Blowup of Nonradial Solutions to Parabolic–Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains

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We consider the initial-boundary value problem of parabolic–elliptic systems on bounded domains in $\mathbb{R}^2$ with smooth boundary which is a mathematical model of chemotaxis. Making a differential inequality on the moment of solutions to the problem, we show the finite-time blowup of nonradial solutions under some condition on the mass and the moment of the initial data.

Keywords: Parabolic–elliptic system; Chemotaxis; Finite-time blowup

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1 INTRODUCTION

This paper is concerned with the finite-time blowup of solutions for two types of parabolic–elliptic system considered in [19,21] which are simplified versions of a parabolic system proposed by Keller and Segel [20]. The parabolic system is called the Keller–Segel model, which is a mathematical model describing aggregation phenomena of cells due to *chemotaxis*, i.e., a phenomenon of the directed movement of cells in response to the gradient of a chemical attractant.

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Throughout this paper, Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The initial-boundary value problem of the parabolic–elliptic system in [21] is described as follows:

$$\text{(P)} \quad \begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \Omega, \quad t > 0, \\ 0 = \Delta v - v + \alpha u & \text{in } \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 & \text{on } \Omega. \end{cases}$$

The initial-boundary value problem in [19] is the following:

$$\text{(JL)} \quad \begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \Omega, \quad t > 0, \\ 0 = \Delta v + \alpha(u - \overline{u_0}) & \text{in } \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 & \text{on } \Omega, \\ \displaystyle\int_\Omega v(x, t)\, \mathrm{d}x = 0 \end{cases}$$

In both these systems, α and χ are positive constants, $\partial/\partial n$ represents the directional derivative along the outward unit normal vector n on $\partial\Omega$. In (JL), $\overline{u_0}$ is the mean value of u_0 defined by

$$\overline{u_0} = \frac{1}{|\Omega|} \int_\Omega u_0 \, \mathrm{d}x.$$

We always assume that

$$u_0 \text{ is smooth on } \bar\Omega, \ u_0 \geq 0, \ \not\equiv 0 \quad \text{on } \Omega.$$

Under this condition there exists $T > 0$ such that (P) admits a unique classical solution (u, v) on $\bar\Omega \times [0, T]$, which satisfies

$$u(x, t) > 0, \ v(x, t) > 0 \quad \text{on } \bar\Omega \times (0, T].$$

If the maximal existence time $T_{\max}$ of (u, v) is finite, then

$$\lim_{t \to T_{\max}} \sup \|u(t)\|_{L^\infty(\Omega)} = +\infty, \tag{1.1}$$

by which we mean that the solution blows up in finite time. For these results, see [21]. The same results are also valid for solutions of (JL) except for the positivity of v. We remark that lim sup in (1.1) for solutions of (P) can be replaced with lim (see [23,27]).

The finite-time blowup of solutions to the Keller−Segel model was conjectured in [4,5,25]. They conjectured that solutions of the Keller−Segel model may blow up in finite time with δ-function singularities. Finite-time blowup with δ-function singularities is referred to as *chemotactic collapse*. In two dimensions, to the best of our knowledge, the first result on finite-time blowup was shown in [19] for radial solutions of (JL) on a disk. In [21] he considered (P) on a disk Ω in $\mathbb{R}^2$, and showed that under the condition $\int_\Omega u_0 \, dx > 8\pi/(\alpha\chi)$ the radial solution of (P) blows up in finite time if $\int_\Omega u_0(x)|x|^2 \, dx$ is sufficiently small, but under the condition $\int_\Omega u_0 \, dx < 8\pi/(\alpha\chi)$ the radial solution exists globally in time. The possibility of blowup in three or more dimensions was also studied. The same results are also valid for (JL). Concerning chemotactic collapse, in [11−13] they showed that chemotactic collapse actually occurs in two dimensions. For further studies, see [14,15]. It is obtained that in [23,27] finite-time blowup in two dimensions necessarily leads to chemotactic collapse at each isolated blowup point, and that in [27] the number of blowup points of solutions to (P) is finite.

We refer to [29] for the local existence of solutions of more general parabolic systems including the Keller−Segel model, to [2,9,24] for the global existence of the Keller−Segel model, and to [16,17] for blowup. For related results to the Keller−Segel model we also refer to [6,7,22,26].

We remark that parabolic−elliptic systems similar to (P) appear as models for gravitational interaction of particles (for instance, see [1,3,8,28] and references therein). In [1,3] they studied the nonexistence of solutions globally in time for star-shaped domains in $\mathbb{R}^n (n \geq 2)$.

