+1/q}\|f'\|_p.$$

Remark 3   If we want to approximate the integral $\int_a^b f(x)\,dx$ by Simpson's formula $A_{S,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := \left[\frac{1}{6\varepsilon}\left(\frac{2^{q+1}+1}{3(q+1)}\right)^{1/q}(b-a)^{1+1/q}\|f'\|_p\right]+1$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

Comments   If the mapping $f:[a, b] \to \mathbb{R}$ is neither four time differentiable nor the fourth derivative is bounded on (a, b), then we cannot apply the classical estimation in Simpson's formula using the fourth derivative. But if we assume that $f' \in L_p(a, b)$, then we can use the formula (4.4) instead.

We give here a class of mappings whose first derivatives belong to $L_p(a, b)$ but having the fourth derivatives unbounded on the given interval.

Let $f_s:[a, b] \to \mathbb{R}, f_s(x) := (x-a)^s$ where $s \in (3, 4)$. Then obviously

$$f_s'(x) := s(x-a)^{s-1}, \quad x \in (a, b)$$

and

$$f_s^{(4)}(x) = \frac{s(s-1)(s-2)(s-3)}{(x-a)^{4-s}}, \quad x \in (a, b).$$

It is clear that $\lim_{x\to a+} f_s^{(4)}(x) = +\infty$, but

$$\|f_s'\|_p = s \cdot \frac{(b-a)^{s-1+(1/p)}}{((s-1)p+1)^{1/p}} < \infty.$$

4.2. Applications for Special Means

(See Section 2.2 for the definition of the means.)

1. Let $f\colon [a, b] \to \mathbb{R}$ $(0 < a < b)$, $f(x) = x^s$, $s \in \mathbb{R} \setminus \{-1, 0\}$. Then

$$\frac{1}{b-a} \int_a^b f(x)\, dx = L_s^s(a, b),$$

$$f\left(\frac{a+b}{2}\right) = A^s(a, b),$$

$$\frac{f(a) + f(b)}{2} = A(a^s, b^s)$$

and

$$\|f'\|_p = |s| L_{(s-1)p}^{s-1}(b - a)^{1/p}.$$

Using the inequality (4.1) we get

$$\left| L_s^s(a, b) - \tfrac{1}{3} A(a^s, b^s) - \tfrac{2}{3} A^s(a, b) \right|$$

$$\leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{1/q} |s| L_{(s-1)p}^{s-1}(a, b)(b - a),$$

where $(1/p) + 1/q = 1$, $p > 1$.

2. Let $f\colon [a, b] \to \mathbb{R}$ $(0 < a < b)$, $f(x) = 1/x$. Then

$$\frac{1}{b-a} \int_a^b f(x)\, dx = L^{-1}(a, b),$$

$$f\left(\frac{a+b}{2}\right) = A^{-1}(a, b),$$

$$\frac{f(a) + f(b)}{2} = H^{-1}(a, b)$$

and

$$\|f'\|_p = L_{-2p}^{-2}(a, b)(b - a)^{1/p}.$$

Using the inequality (4.1) we get

$$|3HA - LA - 2LH| \le \frac{1}{2} AHL \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{1/q} L_{-2p}^{-2}(b-a)^{1/p}.$$

3. Let $f: [a, b] \to \mathbb{R}$ $(0 < a < b), f(x) = \ln x$. Then

$$\frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x = \ln I(a, b),$$

$$f\left(\frac{a+b}{2}\right) = \ln A(a, b),$$

$$\frac{f(a) + f(b)}{2} = \ln A(a, b)$$

and

$$\|f'\|_p = L_{-p}^{-1}(a, b)(b-a)^{1/p}.$$

Using the inequality (4.1) we obtain

$$\left| \ln \left[\frac{I}{G^{1/3} A^{2/3}} \right] \right| \le \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{1/q} L_{-p}^{-1}(a, b)(b-a).$$

5. GRÜSS INEQUALITY FOR THE SIMPSON FORMULA

5.1. Some Preliminary Results

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss' inequality [5, p. 296]:

THEOREM 4   *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\varphi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in (a, b)$; φ, Φ, γ and Γ are constants.*

Then we have the inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)\,dx - \frac{1}{b-a} \int_a^b f(x)\,dx \cdot \frac{1}{b-a} \int_a^b g(x)\,dx \right|$$

$$\leq \tfrac{1}{4}(\phi - \varphi)(\Gamma - \gamma), \tag{5.1}$$

and the inequality is sharp in the sense that the constant 1/4 cannot be replaced by a smaller one.

In 1938, Ostrowski (cf., for example [1, p. 468]) proved the following inequality which gives an approximation of the integral $1/(b-a) \int_a^b f(t)\,dt$ as follows:

THEOREM 5   *Let $f:[a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)\,dt| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty, \tag{5.2}$$

for all $x \in (a,b)$.

In the recent paper [6], Dragomir and Wang proved the following version of Ostrowski's inequality by using the Grüss inequality (5.1).

THEOREM 6   *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in the interior of I and let $a, b \in \mathrm{int}(I)$ with $a < b$. If $f' \in L_1[a,b]$ and*

$$\gamma \leq f'(x) \leq \Gamma$$

for all $x \in [a,b]$, then we have the following inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a} \cdot \left(x - \frac{a+b}{2} \right) \right|$$

$$\leq \tfrac{1}{4}(b-a)(\Gamma - \gamma), \tag{5.3}$$

for all $x \in [a,b]$.

They also applied this result for special means and in Numerical Integration obtaining some quadrature formulae generalizing the mid-point

quadrature rule and the trapezoid rule. Note that the error bounds they obtained are in terms of the first derivative which are particularly useful in the case when f'' does not exist or is very large at some points in $[a, b]$.

For other related results see the papers [7–37].

