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Entropy of $C(X)$-Valued Operators and Diverse Applications

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We investigate how the metric entropy of $C(X)$-valued operators influences the entropy behaviour of special operators, such as integral or matrix operators. Various applications are given, to the eigenvalue distributions of operators and to the metric entropy of convex hulls of precompact sets in Banach spaces, for example. In particular, we provide metric entropy conditions on operators sufficient to ensure that the operators are in certain Schatten classes.

Keywords: Entropy; Integral operators; Schatten classes

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1 INTRODUCTION

Let (X, d) be a metric space and $B(s_0, \varepsilon) := \{s \in X\colon d(s_0, s) \leq \varepsilon\}$ the closed ε-ball in X with centre s_0. Given any bounded set $M \subset X$ and any $\varepsilon > 0$, let $N(M, \varepsilon)$ be the *covering number* of M by ε-balls of X; that is,

$$N(M,\varepsilon) = \inf\left\{N\in\mathbb{N}\colon \text{there are } s_1,\dots,s_N \in X \text{ such that } M \subset \bigcup_{k=1}^{N} B(s_k,\varepsilon)\right\}.$$

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The *entropy numbers* of M are

$$\varepsilon_n(M) := \inf\{\varepsilon > 0\colon N(M,\varepsilon) \leq n\} \quad (n \in \mathbb{N}),$$

and the *dyadic entropy numbers* are

$$e_n(M) := \varepsilon_{2^{n-1}}(M) \quad (n \in \mathbb{N}).$$

For a (bounded linear) operator $T\colon E \to F$ between two Banach spaces E and F, the nth dyadic entropy number $e_n(T)$ of T is defined by

$$e_n(T) := e_n(T(B_E)),$$

where B_E is the closed unit ball in E.

Our study of $C(X)$-valued operators, where (X,d) is a compact metric space, is motivated to a considerable extent by the universality of the Banach space $C[a,b]$ of continuous functions on the closed, bounded interval $[a,b]$. Indeed, as is pointed out in [6], Chapter 5, the universality of $C[a,b]$ implies universality of the class of $C[a,b]$-valued operators in the following sense: given a compact linear operator $T\colon E \to F$ between arbitrary Banach spaces E and F, there is a compact linear operator $S\colon E \to C[a,b]$ such that the (dyadic) entropy numbers $e_n(T)$ of T are equivalent to those of S in the sense that

$$\tfrac{1}{2}e_n(S) \leq e_n(T) \leq 2e_n(S) \quad (n \in \mathbb{N}).$$

This explains why we study the asymptotic behaviour of entropy numbers of operators $T\colon E \to C(X)$ with values in the space $C(X)$ of continuous functions over a compact metric space. We shall see how the geometry of the Banach space E, the entropy numbers $\varepsilon_n(X)$ of the underlying compact metric space X and the smoothness of the operator T in terms of the modulus of continuity $\omega(T,\delta)$ of T,

$$\omega(T,\delta) := \sup_{\|x\|\leq 1} \sup\{|(Tx)(s) - (Tx)(t)|\colon s,t \in X, d(s,t) \leq \delta\},$$

all affect the estimates of the (dyadic) entropy numbers $e_n(T)$ of T. For some proofs we also consider the entropy numbers of operators

$T: E \to \ell_\infty(M)$ from a Banach space E to the space $\ell_\infty(M)$ of all bounded number families $(\xi_t)_{t\in M}$ over an index set M and with the norm

$$\|(\xi_t)\| := \sup_{t\in M} |\xi_t|.$$

Moreover, we study the 'dual' situation, corresponding to maps $T: \ell_1(M) \to E$ from the Banach space $\ell_1(M)$ of all summable families of numbers $(\xi_t)_{t\in M}$ over the index set M and with norm given by

$$\|(\xi_t)\| := \sum_{t\in M} |\xi_t|,$$

to the Banach space E.

In Section 2 we investigate the entropy behaviour of operators $T: E \to C(X)$, where the smoothness property of T is given by a Lipschitz condition, by which we mean that

$$\sup_{\delta>0} \omega(T,\delta)/\delta < \infty.$$

Such operators are called Lipschitz-continuous. We give a universal result for the entropy numbers $e_n(T)$ of Lipschitz-continuous operators T, where the geometry of the underlying Banach space E is provided by so-called 'local estimates' of the entropy numbers $e_k(S: E \to \ell_\infty^n)$ of finite rank operators or in terms of Banach spaces of type p, and the rate of decrease of the sequence $\varepsilon_n(X)$ is of type $n^{-\sigma}\log^\gamma(n+1)$, for some $\sigma > 0$ and $\gamma \geq 0$. This result is applied to integral operators and operators defined by abstract kernels. We study the question of how entropy conditions on the kernel influence the entropy behaviour of the corresponding integral or matrix operator, and obtain information about eigenvalue distributions. Moreover, entropy conditions on the kernel are given which ensure that the induced integral operator belongs to the Schatten class $\mathcal{S}_{q,\infty}$.