The possibility of finite-time blowup for (P) and (JL) as well as the Keller−Segel model has been shown only for radial solutions so far. In this paper we study the finite-time blowup of nonradial solutions (u, v) to either of (P) and (JL) in two-dimensional domains without assuming that Ω is star-shaped. In Section 3, the finite-time blowup of nonradial solutions is shown under the condition $\int_\Omega u_0 \, dx > 8\pi/(\alpha\chi)$ when q is an interior point of Ω, provided that $\int_\Omega u_0(x)|x - q| \, dx$ is sufficiently small. When q is on $\partial\Omega$, the occurrence of the finite-time blowup of solutions requires the condition $\int_\Omega u_0 \, dx > 4\pi/(\alpha\chi)$, since the solution (u, v) exists

globally in time under the condition $\int_\Omega u_0 \, dx < 4\pi/(\alpha\chi)$ (see [2,9,24]). Under a restricted condition on $\partial\Omega$, we show the finite-time blowup of nonradial solutions, provided that $\int_\Omega u_0 \, dx > 4\pi/(\alpha\chi)$ and $\int_\Omega u_0(x)|x - q|^2 \, dx$ is sufficiently small.

2   GREEN FUNCTIONS AND RELATED INEQUALITIES

Let $B = \{x \in \mathbb{R}^2 : |x| < L\}$ $(0 < L < \infty)$, and given $f \in L^p(B)$ $(1 \leq p \leq +\infty)$ consider the Dirichlet boundary value problems of $-\Delta$ on B:

$$-\Delta w = f \ \text{ in } B,$$
$$w = 0 \ \text{ on } \partial B.$$

The solution w is expressed as

$$w(x) = \int_\Omega G(x, y) f(y) \, dy \quad \text{for } x \in B,$$

where $G(x, y)$ is the Green function of $-\Delta$ on B with homogeneous Dirichlet boundary conditions. We remark that the Green function $G(x, y)$ has the following representation (for instance, see [10,18]):

$$G(x, y) = N(x - y) + K(x, y),$$

where

$$N(x - y) = -\frac{1}{2\pi} \log |x - y|$$

and $K(x, y)$ is the compensating function. It is known that

 (i) $G(x, y) = G(y, x)$ for $x, y \in \bar{B}$,
 (ii) $K \in C^2(B \times \bar{B})$,
 (iii) $|\nabla_x G(x, y)| \leq C/|x - y|$ on $B \times B$ for some constant $C > 0$.

Given r_1 and r_2 with $0 < r_1 < r_2$, define the function ϕ on $[0, \infty)$ belonging to $C^1([0, \infty)) \cap W^{2,\infty}((0, \infty))$ by

$$\phi(r) = \begin{cases} r^2 & \text{if } 0 \leq r \leq r_1, \\ a_1 r^2 + a_2 r + a_3 & \text{if } r_1 \leq r \leq r_2, \\ r_1 r_2 & \text{if } r > r_2, \end{cases} \tag{2.1}$$

where

$$a_1 = -\frac{r_1}{r_2 - r_1}, \quad a_2 = \frac{2r_1r_2}{r_2 - r_1}, \quad a_3 = -\frac{r_1^2 r_2}{r_2 - r_1}.$$

Define $\Phi \in C^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$ by

$$\Phi(x) = \phi(|x|),$$

which satisfies the following:

$$\nabla\Phi(x) = \begin{cases} 2x & \text{if } |x| \leq r_1, \\ \dfrac{2r_1}{r_2 - r_1}(r_2 - |x|)\dfrac{x}{|x|} & \text{if } r_1 < |x| \leq r_2, \\ 0 & \text{if } |x - q| > r_2, \end{cases} \qquad (2.2)$$

$$|\nabla\Phi(x)| \leq 2(\Phi(x))^{1/2}, \qquad (2.3)$$

$$\Delta\Phi(x) = 4 \text{ for } |x| \leq r_1, \quad \Delta\Phi(x) \leq 2 \text{ for } |x| > r_1. \qquad (2.4)$$

Put $B_j = \{x \in \mathbb{R}^2 : |x| < r_j\}$.

LEMMA 2.1 *It holds that for $(x, y) \in B_1 \times B_1$,*

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot \nabla N(x - y) = -\frac{1}{\pi} \qquad (2.5)$$

and for $(x, y) \notin B_1 \times B_1$,

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot \nabla N(x - y) \leq \frac{r_1}{\pi(r_2 - r_1)}. \qquad (2.6)$$

Proof It is easy to get (2.5), since $\nabla\Phi(x) = 2x$ on B_1 and

$$\nabla N(x - y) = -\frac{1}{2\pi}\frac{x - y}{|x - y|^2}.$$

To obtain (2.6), it suffices to show that for $(x, y) \notin B_1 \times B_1$,

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y - x) \leq \frac{2r_1}{r_2 - r_1}|x - y|^2. \qquad (2.7)$$

For $(x, y) \in B_1 \times (B_2 \backslash B_1)$, using (2.2) and

$$2x \cdot (y - x) = (|y|^2 - |x|^2) - |x - y|^2,$$
$$2y \cdot (y - x) = (|y|^2 - |x|^2) + |x - y|^2 \tag{2.8}$$

and noting $|y| - r_1 \le |y| - |x| \le |y - x|$, we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y - x)$$
$$= \left\{ 2x - \frac{2r_1}{r_2 - r_1}(r_2 - |y|)\frac{y}{|y|} \right\} \cdot (y - x)$$
$$= \frac{r_2(|y| - r_1)}{(r_2 - r_1)|y|}(|y|^2 - |x|^2) - \left\{ \frac{r_2(|y| + r_1)}{(r_2 - r_1)|y|} - \frac{2r_1}{r_2 - r_1} \right\}|y - x|^2$$
$$\le \frac{r_2(|y| + |x|)}{(r_2 - r_1)|y|}(|y| - |x|)^2 - \left\{ \frac{r_2(|y| + |x|)}{(r_2 - r_1)|y|} - \frac{2r_1}{r_2 - r_1} \right\}|y - x|^2$$
$$\le \frac{2r_1}{r_2 - r_1}|y - x|^2.$$