In this section of our paper we give a generalization of the above inequality which contains as a particular case the classical *Simpson formula*. Application for special means and in *Numerical Integration* are also give.

5.2. An Integral Inequality of Grüss Type

For any real numbers A, B, let us consider the function [21]

$$p(t) \equiv p_x(t) = \begin{cases} t - a + A & \text{if } a \le t \le x, \\ t - b + B & \text{if } x < t \le b. \end{cases}$$

It is clear that p_x has the following properties.

(a) It has the jump

$$[p]_x = (B - A) - (b - a)$$

at point $t = x$ and

$$\frac{\mathrm{d}p_x(t)}{\mathrm{d}t} = 1 + [p]_x \delta(t - x).$$

(b) Let $M_x := \sup_{t \in (a,b)} p_x(t)$ and $m_x := \inf_{t \in (a,b)} p_x(t)$. Then the difference $M_x - m_x$ can be evaluated as follows:

(1) For $B - A \le 0$, we have

$$M_x - m_x = -[p]_x.$$

(2) For $B - A > 0$, the following three cases are possible
   (i) If $0 \le B - A \le \frac{1}{2}(b - a)$, then

$$M_x - m_x = \begin{cases} -x + b & \text{for } a \le x \le a + (B - A); \\ -[p]_x & \text{for } a + (B - A) < x \le b - (B - A); \\ x - a & \text{for } b - (B - A) < x \le b. \end{cases}$$

(ii) If $\frac{1}{2}(a - b) < B - A \leq (b - a)$, then

$$M_x - m_x = \begin{cases} -x + b & \text{for } a \leq x < b - (B - A); \\ B - A & \text{for } b - (B - A) \leq x < a + (B - A); \\ x - a & \text{for } q + (B - A) \leq x \leq b. \end{cases}$$

(iii) If $B - A > b - a$, then

$$M_x - m_x = [p]_x.$$

The following inequality of Ostrowski type holds [21]

THEOREM 7   *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative satisfies the assumption*

$$\gamma \leq f'(t) \leq \Gamma \quad \text{for all } t \in (a, b), \tag{5.4}$$

where γ, Γ are given real numbers. Then we have the inequality:

$$\left| (C - A)f(a) + (b - a - B + A)f(x) + (B - C)f(b) - \int_a^b f(t)\, dt \right|$$
$$\leq \frac{1}{4}(\Gamma - \gamma)(M_x - m_x)(b - a), \tag{5.5}$$

where

$$C_x := \frac{1}{2(b - a)}[(x - a)(x - a + 2A) - (x - b)(x - b + 2B)],$$

and A, B, M_x and m_x are as above, $x \in [a, b]$.

Proof   Using the Grüss inequality (5.1), we can state that

$$\left| \frac{1}{b - a} \int_a^b p_x(t) f'(t)\, dt - \frac{f(b) - f(a)}{b - a} \cdot \frac{1}{b - a} \int_a^b p_x(t)\, dt \right|$$
$$\leq \frac{1}{4}(\Gamma - \gamma)(M_x - m_x), \tag{5.6}$$

for all $x \in (a, b)$.

Integrating the first term by parts we obtain:

$$\int_a^b p_x(t) f'(t)\, dt = Bf(b) - Af(a) - \int_a^b f(t)\, dt + [p]_x f(x). \quad (5.7)$$

Also, as

$$\int_a^b p_x(t)\, dt = \tfrac{1}{2}[(x-a)(x-a+2A) - (x-b)(x-b+2B)],$$

then (5.6) gives the inequality:

$$\left| \frac{1}{b-a}\left[Bf(b) - Af(a) - \int_a^b f(t)\, dt + [p]_x f(x)\right] - C_x \cdot \frac{f(b)-f(a)}{b-a}\right|$$

$$\leq \tfrac{1}{4}(\Gamma - \gamma)(M_x - m_x),$$

which is clearly equivalent with the desired result (5.5).

Remark 4   Setting in (5.5), $A = B = 0$ and taking into account, by the property (b), that $M_x - m_x = b - a$, we obtain the inequality (5.3) by Dragomir and Wang.

The following corollary is interesting:

COROLLARY 9   *Let A, B be real numbers so that $0 \leq B - A \leq (b-a)/2$. If f is as above, then we have the inequality:*

$$\left| \frac{B-A}{2}f(a) + [b - a - (B-A)]f\left(\frac{a+b}{2}\right) + \frac{B-A}{2}f(b) - \int_a^b f(t)\, dt\right|$$

$$\leq \tfrac{1}{4}(\Gamma - \gamma)(b - a - B + A)(b - a). \quad (5.8)$$

Proof   Consider $x = (a+b)/2$. Then, from (5.5)

$$x - a = \frac{b-a}{2},$$

$$x - b = -\frac{b-a}{2}$$

and

$$C_x = \frac{A+B}{2}, \quad x \in [a + (B - A), b - (B - A)].$$

By property (b) we have

$$M_x - m_x = (b - a) - (B - A).$$

Applying Theorem 7 for $x = (a + b)/2$, we get easily (5.8).

Remark 5   If we choose in the above corollary $B - A = (b - a)/2$, then we get

$$\left| \frac{1}{2} \cdot \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] (b - a) - \int_a^b f(t)\, dt \right| \leq \frac{1}{8} (\Gamma - \gamma)(b - a)^2,$$

(5.9)

which is the arithmetic mean of the mid-point and trapezoid formulae.

Remark 6   If we choose in (5.8) $B = A$, then we get the mid-point inequality:

$$\left| (b - a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right| \leq \frac{1}{4} (\Gamma - \gamma)(b - a)^2 \qquad (5.10)$$

discovered by Dragomir and Wang in the paper [6] (see Corollary 2).