Similar problems in the 'dual' situation for maps $T: \ell_1(X) \to E$ are treated in Section 3. As an application, we obtain optimal results in Section 4 about the entropy behaviour of convex hulls of precompact sets X in a Banach space of type p when the entropy numbers $\varepsilon_n(X)$ are of order $n^{-\sigma}\log^\gamma(n+1)$, where $\sigma > 0$ and $\gamma \geq 0$.

2 ENTROPY OF $C(X)$-VALUED LIPSCHITZ-CONTINUOUS OPERATORS

As already explained in the Introduction, the entropy behaviour of a compact linear operator is reflected by that of a $C(X)$-valued operator on a compact metric space (X, d). Thus for our purpose, $C(X)$-valued operators are universal. By the Arzelà–Ascoli theorem, we know that an operator $T: E \to C(X)$ from a Banach space E to the space $C(X)$ of all continuous functions on a compact metric space X is compact if, and only if,

$$\lim_{\delta \to 0+} \omega(T, \delta) = 0,$$

where $\omega(T, \delta)$ is the modulus of continuity of T defined in Section 1. A stronger condition is that of Hölder continuity (cf. [6, 5.6]): an operator $T: E \to C(X)$ is called Hölder-continuous of type α, $0 < \alpha \leq 1$, if

$$|T|_\alpha := \sup_{\delta > 0} \frac{\omega(T, \delta)}{\delta^\alpha} < \infty.$$

When $\alpha = 1$ such an operator is said to be Lipschitz-continuous. The set $\mathcal{L}\mathrm{ip}_\alpha(E, C(X))$ of all operators from E to $C(X)$ which are Hölder-continuous of type α becomes a Banach space under the norm

$$\mathrm{Lip}_\alpha(T) := \max\{\|T\|, |T_\alpha|\},$$

where $\|T\|$ stands for the operator norm of T. When $\alpha = 1$ we write

$$[\mathcal{L}\mathrm{ip}(E, C(X)), \mathrm{Lip}] := [\mathcal{L}\mathrm{ip}_1(E, C(X)), \mathrm{Lip}_1].$$

If we equip the metric space (X, d) with the metric d^α $(0 < \alpha \leq 1)$ we reduce an operator which is Hölder-continuous of type α to a Lipschitz-continuous operator:

$$\mathrm{Lip}(T: E \to C(X, d^\alpha)) = \mathrm{Lip}_\alpha(T: E \to C(X, d)).$$

This reduction will be crucial in the present section.

Before stating a universal theorem about the entropy of Lipschitz-continuous operators we recall the notion of *type*. A Banach space E is said to have type p $(1 \leq p \leq 2)$ if there is a constant C such that for all finite sets $(x_i)_{i=1}^n$ in E,

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\| \mathrm{d}t \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

where $(r_i)_{i=1}^\infty$ are the Rademacher functions; that is, $r_i(t) = \operatorname{sgn}(\sin(2^n \pi t))$. The smallest constant C such that the above inequality is satisfied is written $\tau_p(E)$ and called the type p constant of E. The script $\mathcal{L}_q$-spaces $(1 \leq q < \infty)$ of Lindenstrauss and Pełczynski are of type $p = \min\{2, q\}$. For example, the Lebesgue spaces $L_q(\Omega, \mu)$, the Sobolev spaces $W_q^s((0,1)^N)$ and the Besov spaces $B_q^s((0,1)^N)$ are all script $\mathcal{L}_q$-spaces, and so are of type $p = \min\{2, q\}$ for $1 \leq q < \infty$. For more information about type p spaces we refer to [15].

Now we state the promised universal theorem about Lipschitz-continuous operators. This is a slight extension of the corresponding Theorem 5.10.1 in [6] (cf. [14]), and we omit the proof as it follows the same line as that of Theorem 5.10.1.