For $(x, y) \in B_1 \times (\mathbb{R}^2 \backslash B_2)$, noting $|x - y| \ge r_2 - r_1$, we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y - x) = 2x \cdot (y - x) \le 2|x||x - y|$$
$$\le \frac{2|x|}{|x - y|}|x - y|^2 \le \frac{2r_1}{r_2 - r_1}|x - y|^2.$$

For $(x, y) \in (B_2 \backslash B_1) \times (B_2 \backslash B_1)$, by (2.2) and (2.8) we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y - x)$$
$$= \frac{r_1}{r_2 - r_1}\left\{ \frac{r_2 - |x|}{|x|}2x \cdot (y - x) - \frac{(r_2 - |y|)}{|y|}2y \cdot (y - x) \right\}$$
$$= \frac{r_1}{r_2 - r_1}\left\{ \frac{r_2(|y| - |x|)}{|x||y|}(|y|^2 - |x|^2) - \left(\frac{r_2(|x| + |y|)}{|x||y|} - 2 \right)|x - y|^2 \right\}$$
$$\le \frac{r_1}{r_2 - r_1}\left\{ \frac{r_2(|x| + |y|)}{|x||y|}|y - x|^2 - \left(\frac{r_2(|x| + |y|)}{|x||y|} - 2 \right)|x - y|^2 \right\}$$
$$= \frac{2r_1}{r_2 - r_1}|x - y|^2.$$

For $(x, y) \in (B_2 \backslash B_1) \times (\mathbb{R}^2 \backslash B_2)$, since $|y| \geq r_2$, we have

$$\{\nabla \Phi(x) - \nabla \Phi(y)\} \cdot (y - x) = \frac{2r_1}{r_2 - r_1} (r_2 - |x|) \frac{x}{|x|} \cdot (y - x)$$

$$\leq \frac{2r_1}{r_2 - r_1} (|y| - |x|)|y - x|$$

$$\leq \frac{2r_1}{r_2 - r_1} |x - y|^2.$$

For $(x, y) \in (\mathbb{R}^2 \backslash B_2) \times (\mathbb{R}^2 \backslash B_2)$, since $\nabla \Phi(x) = \nabla \Phi(y) = 0$, we have (2.7). Other cases are reduced to the cases above, since

$$\{\nabla \Phi(x) - \nabla \Phi(y)\} \cdot (y - x) = \{\nabla \Phi(y) - \nabla \Phi(x)\} \cdot (x - y).$$

Hence, we have completed the proof of Lemma 2.1.

Given $f \in C(\bar{\Omega})$ let $w \in C^2(\bar{\Omega})$ satisfy

$$-\Delta w + \gamma w = f \quad \text{in } \Omega,$$

$$\frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial \Omega,$$

where $\gamma (\geq 0)$ is a constant. If $\gamma = 0$, we assume that

$$\int_\Omega f \, dx = \int_\Omega w \, dx = 0.$$

Then the function w is expressed as

$$w(x) = \int_\Omega G(x, y) f(y) \, dy \quad \text{for } x \in \Omega,$$

where $G(x, y)$ is the Green function of $-\Delta + \gamma$ on Ω with homogeneous Neumann boundary conditions (see [10,18]). $G(x, y)$ satisfies

$$|G(x, y)| \leq C \left(1 + \log^+ \frac{1}{|x - y|} \right), \quad |\nabla_x G(x, y)| \leq \frac{C}{|x - y|}, \quad (2.9)$$

where C is a positive constant and $\log^+ a = \max\{\log a, 0\}$ for $a > 0$. It follows from (2.9) and Young's inequality for convolutions that

$$\|w\|_{L^p(\Omega)} \leq C_p \|f\|_{L^1(\Omega)} \quad (1 \leq p < \infty), \quad (2.10)$$

$$\|\nabla w\|_{L^q(\Omega)} \leq C_q \|f\|_{L^1(\Omega)} \quad (1 \leq q < 2), \quad (2.11)$$

where C_p (resp. C_q) is a positive constant depending on p (resp. q).

3 FINITE-TIME BLOWUP OF SOLUTIONS

Our first result is the following theorem.

THEOREM 3.1 *Let $q \in \Omega$. Assume that $\int_\Omega u_0 \, dx > 8\pi/(\alpha\chi)$. If $\int_\Omega u_0(x)|x - q|^2 \, dx$ is sufficiently small, then the maximal existence time of the solution (u, v) to either of* (P) *and* (JL) *corresponding to the initial function u_0 is finite, that is, the solution blows up in finite time.*

We next mention finite-time blowup when q is on the boundary $\partial\Omega$.