Remark 7   If we choose in (5.8) $B - A = (b - a)/3$, then we obtain the celebrated Simpson's formula:

$$\left| \frac{b-a}{6} \cdot \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)\, dt \right| \leq \frac{1}{6} (\Gamma - \gamma)(b - a)^2,$$

(5.11)

for which we have an estimation in terms of the first derivative not as in the classical case in which the fourth derivative is required as follows:

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)\, dt \right| \leq \frac{\|f^{(4)}\|_\infty}{2880} (b - a)^5.$$

(5.12)

The method of evaluation of the error for the Simpson rule considered above can be applied for any quadrature formula of Newton–Cotes type.

For example, to get the analogous evaluation of the error for the Newton–Cotes rule of order 3 it is sufficient to replace the function $p_x(t)$ in (2.3) by the function

$$p_x(t) := \begin{cases} t - a - A & \text{if } a \le t \le a + h; \\ t - \dfrac{a+b}{2} + \dfrac{A+B}{2} & \text{if } a + h < t \le b - h; \\ t - b - B & \text{if } b - h < t \le b; \end{cases}$$

where $B - A = (b - a)/4$, $h = (b - a)/3$.

5.3. Applications for Special Means

(See Section 2.2 for the definition of the means.)

1. Consider the mapping $f(x) = x^p (p > 1)$, $x > 0$. Then

$$\Gamma - \gamma = (a - b)(p - 1)L_{p-2}^{p-2}$$

for $a, b \in \mathbb{R}$ with $0 < a < b$. Consequently, we have the inequality:

$$\left| \frac{2}{3} A^p(a, b) + \frac{1}{2} A(a^p, b^p) - L_p^p(a, b) \right| \le \frac{1}{6}(b - a)^2(p - 1)L_{p-2}^{p-2}.$$

2. Consider the mapping $f(x) = 1/x$, $x > 0$. Then

$$\Gamma - \gamma = \frac{b^2 - a^2}{a^2 b^2} = 2 \cdot \frac{(b - a)A(a, b)}{G^4(a, b)}$$

for $0 < a < b$. Consequently we have the inequality:

$$\left| \frac{2}{3} A^{-1}(a, b) + \frac{1}{3} H^{-1}(a, b) - L^{-1}(a, b) \right| \le \frac{1}{3}(b - a)^2 \frac{A(a, b)}{G^4(a, b)}$$

which is equivalent to

$$\left| \frac{2}{3} HL + \frac{1}{3} AL - AH \right| \le \frac{1}{3}(b - a)^2 \frac{A^2 HL}{G^4}.$$

3. Consider the mapping $f(x) = \ln x$, $x > 0$. Then we have

$$\Gamma - \gamma = \frac{b - a}{G^2}$$

for $a, b \in \mathbb{R}$ with $0 < a < b$. Consequently, we have the inequality:

$$\left| \frac{2}{3} \ln A + \frac{1}{3} \ln G - \ln I \right| \leq \frac{1}{6} \frac{(b - a)^2}{G^2},$$

which is equivalent to

$$\left| \ln \left(\frac{A^{2/3} G^{1/3}}{I} \right) \right| \leq \frac{1}{6} \cdot \frac{(b - a)^2}{G^2}.$$

5.4. Estimation of Error Bounds in Simpson's Rule

The following theorem holds.

THEOREM 8   *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping (a, b) whose derivative satisfies the condition*

$$\gamma \leq f'(t) \leq \Gamma \quad \text{for all } t \in (a, b);$$

where γ, Γ are given real numbers. Then we have

$$\int_a^b f(t) \, dt = S_n(I_n, f) + R_n(I_n, f), \tag{5.13}$$

where

$$S_n(I_n, f) = \frac{1}{3} \sum_{i=0}^{n-1} h_i [f(x_i) + 4f(x_i + h_i) + f(x_{i+1})], \tag{5.14}$$

I_h is the partition given by

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

$h_i := \frac{1}{2}(x_{i+1} - x_i), i = 0, \ldots, n-1$ *and the remainder term $R_n(I_n, f)$ satisfies the estimation:*

$$|R_n(I_n, f)| \leq \frac{2}{3}(\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2. \tag{5.15}$$

Proof   Let us set in (5.11)

$$a = x_i, \ b = x_{i+1}, \ 2h_i = x_{i+1} - x_i \ \text{and} \ x_i + h_i = \frac{1}{2}(x_i + x_{i+1}),$$

where $i = 0, \ldots, n-1$.
   Then we have the estimation:

$$\left| \frac{h_i}{3}[f(x_i) + 4f(x_i + h_i) + f(x_{i+1})] - \int_{x_i}^{x_{i+1}} f(t) \, dt \right| \leq \frac{2}{3}(\Gamma - \gamma)h_i^2,$$

for all $i = 0, \ldots, n-1$.
   After summing and using the triangle inequality, we obtain

$$\left| \sum_{i=0}^{n-1} \frac{h_i}{3}[f(x_i) + 4f(x_i + h_i) + f(x_{i+1})] - \int_a^b f(t) \, dt \right| \leq \frac{2}{3}(\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2,$$

which proves the required estimation.

COROLLARY 10   *Under the above assumptions and if we put $\|f'\|_\infty :=$ $\sup_{t \in (a,b)} |f'(t)| < \infty$, then we have the following estimation of the remainder term in Simpson's formula*

$$|R_n(I_n, f)| \leq \frac{4}{3}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{5.16}$$

   The classical error estimates based on the Taylor expansion for Simpson's rule involve the fourth derivative $\|f^{(4)}\|_\infty$. In the case that $f^{(4)}$ does not exist or is very large at some points in $[a, b]$, the classical estimates cannot be applied, and thus (5.15) and (5.16) provide alternative error estimates for the Simpson's rule.