THEOREM 2.1 *Let (X, d) be a compact metric space such that for some constants $\sigma > 0$ and $\gamma \geq 0$, the entropy numbers $\varepsilon_n(X)$ satisfy*

$$\sup_{n \in \mathbb{N}} n^\sigma \log^{-\gamma}(n+1) \varepsilon_n(X) < \infty. \tag{2.1}$$

Moreover, let E be a Banach space with the property that there exists $\beta > 0$ such that for each $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ with the so-called 'local estimate' holding:

$$e_k(S) \leq c(\varepsilon) \|S\| k^{-\beta} (n/k)^\varepsilon \tag{2.2}$$

for all $n \in \mathbb{N}$, all $k \in [1, n] \cap \mathbb{N}$ and all operators $S : E \to \ell_\infty^n$. Then for the dyadic entropy numbers $e_n(T)$ of any operator $T \in \mathcal{L}\mathrm{ip}(E, C(X))$ we have the estimate

$$\sup_{n \in \mathbb{N}} n^{\beta+\sigma} \log^{-\gamma}(n+1) e_n(T) < \infty. \tag{2.3}$$

Remarks

(i) If $T \in \mathcal{L}ip_\alpha(E, C(X))$ is a Hölder-continuous operator of type α $(0 < \alpha \leq 1)$, where E and X are as in Theorem 2.1, we conclude that

$$\sup_{n\in\mathbb{N}} n^{\beta+\alpha\sigma} \log^{-\alpha\gamma}(n+1) e_n(T) < \infty.$$

This follows easily from Theorem 2.1 by using the formulae

$$\varepsilon_n^\alpha(X, d) = \varepsilon_n(X, d^\alpha)$$

and

$$\mathrm{Lip}(T: E \to C(X, d^\alpha)) = \mathrm{Lip}_\alpha(T: E \to C(X, d)).$$

(ii) If E is a Banach space with dual E' of type $p > 1$, then in the local estimate (2.2) of Theorem 2.1 we may take $\beta = 1 - 1/p$ (see [5, Theorem 1.8]), and thus obtain for the entropy numbers $e_n(T)$ of an operator $T \in \mathcal{L}ip(E, C(X))$ the estimate

$$\sup_{n\in\mathbb{N}} n^{1-1/p+\sigma} \log^{-\gamma}(n+1) e_n(T) < \infty.$$

In particular, if E is an $\mathcal{L}_q$-space, we have $\beta = \min\{1/2, 1/q\}$ for $1 < q < \infty$ and

$$\sup_{n\in\mathbb{N}} n^{\min\{1/2,1/q\}+\sigma} \log^{-\gamma}(n+1) e_n(T) < \infty.$$

(iii) Theorem 2.1 remains true for Lipschitz-continuous operators $T \in \mathcal{L}ip(E, \ell_\infty(X))$ where (X, d) is a precompact metric space satisfying the entropy condition (2.1).

(iv) Theorem 2.1 is optimal. Indeed for the Sobolev embedding

$$I: W_p^s(Q) \to C(\bar{Q}), \quad Q = (0,1)^N,$$

where $1 \leq p < \infty$ and $s > N/p$, we know that

$$e_n(I) \asymp n^{-s/N};$$

see [9], 3.3. The upper estimate in this, when $2 \leq p < \infty$ and $1/p < s/N < 1/p + 1/N$, can be obtained from Theorem 2.1. For we can factorise I in the form

$$I: W_p^s(Q) \xrightarrow{I_0} C^{s_0}(\bar{Q}) \xrightarrow{I_1} C(\bar{Q}),$$

where $s - s_0 = N/p$ and $0 < s_0 \leq 1$. This means that

$$I \in \mathcal{L}\mathrm{ip}_{s_0}(W_p^s(Q) \to C(\bar{Q})).$$

Since $\varepsilon_n(\bar{Q}) \asymp n^{-1/N}$ and the dual of $W_p^s(Q)$ is a script $\mathcal{L}_{p'}$-space, we see from Theorem 2.1 and Remarks (i) and (ii) that

$$\sup_{n \in \mathbb{N}} n^{1/p + s_0/N} e_n(I) = \sup_{n \in \mathbb{N}} n^{s/N} e_n(I) < \infty.$$

(v) An interesting generalisation of the classical Sobolev spaces is provided by the spaces $W_p^1(X, d, \mu)$ recently introduced by Hajłasz [11]. Here $1 < p < \infty$, (X, d) is a compact metric space with finite diameter and μ is a finite positive Borel measure. A particularly important case occurs when X is a compact subset of $\mathbb{R}^n$ which is strictly s-regular ($0 \leq s \leq n$) in the sense that there are positive constants c_1 and c_2 such that for all $x \in X$ and all r with $0 < r \leq \operatorname{diam} X$,