THEOREM 3.2 *Assume that $\partial\Omega$ has a line segment l_0, and that Ω lies on one side of a line l containing l_0. Let $q \in l_0$ be such that q is not end-points of l_0. Then, under the condition $\int_\Omega u_0 \, dx > 4\pi/(\alpha\chi)$, the solution (u, v) to either of* (P) *and* (JL) *corresponding to the initial function u_0 blows up in finite, provided that $\int_\Omega u_0(x)|x - q|^2 \, dx$ is sufficiently small.*

3.1 Proofs of the Theorems for (P)

Let (u, v) be the solution of (P) with $u(\cdot, 0) = u_0$. To prove the theorems, we begin with the following key lemma, which is shown by a method similar to that in [1–3,21].

LEMMA 3.1 *Let $q \in \Omega$ and $0 < r_1 < r_2 < \mathrm{dist}(q, \partial\Omega)$, where $\mathrm{dist}(q, \partial\Omega)$ is the distance between q and $\partial\Omega$. Then there are positive constants C_1, C_2 depending only on r_1, r_2 and $\mathrm{dist}(q, \partial\Omega)$ such that for $t \in (0, T_{\max})$,*

$$\frac{d}{dt} \int_\Omega u(x, t)\Phi(x) \, dx$$

$$\leq 4 \int_\Omega u_0 \, dx - \frac{\alpha\chi}{2\pi} \left(\int_\Omega u_0 \, dx \right)^2 + C_1 \left(\int_\Omega u_0 \, dx \right) \left(\int_\Omega u(x, t)\Phi(x) \, dx \right)$$

$$+ C_2 \left(\int_\Omega u_0 \, dx \right)^{3/2} \left(\int_\Omega u(x, t)\Phi(x) \, dx \right)^{1/2}, \tag{3.1}$$

where $\Phi(x) = \phi(|x - q|)$, ϕ is the one defined by (2.1) *in Section 2.*

Proof For simplicity, we may assume that the point q is the origin by the translation $x \mapsto x - q$, since two equations of (P) are invariant under translations. Multiply $u_t = \nabla \cdot (\nabla u - \chi u \nabla v)$ by Φ and integrate over Ω. Since $\partial\Phi/\partial n = 0$ on $\partial\Omega$ because of $r_2 < \mathrm{dist}(q, \partial\Omega)$ and the definition of

ϕ, integration by parts yields that

$$\frac{d}{dt} \int_\Omega u(x,t)\Phi(x)\,dx$$

$$= \int_\Omega u(x,t)\Delta\Phi(x)\,dx + \chi \int_\Omega u(x,t)\nabla\Phi(x)\cdot\nabla v(x,t)\,dx.$$

By $\Delta\Phi \le 4$ on Ω (see (2.4)) and $\int_\Omega u(x,t)\,dx = \int_\Omega u_0\,dx$,

$$\int_\Omega u(x,t)\Delta\Phi(x)\,dx \le 4 \int_\Omega u_0\,dx.$$

Then,

$$\frac{d}{dt} \int_\Omega u(x,t)\Phi(x)\,dx \le 4 \int_\Omega u_0\,dx + \chi \int_\Omega u(x,t)\nabla\Phi(x)\cdot\nabla v(x,t)\,dx.$$

$$(3.2)$$

Let us take r_3, r_4 such that

$$r_2 < r_3 < r_4 < \operatorname{dist}(q, \partial\Omega),$$

and the function $\eta \in C_0^\infty(\mathbb{R}^2)$ such that

$$0 \le \eta \le 1, \quad \eta(x) = \begin{cases} 1 & \text{if } |x| < r_3, \\ 0 & \text{if } |x| \ge r_4. \end{cases}$$

Put

$$B = \{x \in \mathbb{R}^2 : |x| < r_4\} \subset \Omega,$$

$$B_j = \{x \in \mathbb{R}^2 : |x| < r_j\} \quad \text{for } 1, 2, 3.$$

By the second equation of (P), the function $w(x,t) = \eta(x)v(x,t)$ satisfies

$$-\Delta w = \alpha\eta u + g - \eta v \quad \text{in } B,$$
$$w = 0 \qquad\qquad\qquad \text{on } \partial B$$

for each $t \in (0, T_{\max})$, where

$$g = -2\nabla\eta \cdot \nabla v - (\Delta\eta)v.$$

Then w is expressed as

$$w(x, t) = \int_B G(x, y)\{\alpha\eta(y)u(y, t) + g(y, t) - \eta(y)v(y, t)\}\,dy, \quad (3.3)$$

where $G(x, y)$ is the Green function of $-\Delta$ on B with homogeneous Dirichlet boundary conditions. As mentioned in Section 2, $G(x, y)$ has the following representation:

$$G(x, y) = N(x - y) + K(x, y), \quad (3.4)$$

where

$$N(x - y) = -\frac{1}{2\pi}\log|x - y|$$

and $K \in C^2(B \times \bar{B})$. Since $v = w$ on B_3 and $\nabla\Phi = 0$ outside of B_2, by (3.3) the second integral on the right-hand side of (3.2) is expressed as