6. A CONVEX COMBINATION

The following generalization of Ostrowski's inequality holds [19]:

THEOREM 9   Let $f:[a,b] \to R$ be absolutely continuous on $[a,b]$, and whose derivative $f':[a,b] \to R$ is bounded on $[a,b]$. Denote $\|f'\|_\infty :=$ ess $\sup_{t \in [a,b]} |f'(x)| < \infty$. Then

$$\left| \int_a^b f(t)\, dt - \left[f(x) \cdot (1-\delta) + \frac{f(a)+f(b)}{2} \cdot \delta \right] (b-a) \right|$$

$$\leq \left[\frac{1}{4}(b-a)^2 \left[\delta^2 + (\delta-1)^2 \right] + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty \quad (6.1)$$

for all $\delta \in [0,1]$ and $a + \delta \cdot (b-a)/2 \leq x \leq b - \delta \cdot (b-a)/2$.

Proof   Let us define the mapping $p:[a,b]^2 \to R$ given by

$$p(x,t) := \begin{cases} t - \left[a + \delta \cdot \dfrac{b-a}{2} \right], & t \in [a,x], \\[2ex] t - \left[b - \delta \cdot \dfrac{b-a}{2} \right], & t \in (x,b]. \end{cases}$$

Integrating by parts, we have

$$\int_a^b p(x,t) f'(t)\, dt$$

$$= \int_a^x \left(t - \left[a + \delta \cdot \frac{b-a}{2} \right] \right) f'(t)\, dt + \int_x^b \left(t - \left[b - \delta \cdot \frac{b-a}{2} \right] \right) f'(t)\, dt$$

$$= \delta \cdot (b-a) \frac{(f(a)+f(b))}{2} + (1-\delta) \cdot f(x) - \int_a^b f(t)\, dt. \quad (6.2)$$

On the other hand

$$\left| \int_a^b p(x,t) f'(t)\, dt \right| \leq \int_a^b |p(x,t)|\, |f'(t)|\, dt \leq \|f'\|_\infty \int_a^b |p(x,t)|\, dt$$

$$= \|f'\|_\infty \left[\int_a^x \left| t - \left(a + \delta \cdot \frac{b-a}{2} \right) \right| dt + \int_x^b \left| t - \left(b - \delta \cdot \frac{b-a}{2} \right) \right| dt \right]$$

$$=: \|f'\|_\infty L.$$

Now, let us observe that

$$\int_p^r |t - q|\, dt = \int_p^q (q - t)\, dt + \int_q^r (t - q)\, dt$$

$$= \frac{1}{2}\left[(q - p)^2 + (r - q)^2\right] = \frac{1}{4}(p - r)^2 + \left(q - \frac{r + p}{2}\right)^2$$

for all r, p, q such that $p \leq q \leq r$.

Using the previous identity, we have that

$$\int_a^x \left| t - \left(a + \delta \cdot \frac{b - a}{2}\right)\right|\, dt = \frac{1}{4}(x - a)^2 + \left(a + \delta \cdot \frac{b - a}{2} - \frac{a + x}{2}\right)^2$$

and

$$\int_x^b \left| t - \left(b - \delta \cdot \frac{b - a}{2}\right)\right|\, dt = \frac{1}{4}(b - x)^2 + \left(b - \delta \cdot \frac{b - a}{2} - \frac{x + b}{2}\right)^2.$$

Then we get

$$L = \frac{1}{2} \cdot \frac{(x - a)^2 + (b - x)^2}{2} + \left(\delta \cdot \frac{b - a}{2} - \frac{x - a}{2}\right)^2 + \left(\frac{b - x}{2} - \delta \cdot \frac{b - a}{2}\right)^2$$

$$= \frac{(b - a)^2}{4} \cdot \left[\delta^2 + (\delta - 1)^2\right] + \left(x - \frac{a + b}{2}\right)^2$$

and the theorem is thus proved.

Remark 8   (a) If we choose in (6.1), $\delta = 0$, we get Ostrowski's inequality.
(b) If we choose in (6.1), $\delta = 1$ and $x = (a + b)/2$ we get the trapezoid inequality:

$$\left|\int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b - a)\right| \leq \frac{1}{4}(b - a)^2 \|f'\|_\infty. \qquad (6.3)$$

COROLLARY 11   *Under the above assumptions, we have the inequality:*

$$\left|\int_a^b f(t)\, dt - \frac{1}{2}\left[f(x) + \frac{f(a) + f(b)}{2}\right](b - a)\right|$$

$$\leq \left[\frac{1}{8}(b - a)^2 + \left(x - \frac{a + b}{2}\right)^2\right]\|f'\|_\infty$$

for all $x \in [(b + 3a)/4, (a + 3b)/4]$, and, in particular, the following mixture of the trapezoid inequality and mid-point inequality:

$$\left| \int_a^b f(t)\, dt - \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] (b - a) \right| \leq \frac{1}{8} (b - a)^2 \|f'\|_\infty.$$

$$(6.4)$$

Finally, we also have the following generalization of Simpson's inequality:

COROLLARY 12 *Under the above assumptions, we have*

$$\left| \int_a^b f(t)\, dt - \frac{1}{6} [f(a) + 4f(x) + f(b)](b - a) \right|$$

$$\leq \left[\frac{5}{36} (b - a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty$$

for all $x \in [(b + 5a)/4, (a + 5b)/4]$, and, in particular, the Simpson's inequality:

$$\left| \int_a^b f(t)\, dt - \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] (b - a) \right| \leq \frac{5}{36} (b - a)^2 \|f'\|_\infty.$$

$$(6.5)$$

6.1. Applications in Numerical Integration

The following approximation of the integral $\int_a^b f(x)\, dx$ holds [19].