$$c_1 r^s \leq \mu(B(x, r) \cap X) \leq c_2 r^s,$$

μ being Hausdorff s-measure. The Cantor set in $\mathbb{R}^n$ is such a set X, with $s = \log(3^n - 1)/\log 3$. It turns out (see [11, Theorem 6]) that if X is strictly s-regular and $p > s$, then $W_p^1(X, d, \mu)$ is embedded in $C(X)$. Application of Theorem 2.1, as in (i) and (ii) above, now shows that the dyadic entropy numbers $e_n(I)$ of this embedding map satisfy

$$\sup_{n \in \mathbb{N}} n^{\min\{1/2, 1/p\} + 1/s} e_n(I) < \infty.$$

We now give diverse applications of Theorem 2.1 to operators generated by abstract kernels. For this purpose we need some more concepts. Let $C(X, Z)$ be the set of all continuous, Z-valued functions on a compact metric space (X, d), Z being an arbitrary Banach space. Evidently $C(X, Z)$ becomes a Banach space when given the norm

$$\|K\| := \sup_{s \in X} \|K(s)\|_Z.$$

Just as for scalar-valued bounded functions f on X we introduce the modulus of continuity

$$\omega_Z(K, \delta) := \sup\{\|K(s) - K(t)\|_Z \colon s, t \in X,\ d(s,t) \le \delta\}, \tag{2.4}$$

for $0 \le \delta < \infty$. This is well-defined for arbitrary bounded Z-valued functions K on X; such a K is continuous if, and only if,

$$\lim_{\delta \to 0+} \omega_Z(K, \delta) = 0.$$

By means of this modulus of continuity we introduce classes $C^\alpha(X, Z)$ of Hölder-continuous Z-valued functions of type α on X, for $0 < \alpha \le 1$: $C^\alpha(X, Z)$ is the subset of $C(X, Z)$ consisting of those K for which

$$|K|_{Z,\alpha} := \sup_{\delta > 0} \omega_Z(K, \delta)/\delta^\alpha \tag{2.5}$$

is finite. The class $C^\alpha(X, Z)$ can be shown to be a Banach space when given the norm

$$\|K\|_{Z,\alpha} := \max\{\|K\|, |K|_{Z,\alpha}\}. \tag{2.6}$$

When $Z = E'$, the dual of a Banach space E, the element $K \in C(X, E')$ can be used to generate operators $T_K \colon E \to C(X)$ by the rule

$$(T_K x)(s) := \langle x, K(s) \rangle, \quad x \in E,\ s \in X, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E'. These operators are linear and bounded, and

$$\|T_K\| = \|K\|.$$

Moreover, it is easy to see that

$$\omega(T_K, \delta) = \omega_{E'}(K, \delta). \tag{2.8}$$

Since $\lim_{\delta \to 0+} \omega_{E'}(K, \delta) = 0$ it follows that T_K is compact, for all $K \in C(X, E')$.

On the other hand, any compact operator $T: E \to C(X)$ can be generated by an 'abstract kernel' $K \in C(X, E')$ in the sense of (2.7) if we put

$$K(s) := T'\delta_s, \quad s \in X, \tag{2.9}$$

where δ_s is the Dirac functional on $C(X)$: $\langle f, \delta_s \rangle = f(s)$ for $s \in X, f \in C(X)$. We again have

$$\|K\| = \|T\|, \quad \omega_{E'}(K, \delta) = \omega(T, \delta); \tag{2.10}$$

and since T is compact,

$$\lim_{\delta \to 0+} \omega_{E'}(K, \delta) = 0,$$

so that $K \in C(X, E')$. By (2.7) and (2.10) the operator T_K generated by K coincides with the original operator T since

$$(T_K x)(s) = \langle x, T'\delta_s \rangle = \langle Tx, \delta_s \rangle = (Tx)(s).$$

We summarise these considerations in the following well-known proposition (cf. [6, Proposition 5.13.1]):

PROPOSITION 2.2 *Let (X, d) be a compact metric space and E a Banach space. Let $\mathcal{K}(E, C(X))$ be the Banach space of all compact linear operators from E to $C(X)$. Then the map*

$$\phi : \mathcal{K}(E, C(X)) \to C(X, E')$$

defined by

$$\phi(T)(s) := T'\delta_s, \quad s \in X, \ T \in \mathcal{K}(E, C(X)),$$

is a metric isomorphism from $\mathcal{K}(E, C(X))$ onto $C(X, E')$ as well as a metric isomorphism from the subclass $\mathcal{L}\mathrm{ip}_\alpha(E, C(X))$ of $\mathcal{K}(E, C(X))$ onto the subclass $C^\alpha(X, E')$ of $C(X, E')$, for $0 < \alpha \leq 1$.

Now we can reformulate Theorem 2.1 for operators generated by abstract kernels.