$$\int_\Omega u(x, t)\nabla\Phi(x) \cdot \nabla v(x, t)\,dx$$

$$= \alpha \int_{B_2} \int_B u(x, t)u(y, t)\eta(y)\nabla\Phi(x) \cdot \nabla_x G(x, y)\,dy\,dx$$

$$+ \int_{B_2} \int_B u(x, t)g(y, t)\nabla\Phi(x) \cdot \nabla_x G(x, y)\,dy\,dx$$

$$- \int_{B_2} \int_B u(x, t)\eta(y)v(y, t)\nabla\Phi(x) \cdot \nabla_x G(x, y)\,dy\,dx$$

$$= I + II + III. \quad (3.5)$$

By (3.4) and $\eta = 1$ on B_3, I is divided into three parts as follows:

$$I = \alpha \int_{B_2} \int_{B_3} u(x, t)u(y, t)\nabla\Phi(x) \cdot \nabla N(x - y)\,dy\,dx$$

$$+ \alpha \int_{B_2} \int_{B \backslash B_3} u(x, t)u(y, t)\eta(y)\nabla\Phi(x) \cdot \nabla N(x - y)\,dy\,dx$$

$$+ \alpha \int_{B_2} \int_B u(x, t)\eta(y)\nabla\Phi(x) \cdot \nabla_x K(x, y)\,dy\,dx$$

$$= I_1 + I_2 + I_3.$$

Using $\nabla\Phi = 0$ outside of B_2, we rewrite I_1 as

$$I_1 = \alpha \int_{B_3} \int_{B_3} u(x,t)u(y,t)\nabla\Phi(x) \cdot \nabla N(x-y)\,dy\,dx,$$

and by symmetry properties of the integral

$$\int_{B_3} \int_{B_3} u(x,t)u(y,t)\nabla\Phi(x) \cdot \nabla N(x-y)\,dy\,dx$$

$$= -\int_{B_3} \int_{B_3} u(x,t)u(y,t)\nabla\Phi(y) \cdot \nabla N(x-y)\,dy\,dx,$$

we have

$$I_1 = \frac{\alpha}{2} \int_{B_3} \int_{B_3} u(x,t)u(y,t)\{\nabla\Phi(x) - \nabla\Phi(y)\}\nabla N(x-y)\,dy\,dx.$$

Such a representation of I_1 is used in [3, Theorem 2(v)]. Applying Lemma 2.1 to the relation above yields that

$$I_1 \leq -\frac{\alpha}{2\pi} \int_{B_1} \int_{B_1} u(x,t)u(y,t)\,dy\,dx$$

$$+ \frac{\alpha r_1}{2\pi(r_2 - r_1)} \iint_{(B\times B)\setminus(B_1\times B_1)} u(x,t)u(y,t)\,dy\,dx. \qquad (3.6)$$

The first term on the right-hand side of (3.6) is estimated as

$$-\frac{\alpha}{2\pi} \int_{B_1} \int_{B_1} u(x,t)u(y,t)\,dy\,dx = -\frac{\alpha}{2\pi}\left(\int_{B_1} u(x,t)\,dx\right)^2$$

$$= -\frac{\alpha}{2\pi}\left(\int_{\Omega} u(x,t)\,dx - \int_{\Omega\setminus B_1} u(x,t)\,dx\right)^2$$

$$\leq -\frac{\alpha}{2\pi}\left(\int_{\Omega} u(x,t)\,dx\right)^2 + \frac{\alpha}{\pi}\left(\int_{\Omega} u(x,t)\,dx\right)\left(\int_{\Omega\setminus B_1} u(x,t)\,dx\right)$$

$$\leq -\frac{\alpha}{2\pi}\left(\int_{\Omega} u_0\,dx\right)^2 + \frac{\alpha}{\pi r_1^2}\left(\int_{\Omega} u_0\,dx\right)\left(\int_{\Omega} u(x,t)\Phi(x)\,dx\right).$$

Here, we used

$$r_1^2 \leq \Phi(x) \quad \text{for } x \notin B_1.$$

The second term on the right-hand side of (3.6) is estimated as

$$\frac{\alpha r_1}{2\pi(r_2 - r_1)} \int\!\!\int_{(B \times B)\backslash(B_1 \times B_1)} u(x,t)u(y,t)\,dy\,dx$$

$$\leq \frac{\alpha}{\pi(r_2 - r_1)r_1} \left(\int_\Omega u_0\,dx \right) \left(\int_\Omega u(x,t)\Phi(x)\,dx \right).$$

Hence, the term I_1 is estimated as

$$I_1 \leq -\frac{\alpha}{2\pi} \left(\int_\Omega u_0\,dx \right)^2 + \frac{\alpha r_2}{\pi(r_2 - r_1)r_1^2} \left(\int_\Omega u_0\,dx \right) \left(\int_\Omega u(x,t)\Phi(x)\,dx \right).$$