THEOREM 10 *Let $f : [a, b] \to R$ be an absolutely continuous mapping on $[a, b]$ whose derivative is bounded on $[a, b]$. If $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition of $[a, b]$ and $h_i := x_{i+1} - x_i$, $i = 0, \ldots, n - 1$, then we have*

$$\int_a^b f(x)\, dx = A_\delta(I_n, \xi, \delta, f) + R_\delta(I_n, \xi, \delta, f), \qquad (6.6)$$

where

$$A_\delta(I_n, \xi, \delta, f) = (1 - \delta) \sum_{i=0}^{n-1} f(\xi_i)h_i + \delta \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i, \quad (6.7)$$

$\delta \in [0, 1], x_i + \delta \cdot h_i/2 \leq \xi_i \leq x_{i+1} - \delta \cdot h_i/2, i = 0, \ldots, n - 1;$ *and the remainder term satisfies the estimation:*

$$|R_\delta(I_n, \xi, \delta, f)|$$

$$\leq \|f'\|_\infty \left[\frac{1}{4} \left[\delta^2 + (\delta - 1)^2 \right] \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]. \quad (6.8)$$

Proof   Applying Theorem 9 on the interval $[x_i, x_{i+1}]$, $i = 0, \ldots, n - 1$ we get

$$\left| h_i \left[(1 - \delta) \cdot f(\xi_i) + \frac{f(x_i) + f(x_{i+1})}{2} \cdot \delta \right] - \int_{x_i}^{x_{i+1}} f(x) \, dx \right|$$

$$\leq \left[\left[\delta^2 + (\delta - 1)^2 \right] \frac{h_i^2}{4} + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty$$

for all $\delta \in [0, 1]$ and $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \ldots, n - 1$.

Summing over i from 0 to $n - 1$ and using the triangle inequality we get the estimation (6.8).

Remark 9   (a) If we choose $\delta = 0$, then we get the quadrature formula:

$$\int_a^b f(x) \, dx = A_T(I_n, \xi, f) + R_T(I_n, \xi, f), \quad (6.9)$$

where $A_T(I_n, \xi, f)$ is the Riemann sum, i.e.,

$$A_T(I_n, \xi, f) := \sum_{i=0}^{n-1} f(\xi_i)h_i, \quad \xi_i \in [x_i, x_{i+1}], \quad i = 0, \ldots, n - 1;$$

and the remainder term satisfies the estimate (see also [8]):

$$|R_T(I_n, \xi, f)| \leq \|f'\|_\infty \sum_{i=0}^{n-1} \left[\frac{h_i^2}{4} + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]. \quad (6.10)$$

(b) If we choose $\delta = 1$, then we get the trapezoid formula:

$$\int_a^b f(x)\,dx = A_T(I_n,f) + R_T(I_n,f) \tag{6.11}$$

where $A_T(I_n,f)$ is the trapezoidal rule

$$A_T(I_n,f) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i$$

and the remainder terms satisfies the estimation:

$$|R_T(I_n,f)| \leq \frac{\|f'\|_\infty}{4} \sum_{i=0}^{n-1} h_i^2. \tag{6.12}$$

COROLLARY 13   *Under the above assumptions we have*

$$\int_a^b f(x)\,dx = B_T(I_n, \xi, f) + Q_T(I_n, \xi, f), \tag{6.13}$$

where

$$B_T(I_n,\xi,f) = \frac{1}{2}\left[\sum_{i=0}^{n-1} f(\xi_i)h_i + \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i\right],$$

$$\xi_i \in \left[\frac{x_{i+1} + 3x_i}{4}, \frac{x_i + 3x_{i+1}}{4}\right],$$

and the remainder term satisfies the estimation:

$$|Q_T(I_n,\xi,f)| \leq \|f'\|_\infty \left[\frac{1}{8}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]. \tag{6.14}$$

In particular, we have

$$\int_a^b f(x)\,\mathrm{d}x = B_T(I_n, f) + Q_T(I_n, f),\tag{6.15}$$

where

$$B_T(I_n, f) = \frac{1}{2}\left[\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i + \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}\cdot h_i\right]$$

and $Q_T(I_n, f)$ satisfies the estimation:

$$|Q_T(I_n, f)| \le \frac{\|f'\|_\infty}{8}\sum_{i=0}^{n-1} h_i^2.\tag{6.16}$$

Finally, we have the following generalization of Simpson's inequality whose remainder term is estimated by the use of the first derivative only.

COROLLARY 14 *Under the above assumptions we have*

$$\int_a^b f(x)\,\mathrm{d}x = S_T(I_n, \xi, f) + W_T(I_n, \xi, f),\tag{6.17}$$

where

$$S_T(I_n, \xi, f) = \frac{2}{3}\sum_{i=0}^{n-1} f(\xi_i)h_i + \frac{1}{6}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]h_i,$$

$$\xi_i \in \left[\frac{x_{i+1} + 5x_i}{6}, \frac{x_i + 5x_{i+1}}{6}\right],$$

and the remainder term $W_T(I_n, \xi, f)$ satisfies the bound:

$$|W_T(I_n, \xi, f)| \le \|f'\|_\infty\left[\frac{5}{36}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\tag{6.18}$$

and, in particular, the Simpson's rule:

$$\int_a^b f(x)\,dx = S_T(I_n,f) + W_T(I_n,f), \qquad (6.19)$$

where

$$S_T(I_n,f) = \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i + \frac{1}{6}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]h_i$$

and the remainder term satisfies the estimation:

$$|W_T(I_n,f)| \le \frac{5}{36}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \qquad (6.20)$$

6.2. Applications for Special Means

Now, let us reconsider the inequality (6.1) in the following equivalent form:

$$\left|(1-\delta)\cdot f(x) + \frac{f(a)+f(b)}{2}\cdot\delta - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\le \left[(b-a)\left[\frac{\delta^2 + (\delta-1)^2}{4}\right] + \frac{(x-(a+b)/2)^2}{(b-a)}\right]\|f'\|_\infty$$

for all $\delta \in [0,1]$ and $x \in [a,b]$ such that

$$a + \delta\cdot\left(\frac{b-a}{2}\right) \le x \le b - \delta\cdot\left(\frac{b-a}{2}\right).$$

1. Consider the mapping $f:(0,\infty) \to (0,\infty)$, $f(x) = x^p$, $p \in \mathbb{R}\setminus\{-1,0\}$. Then, for $0 < a < b$, we have

$$\|f'\|_\infty = \begin{cases} |p|b^{p-1} & \text{if } p > 1, \\ |p|a^{p-1} & \text{if } p \in (-\infty, 1]\setminus\{-1,0\}, \end{cases}$$

and then, by (6.21), we deduce that

$$|(1 - \delta) \cdot x^p + \delta \cdot A(a^p, b^p) - L_p^p(a,b)|$$

$$\leq \left\{ (b-a) \left[\frac{\delta^2 + (\delta - 1)^2}{4} \right] + \frac{(x-A)^2}{(b-a)} \right\} \delta_p(a,b),$$

where

$$\delta_p(a,b) := \begin{cases} |p|b^{p-1} & \text{if } p > 1, \\ |p|a^{p-1} & \text{if } p \in (-\infty, 1] \backslash \{-1, 0\} \end{cases}$$

and $\delta \in [0, 1]$, $x \in [a + \delta \cdot (b - a)/2, b - \delta \cdot (b - a)/2]$.

2. Consider the mapping $f: (0, \infty) \to (0, \infty)$, $f(x), = 1/x$ and $0 < a < b$. We have:

$$\|f'\|_\infty = \frac{1}{a^2}$$

and then by (6.21), we deduce, for all $\delta \in [0, 1]$, and $a + \delta \cdot (b - a)/2 \leq x \leq b - \delta \cdot (b - a)/2$ that

$$|(1 - \delta)\delta L + Lx\delta - x\delta| \leq \frac{x\delta L}{a^2} \left[(b-a) \left[\frac{\delta^2 + (\delta - 1)^2}{4} \right] + \frac{(x-A)^2}{(b-a)} \right].$$

3. Consider the mapping $f: (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$ and $0 < a < b$. We have

$$\|f'\|_\infty = \frac{1}{a},$$

and then, by (6.21), we deduce that

$$\left| \ln \left[\frac{x^{1-\delta} G^\delta}{I} \right] \right| \leq \frac{1}{a} \left[(b-a) \left[\frac{\delta^2 + (\delta - 1)^2}{4} \right] + \frac{(x-A)^2}{(b-a)} \right],$$

for all $\delta \in [0, 1]$, and $x \in [a + \delta \cdot (b - a)/2, b - \delta \cdot (b - a)/2]$.

7. A GENERALIZATION FOR MONOTONIC MAPPINGS

In [20], Dragomir established the following Ostrowski type inequality for monotonic mappings.

THEOREM 11   *Let $f: [a, b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\,dx \right|$$

$$\leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b \operatorname{sgn}(t-x) f(t)\,dt \right\}$$

$$\leq \frac{1}{b-a} [(x-a)(f(x) - f(a)) + (b-x)(f(b) - f(x))]$$

$$\leq \left[\frac{1}{2} + \frac{|x - (a+b)/2|}{b-a} \right] (f(b) - f(a)).$$

All the inequalities are sharp and the constant $\frac{1}{2}$ is the best possible one.

In this section we shall obtain a generalization of this result which also contains the trapezoid and Simpson type inequalities.

The following result holds [38]:

THEOREM 12   *Let $f: [a, b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a, b]$ and $t_1, t_2, t_3 \in (a, b)$ be such that $t_1 < t_2 < t_3$. Then*

$$\left| \int_a^b f(x)\,dx - [(t_1 - a)f(a) + (b - t_3)f(b) + (t_3 - t_1)f(t)] \right|$$

$$\leq (b - t_3)f(b) + (2t_2 - t_1 - t_3)f(t_2) - (t_1 - a)f(a)$$

$$\quad + \int_a^b T(x)f(x)\,dx$$

$$\leq (b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2))$$

$$\quad + (t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(t_2))$$

$$\leq \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)), \qquad (7.1)$$

where

$$T(x) = \begin{cases} \operatorname{sgn}(t_1 - x), & \text{for } x \in [a, t_2], \\ \operatorname{sgn}(t_3 - x), & \text{for } x \in [t_2, b]. \end{cases}$$

Proof Using integration by parts formula for Riemann–Stieltjes integral we have

$$\int_a^b s(x)\,df(x) = (t_1 - a)f(a) + (b - t_3)f(b) + (t_3 - t_1)f(t_2)$$

$$- \int_a^b f(x)\,d(x),$$

where

$$s(x) = \begin{cases} x - t_1, & x \in [a, t_2], \\ x - t_3, & x \in [t_2, b]. \end{cases}$$

Indeed

$$\int_a^b s(x)\,df(x) = \int_a^{t_2} (x - t_1)\,df(x) + \int_{t_2}^b (x - t_3)\,df(x)$$

$$= (x - t_1)f(x)\big|_a^{t_2} + (x - t_3)f(t)\big|_{t_2}^b - \int_a^b f(x)\,d(x)$$

$$= (t_1 - a)f(a) + (b - t_3)f(b)$$

$$+ (t_3 - t_1)f(t_2) - \int_a^b f(x)\,dx.$$

Assume that $A_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(A_n) \to 0$ as $n \to \infty$, where $\nu(A_n) := \max_{i=0,\ldots,n-1}\left(x_{i-1}^{(n)} - x_i^{(n)}\right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. If $p : [a, b] \to \mathbb{R}$ is a continuous mapping on $[a, b]$ and v is monotonic nondecreasing on $[a, b]$, then