Since $|x - y| \geq r_3 - r_2$ for $(x, y) \in B_2 \times (B \backslash B_3)$, I_2 is estimated as

$$I_2 \leq \frac{\alpha}{2\pi} \int_{B_2} \int_{B \backslash B_3} u(x,t)u(y,t)|\nabla\Phi(x)|\frac{1}{|x-y|}\,dy\,dx$$

$$\leq \frac{\alpha}{\pi(r_3 - r_2)} \int_{B_2} \int_{B \backslash B_3} u(x,t)u(y,t)(\Phi(x))^{1/2}\,dy\,dx$$

$$\leq \frac{\alpha}{\pi(r_3 - r_2)} \left(\int_\Omega u(y,t)\,dy \right) \left(\int_\Omega u(x,t)(\Phi(x))^{1/2}\,dx \right)$$

$$\leq \frac{\alpha}{\pi(r_3 - r_2)} \left(\int_\Omega u_0\,dx \right)^{3/2} \left(\int_\Omega u(x,t)\Phi(x)\,dx \right)^{1/2}.$$

Here, we used (2.3) and

$$\int_\Omega u(x,t)(\Phi(x))^{1/2}\,dx \leq \left(\int_\Omega u_0\,dx \right)^{1/2} \left(\int_\Omega u(x,t)\Phi(x)\,dx \right)^{1/2}.$$

By noting $\nabla_x K \in L^\infty(B_2 \times B)$, I_3 is estimated as

$$I_3 \leq 2\alpha\|\nabla_x K\|_{L^\infty(B_2 \times B)} \int_{B_2} \int_B u(x,t)u(y,t)(\Phi(x))^{1/2}\,dy\,dx$$

$$\leq 2\alpha\|\nabla_x K\|_{L^\infty(B_2 \times B)} \left(\int_\Omega u_0\,dx \right)^{3/2} \left(\int_\Omega u(x,t)\Phi(x)\,dx \right)^{1/2}.$$

Hence,

$$
\begin{aligned}
I \le{}& -\frac{\alpha}{2\pi}\left(\int_\Omega u_0\,\mathrm{d}x\right)^2 \\
&+\frac{\alpha r_2}{\pi(r_2-r_1)r_1^2}\left(\int_\Omega u_0\,\mathrm{d}x\right)\left(\int_\Omega u(x,t)\Phi(x)\,\mathrm{d}x\right) \\
&+\alpha\left\{\frac{1}{\pi(r_3-r_2)}+2\|\nabla_x K\|_{L^\infty(B_2\times B)}\right\}\left(\int_\Omega u_0\,\mathrm{d}x\right)^{3/2} \\
&\times\left(\int_\Omega u(x,t)\Phi(x)\,\mathrm{d}x\right)^{1/2}.
\end{aligned}
\tag{3.7}
$$

To estimate II, we note that

$$
II=\int_{B_2}\int_{B\setminus B_3} u(x,t)g(y,t)\nabla\Phi(x)\cdot\nabla_x G(x,y)\,\mathrm{d}y\,\mathrm{d}x,
$$

since $g=-2\nabla\eta\cdot\nabla v-(\Delta\eta)v=0$ on B_3. For $(x,y)\in B_2\times(B\setminus B_3)$, observe that $|x-y|\ge r_3-r_2$ and

$$
|\nabla_x G(x,y)|\le\frac{C}{|x-y|}\le\frac{C}{r_3-r_2}
$$

with a positive constant C. Then,

$$
\begin{aligned}
II\le{}&\frac{C}{r_3-r_2}\int_{B_2}\int_{B\setminus B_3} u(x,t)|g(y,t)|\,|\nabla\Phi(x)|\,\mathrm{d}y\,\mathrm{d}x \\
\le{}&\frac{C}{r_3-r_2}\left(\int_\Omega u(x,t)(\Phi(x))^{1/2}\,\mathrm{d}x\right)\left(\int_\Omega|g(y,t)|\,\mathrm{d}y\right) \\
\le{}&\frac{C}{r_3-r_2}\left(\int_\Omega u_0\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega u(x,t)\Phi(x)\,\mathrm{d}x\right)^{1/2} \\
&\times\left(\int_\Omega|g(y,t)|\,\mathrm{d}y\right).
\end{aligned}
$$

To calculate further, we apply (2.10) and (2.11) to the second equation of (P) to get

$$
\|v(t)\|_{W^{1,1}(\Omega)}\le C\|u(t)\|_{L^1(\Omega)}=C\|u_0\|_{L^1(\Omega)}.
$$

By this inequality,

$$\int_\Omega |g(y,t)|\,dy \le 2(\|\Delta\eta\|_{L^\infty} + \|\nabla\eta\|_{L^\infty})\|v(t)\|_{W^{1,1}(\Omega)}$$

$$\le C\int_\Omega u_0\,dy,$$

where C is a constant depending on $\|\nabla\eta\|_{L^\infty}$, $\|\Delta\eta\|_{L^\infty}$. Hence,

$$II \le \frac{C}{r_3 - r_2}\left(\int_\Omega u_0\,dx\right)^{3/2}\left(\int_\Omega u(x,t)\Phi(x)\,dx\right)^{1/2}. \qquad (3.8)$$