$$\left|\int_a^b p(x)\,dv(x)\right| = \lim_{\nu(A_n)\to\infty} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right]$$

$$\leq \lim_{\nu(A_n)\to\infty} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right|\left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right|$$

$$= \lim_{\nu(A_n)\to\infty} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right|\left(v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right)$$

$$= \int_a^b |p(x)|\,dv(x). \tag{7.2}$$

Applying the inequality (7.2) for $p(x) = s(x)$ and $v(x) = f(x)$, $x \in [a, b]$ we can state:

$$\left| \int_a^b s(x) \, df(x) \right| \leq \int_a^b |s(x)| \, df(x)$$

$$= \int_a^{t_1} (t_1 - x) \, df(x) + \int_{t_1}^{t_2} (x - t_1) \, df(x)$$

$$+ \int_{t_2}^{t_3} (t_3 - x) \, df(x) + \int_{t_3}^b (x - t_3) \, df(x)$$

$$= (t_1 - x) f(x)|_a^{t_1} + \int_a^{t_1} f(x) \, dx + (x - t_1) f(x)|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} f(x) \, dx + (t_3 - x) f(x)|_{t_2}^{t_3}$$

$$+ \int_{t_2}^{t_3} f(x) \, dx + (x - t_3) f(x)|_{t_3}^b + \int_{t_3}^b f(x) \, dx$$

$$= -(t_1 - a) f(a) + (t_2 - t_1) f(t_2) - (t_3 - t_2) f(t_2)$$

$$+ (b - t_3) f(b) + \int_a^b T(x) f(x) \, dx.$$

which is the first inequality in (7.1).

If $f : [a, b] \to \mathbb{R}$ is monotonic nondecreasing in $[a, b]$, we can also state:

$$\int_a^{t_1} f(x) \, dx \leq f(t_1)(t_1 - a),$$

$$\int_{t_1}^{t_2} f(x) \, dx \geq f(t_2)(t_2 - t_1),$$

$$\int_{t_2}^{t_3} f(x) \, dx \leq f(t_3)(t_3 - t_2),$$

and

$$\int_{t_3}^b f(x) \, dx \geq f(t_3)(b - t_3).$$

So,

$$\int_a^b T(x)f(x)\,dx$$

$$= \int_a^{t_1} f(x)\,dx - \int_{t_1}^{t_2} f(x)\,dx + \int_{t_2}^{t_3} f(x)\,dx - \int_{t_3}^b f(x)\,dx$$

$$\leq f(t_1)(t_1 - a) - f(t_2)(t_2 - t_1) + f(t_3)(t_3 - t_2) - f(t_3)(b - t_3).$$

We have

$$-(t_1 - a)f(a) + (t_2 - t_1)f(t_2) - (t_3 - t_2)f(t_2)$$

$$-(b - t_3)f(b) + \int_a^b T(x)f(x)\,dx$$

$$\leq -(t_1 - a)f(a) + (t_2 - t_1)f(t_2) - (t_3 - t_2)f(t_2) + (b - t_3)f(b)$$

$$+ (t_1 - a)f(t_1) - (t_2 - t_1)f(t_1) + (t_3 - t_2)f(t_3) - (b - t_3)f(t_3)$$

$$= (t_1 - a)(f(t_1) - f(a)) + (t_2 - t_1)(f(t_2) - f(t_1))$$

$$+ (t_3 - t_2)(f(t_3) - f(t_2)) + (b - t_3)(f(b) - f(t_3))$$

$$\leq \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).$$

The theorem is thus proved.

Remark 10   For $t_1 = 0$, $t_2 = x$, $t_3 = b$, a generalized trapezoid inequality is obtained and we get Theorem 11 from the above Theorem.

For $t_1 = t_2 = t_3 = x$ Theorem 12 becomes [38]:

COROLLARY 15   *Let f be defined as in Theorem 12. Then*

$$\left| \int_a^b f(x)\,dt - [(x - a)f(a) + (b - x)f(b)] \right|$$

$$\leq (b - x)f(b) - (x - a)f(a) + \int_a^b \operatorname{sgn}(x - t)f(t)\,dt$$

$$\leq (b - x)(f(b) - f(x)) + (x - a)(f(x) - f(a))$$

$$\leq \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right](f(b) - f(a)). \tag{7.3}$$

All the inequalities in (7.3) *are sharp and the constant $\frac{1}{2}$ is the best possible.*

Proof We only need to prove that the constant $\frac{1}{2}$ is the best possible one. Choose the mapping $f_0 : [a, b] \to \mathbb{R}$ given by

$$f_0(x) = \begin{cases} 0, & \text{if } x \in [a, b), \\ 1, & \text{if } x = b. \end{cases}$$

Then, f_0 is monotonic nondecreasing on $[a, b]$, and for $x = a$ we have

$$\left| \int_a^b f(t) \, dt - [(x - a)f(a) + (b - x)f(b)] \right|$$

$$= (b - x)f(b) - (x - a)f(a) + \int_a^b \text{sgn}(t - x)f(t) \, dt$$

$$= (b - x)(f(b) - f(x)) + (x - a)(f(x) - f(a))$$

$$= (b - a)$$

$$\leq \left[C(b - a) + \left| x - \frac{a + b}{2} \right| \right] (f(b) - f(a))$$

$$= \left(C + \tfrac{1}{2} \right)(b - a)$$

which prove the sharpness of the first two inequalities and the fact that C cannot be less than $\frac{1}{2}$.

For $x = (a + b)/2$ we get the trapezoid inequality [38].

COROLLARY 16 *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a, b]$. Then*

$$\left| \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} \cdot (b - a) \right|$$

$$\leq \frac{1}{2}(b - a)(f(b) - f(a)) - \int_a^b \text{sgn}\left(t - \frac{a + b}{2} \right) f(t) \, dt$$

$$\leq \frac{1}{2}(b - a)(f(b) - f(a)). \tag{7.4}$$

The constant factor $\frac{1}{2}$ is the best in both inequalities.

COROLLARY 17 (see [38])   *Let f be as in Theorem 9 and $p, q \in \mathbb{R}_+$ with*
$p > q$. Then

$$\left| \int_a^b f(x)\,dx - \frac{q}{p+q}(b-a)\left[f(a) + f(b) + \frac{p-q}{q} f\left(\frac{a+b}{2}\right) \right] \right|$$

$$\leq \frac{q}{p+q}(b-a)(f(b) - f(a)) + \int_a^b T_1(x)f(x)\,dx$$

$$\leq \frac{q}{p+q}(b-a)(f(b) - f(a))$$

$$+ \frac{p-3q}{2(p+q)}(b-a)\left[f\left(\frac{pb+qa}{p+q}\right) - f\left(\frac{pa+qb}{p+q}\right) \right]$$

$$\leq \max\left\{ q, \frac{p-q}{2} \right\} \frac{b-a}{p+q}(f(b) - f(a)),$$

where

$$T_1(x) = \begin{cases} \operatorname{sgn}\left(\dfrac{pa+qb}{p+q} - x\right), & \text{if } x \in \left[a, \dfrac{a+b}{2}\right], \\[3mm] \operatorname{sgn}\left(\dfrac{pb+qa}{p+q} - x\right), & \text{if } x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Proof Set in Theorem 12: $t_1 = (pa+qb)/(p+q), t_2 = (a+b)/2, t_3 = (qa+pb)/(p+q)$.

Remark 11   Of special interest is the case $p = 5$ and $q = 1$ where we get
from Corollary 17 the following result of Simpson type;

$$\left| \int_a^b f(x)\,dx - \frac{1}{3}(b-a)\left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right|$$

$$\leq \frac{b-a}{6}(f(b) - f(a)) + \int_a^b T_2(x)f(x)\,dx$$

$$\leq \frac{b-a}{6}\left[f(b) - f(a) + f\left(\frac{5b+a}{6}\right) - f\left(\frac{5a+b}{6}\right) \right]$$

$$\leq \frac{1}{3}(b-a)(f(b) - f(a)),$$

where

$$T_2(x) = \begin{cases} \mathrm{sgn}\left(\dfrac{5a+b}{3} - x\right), & x \in \left[a, \dfrac{a+b}{2}\right], \\[3mm] \mathrm{sgn}\left(\dfrac{a+5b}{3} - x\right), & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Remark 12   For $p \to q$ we get Corollary 16 from Corollary 17.

7.1. An Inequality for the Cumulative Distribution Function

Let X be a random variable taking values in the finite interval $[a, b]$, with cumulative distributions function $F(x) = \mathrm{Pr}(X \leq x)$.

The following result from [20] can be obtained from Theorem 12 (see [38]).

THEOREM 13   *Let X and F be as above. Then we have the inequalities:*

$$\left| \mathrm{Pr}(X \leq x) - \frac{b - E(x)}{b - a} \right|$$

$$\leq \frac{1}{b - a}\left[[2x - (a + b)]\mathrm{Pr}(X \leq x) + \int_a^b \mathrm{sgn}(t - x)F(t)\, \mathrm{d}t \right]$$

$$\leq \frac{1}{b - a}[(b - x)\mathrm{Pr}(X \geq x) + (x - a)\mathrm{Pr}(X \leq x)]$$

$$\leq \frac{1}{2} + \frac{|x - (a + b)/2|}{b - a} \tag{7.5}$$

for all $x \in [a, b]$.

All the inequalities in (7.5) are sharp and the constant $\frac{1}{2}$ is the best possible.

Now we shall prove the following result [38].

THEOREM 14   *Let X and F be as above. Then we have the inequalities:*

$$|E(x) - x| \leq b - x + \int_a^b \mathrm{sgn}\,(x - t)F(t)\, \mathrm{d}t$$

$$\leq (b - x)\mathrm{Pr}(X \geq x) + (x - a)\mathrm{Pr}(X \leq x)$$

$$\leq \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \tag{7.6}$$

for all $x \in [a, b]$.

All the inequalities in (7.6) are sharp and the constant $\frac{1}{2}$ is the best possible.

Proof Apply Corollary 15 for the monotonic nondecreasing mapping $f(t) := F(t)$, $t \in [a, b]$ to get

$$\left| \int_a^b F(t)\, dt - [(x-a)F(a) + (b-x)F(b)] \right|$$

$$\leq (b-x)F(b) + (x-a)F(a) + \int_a^b \mathrm{sgn}(x-t)F(t)\, dt$$

$$\leq (b-x)(F(b) - F(x)) + (x-a)(F(x) - F(a))$$

$$\leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right](F(b) - F(a)) \qquad (7.7)$$

and as

$$F(a) = 0, \quad F(b) = 1$$

by the integration by parts formula for Riemann–Stieltjes integrals

$$E(x) = \int_a^b t\, dF(t) = tF(t)|_a^b - \int_a^b F(t)\, dt$$

$$= bF(b) - aF(a) - \int_a^b F(t)\, dt$$

$$= b - \int_a^b F(t)\, dt.$$

That is,

$$\int_a^b F(t)\, dt = b - E(x).$$

The inequalities (7.7) give the desired estimation (7.6).

COROLLARY 18 (see [38])   *Let X be a random variable taking values in the finite interval [a, b], with cumulative distribution function $F(x) = \Pr(X \leq x)$ and the expectation $E(x)$. Then we have the inequality*

$$\left| E(x) - \frac{a+b}{2} \right| \leq \frac{1}{2}(b-a) - \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2} \right) F(t)\,\mathrm{d}t \leq \frac{1}{2}(b-a).$$

The constant $\frac{1}{2}$ is the best in both inequalities.

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