To estimate III, we rewrite III as

$$III = \int_{B_2} u(x,t)\nabla\Phi(x)\cdot\nabla\psi(x,t)\,dx,$$

where

$$\psi(x,t) = -\int_B \eta(y)v(y,t)G(x,y)\,dy.$$

Noting $|\nabla_x G(x,y)| \le C/|x-y|$ and using Hölder's inequality, we observe that

$$|\nabla\psi(x,t)| \le C\int_B \frac{v(y,t)}{|x-y|}\,dy$$

$$\le C\left(\int_B v^3 dy\right)^{1/3}\left(\int_B |x-y|^{-3/2}\,dy\right)^{2/3}$$

$$\le C\|v(t)\|_{L^3(\Omega)}.$$

Here, we used

$$\sup_{x\in B_2}\left(\int_B |x-y|^{-3/2}\,dy\right)^{2/3} < +\infty.$$

Applying (2.10) to the second equation of (P), by $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ we have

$$\|v(t)\|_{L^3(\Omega)} \le \|u_0\|_{L^1(\Omega)}.$$

Hence,

$$|\nabla\psi(x,t) \le C \int_\Omega u_0 \, dx \quad \text{for } x \in B_2.$$

Therefore,

$$III \le C\left(\int_\Omega u_0 \, dx\right)\left(\int_{B_2} u(x,t)|\nabla\Phi(x)| \, dx\right)$$

$$\le \left(\int_\Omega u_0 \, dx\right)^{3/2}\left(\int_\Omega u(x,t)\Phi(x) \, dx\right)^{1/2}. \tag{3.9}$$

Putting together (3.5), (3.7)–(3.9) yields that

$$\int_\Omega u(x,t)\nabla\Phi(x) \cdot \nabla v(x,t) \, dx$$

$$\le -\frac{\alpha}{2\pi}\left(\int_\Omega u_0 \, dx\right)^2 + C_1\left(\int_\Omega u_0 \, dx\right)\left(\int_\Omega u(x,t)\Phi(x) \, dx\right)$$

$$+ C_2\left(\int_\Omega u_0 \, dx\right)^{3/2}\left(\int_\Omega u(x,t)\Phi(x) \, dx\right)^{1/2}.$$

Hence, substituting this inequality into (3.2) we arrive at (3.1), thereby completing the proof of Lemma 3.1.

We are now in a position to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1   By putting

$$M_\Phi(t) = \int_\Omega u(x,t)\Phi(x) \, dx,$$

the differential inequality (3.1) in Lemma 3.1 is rewritten as

$$\frac{d}{dt}M_\Phi(t) \le H(M_\Phi(t)),$$

where $H(s)$ is the function on $[0,\infty)$ defined by

$$H(s) = \frac{\alpha\chi}{2\pi}\left(\int_\Omega u_0 \, dx\right)\left(\frac{8\pi}{\alpha\chi} - \int_\Omega u_0 \, dx\right)$$

$$+ C_1\left(\int_\Omega u_0 \, dx\right)s + C_2\left(\int_\Omega u_0 \, dx\right)^{3/2}s^{1/2}.$$

We note that $H(0) < 0$ under the condition $\int_\Omega u_0(x)\,\mathrm{d}x > 8\pi/(\alpha\chi)$ and the function $s \mapsto H(s)$ is nondecreasing. There exists a unique positive root a_+ of $H(s) = 0$ such that $H(s) < 0$ for $s < a_+$ and $H(s) > 0$ for $s > a_+$. Hence, if the solution (u, v) exists for all $t > 0$, then the function $t \mapsto \int_\Omega u(x, t) \times \Phi(x)\,\mathrm{d}x$ must vanish in finite time provided $\int_\Omega u(x, t)\Phi(x)\,\mathrm{d}x < a_+$. This is a contradiction to the positivity of $\int_\Omega u(x, t)\Phi(x)\,\mathrm{d}x$. Thus, the proof of Theorem 3.1 is complete.

We next give the proof of Theorem 3.2.

Proof of Theorem 3.2   We may assume that q is the origin and

$$l_0 = \{(x_1, x_2): \; a < x_1 < b, \; x_2 = 0\} \subset l = \{(x_1, x_2): \; x_2 = 0\}$$

by a parallel translation and a rotation of coordinates, since two equations of (P) are invariant under those transformations. Given r_1, r_2 satisfying

$$0 < r_1 < r_2 < \min\{|a|, b, \mathrm{dist}(0, \partial\Omega \backslash l_0)\},$$

let us take Φ as in Section 2. Noting $\nabla\Phi(x) = 0$ for $|x| > r_2$ and observing that

$$\frac{\partial\Phi}{\partial \boldsymbol{n}}(x) = \phi'(|x|)\frac{x}{|x|} \cdot (0, -1) = 0 \quad \text{for } x = (x_1, 0) \in l_0,$$

we see that

$$\frac{\partial\Phi}{\partial \boldsymbol{n}}(x) = 0 \;\; \text{on } \partial\Omega.$$

Then, as in Section 2 it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x, t)\Phi(x)\,\mathrm{d}x$$

$$\leq 4\int_\Omega u_0\,\mathrm{d}x + \chi\int_\Omega\int_\Omega u(x, t)\nabla\Phi(x) \cdot \nabla v(x, t)\,\mathrm{d}y\,\mathrm{d}x. \qquad (3.10)$$

Let Ω_* be the reflection of Ω with respect to the x_1-axes and put

$$\Omega^* = \Omega \cup \Omega_* \cup l_0.$$

For each $t \in [0, T_{\max})$, define the function $(u^*(\cdot, t), v^*(\cdot, t))$ on $\overline{\Omega^*}$ by

$$(u^*(x, t), v^*(x, t)) = \begin{cases} (u(x, t), v(x, t)) & \text{if } x \in \bar{\Omega}, \\ (u(x^*, t), v(x^*, t)) & \text{if } x \in \bar{\Omega}_*, \end{cases}$$

where $x^* = (x_1, -x_2)$ for $x = (x_1, x_2)$. We then see that

$$-\Delta v^* + v^* = \alpha u^* \quad \text{in } \Omega^*.$$

To estimate the second term of (3.10), we observe that

$$\int_\Omega u(x, t) \nabla \Phi(x) \cdot \nabla v(x, t) \, dx$$
$$= \frac{1}{2} \int_{\Omega^*} u^*(x, t) \nabla \Phi(x) \cdot \nabla v^*(x, t) \, dx.$$

Since the origin is an interior point of Ω^*, by the same method as in the proof of Lemma 3.1, we obtain

$$\int_{\Omega^*} u^*(x, t) \nabla \Phi(x) \cdot \nabla v^*(x, t) \, dx$$
$$\leq -\frac{\alpha}{2\pi} \left(\int_{\Omega^*} u_0^* \, dx \right)^2 + C_1 \left(\int_{\Omega^*} u_0^* \, dx \right) \left(\int_{\Omega^*} u^*(x, t) \Phi(x) \, dx \right)$$
$$+ C_2 \left(\int_{\Omega^*} u_0^* \, dx \right)^{3/2} \left(\int_{\Omega^*} u^*(x, t) \Phi(x) \, dx \right)^{1/2}$$
$$= -\frac{2\alpha}{\pi} \left(\int_\Omega u_0 \, dx \right)^2 + 4C_1 \left(\int_\Omega u_0 \, dx \right) \left(\int_\Omega u(x, t) \Phi(x) \, dx \right)$$
$$+ 4C_2 \left(\int_\Omega u_0 \, dx \right)^{3/2} \left(\int_\Omega u(x, t) \Phi(x) \, dx \right)^{1/2}. \tag{3.11}$$

Putting together (3.10) and (3.11) yields that

$$\frac{d}{dt} \int_\Omega u(x, t) \Phi(x) \, dx$$
$$\leq 4 \int_\Omega u_0 \, dx - \frac{\alpha \chi}{\pi} \left(\int_\Omega u_0 \, dx \right)^2$$
$$+ 4C_1 \left(\int_\Omega u_0 \, dx \right) \left(\int_\Omega u(x, t) \Phi(x) \, dx \right)$$
$$+ 4C_2 \left(\int_\Omega u_0 \, dx \right)^{3/2} \left(\int_\Omega u(x, t) \Phi(x) \, dx \right)^{1/2}.$$

This differential inequality gives the conclusion of Theorem 3.2.

3.2   Remark on the Proofs of the Theorems for (JL)

As in the preceding subsection, it suffices to show that (3.1) in Lemma 3.1 is valid for the solution (u, v) of (JL) with $u(\,\cdot\,, 0) = u_0$. To estimate the second term on the right-hand side of (3.2) for the solution of (JL), we remark that the function $w(x, t) = \eta(x)v(x, t)$ in the proof of Lemma 3.1 satisfies

$$-\Delta w = \alpha\eta u + g - \alpha\overline{u_0}\eta \quad \text{in } B,$$
$$w = 0 \qquad\qquad \text{on } \partial B,$$

where $g = -2\nabla\eta \cdot \nabla v - (\Delta\eta)v$. Let $G(x, y)$ be the same Green function of $-\Delta$ as in the previous subsection. Then, as before we have

$$\int_\Omega u(x, t)\nabla\Phi(x) \cdot \nabla v(x, t)\,\mathrm{d}x = I + II + III,$$

where I and II are the same ones as in (3.5), and the term III is the following:

$$III = \int_{B_2} u(x, t)\nabla\Phi(x) \cdot \nabla\psi(x)\,\mathrm{d}x,$$
$$\psi(x) = -\alpha\overline{u_0}\int_\Omega G(x, y)\eta(y)\,\mathrm{d}y.$$

The term I is estimated as (3.7). Similarly, (3.8) remains valid, since $\|v(t)\|_{W^{1,1}(\Omega)} \leq C\|u_0\|_{L^1(\Omega)}$ by (2.10) and (2.11). Noting $\overline{u_0} = 1/|\Omega| \int_\Omega u_0\,\mathrm{d}x$, we observe that

$$\|\nabla\psi\|_{L^\infty(\Omega)} \leq C\int_\Omega u_0\,\mathrm{d}x$$

with a positive constant C, and we see that III is estimated as (3.9). Therefore, (3.1) in Lemma 3.1 holds.

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