THEOREM 2.3 *Let (X, d) be a compact metric space and E a Banach space satisfying conditions* (2.1) *and* (2.2) *of Theorem* 2.1, *respectively. Let $K \in C^1(X, E')$ be a kernel of Lipschitz type* (*that is, of Hölder type* 1), *and let $T_K: E \rightarrow C(X)$ be the corresponding induced operator given by*

$$(Tx)(s) = \langle x, K(s) \rangle, \quad x \in E, \ s \in X.$$

Then for the dyadic entropy numbers $e_n(T_K)$ we have

$$\sup_{n \in \mathbb{N}} n^{\beta+\delta} \log^{-\gamma}(n+1) e_n(T_K) < \infty.$$

Moreover, Remarks (i)–(iv) *following Theorem* 2.1 *can be carried over to this new setting.*

As a first consequence of Theorem 2.3 we give the following useful theorem.

THEOREM 2.4 *Let E be a Banach space satisfying the entropy condition* (2.2) *of Theorem* 2.1, *let X be an arbitrary index set and let $K \in \ell_\infty(X, E')$ be a bounded function from X to the dual Banach space E'. Define an operator $T_K: E \rightarrow \ell_\infty(X)$ by the rule*

$$(T_K x)(s) = \langle x, K(s) \rangle \quad \text{for } x \in E, \ s \in X.$$

If the image $\operatorname{Im} K = K(X)$ *of K satisfies the condition*

$$\sup_{n \in \mathbb{N}} n^{\sigma} \log^{-\gamma}(n+1) \varepsilon_n(\operatorname{Im} K) < \infty \tag{2.11}$$

for some $\sigma > 0$ and $\gamma \geq 0$, then for the dyadic entropy numbers $e_n(T_K)$ of the induced operator T_K we have

$$\sup_{n \in \mathbb{N}} n^{\beta+\sigma} \log^{-\gamma}(n+1) e_n(T_K) < \infty. \tag{2.12}$$

In particular, if E' is of type $p > 1$, then $\beta = 1 - 1/p$.

Proof Without loss of generality we assume $\operatorname{Im} K$ to be compact. Let $[t] := \{s \in X: K(s) = K(t)\}$, $t \in X$. On the set

$$\hat{X} := \{[t]: t \in X\}$$

we introduce a metric d by the rule

$$d([s],[t]) := \|K(s) - K(t)\|.$$

Thus $\varepsilon_n(\hat{X}) \leq 2\varepsilon_n(\operatorname{Im} K)$, $n \in \mathbb{N}$. Next, define $S: E \to \ell_\infty(\hat{X})$ by

$$(Sx)([s]) := (T_K x)(s),$$

so that $e_n(T_K) = e_n(S)$, $n \in \mathbb{N}$. Since

$$\begin{aligned}|(Sx)([s]) - (Sx)([t])| &= |(T_K x)(s) - (T_K x)(t)| \\ &= |\langle x, K(s) - K(t)\rangle| \leq \|x\|\|K(s) - K(t)\| \\ &\leq \|x\| d([s],[t]),\end{aligned}$$

we see that

$$S \in \mathcal{L}\mathrm{ip}(E, \ell_\infty(\hat{X})).$$

Theorem 2.3 and Remark (iii) after Theorem 2.1 now give the desired estimate (2.12). The remaining part of the theorem follows from Remark (ii) after Theorem 2.1.

The rest of this section is devoted to consequences of the previous theorems.

Example 2.5 Let $1 < q < \infty$, let (Y, ν) be a measure space, let X be any set and define the integral operator $T_{K,\nu}: L_q(Y,\nu) \to \ell_\infty(X)$ by

$$(T_{K,\nu} f)(s) := \int_Y K(s,t) f(t)\, \mathrm{d}\nu(t), \quad s \in X, \tag{2.13}$$

where the kernel K satisfies the condition

$$\sup_{s \in X} \|K(s,\cdot)\|_{L_{q'}(Y,\nu)} < \infty. \tag{2.14}$$

This operator can be handled in our general framework. Indeed, let $K(s) := K(s, \cdot)$, $s \in X$; then the kernel K can be considered as an abstract kernel in $\ell_\infty(X, L_{q'}(Y, \nu))$. If we additionally assume that the entropy numbers $\varepsilon_n(\mathrm{Im}(K))$ of the image

$$\mathrm{Im}(K) := \{K(s, \cdot) \colon s \in X\} \subset L_{q'}(Y, \nu) \tag{2.15}$$

of K satisfy the condition

$$\sup_{n \in \mathbb{N}} n^\sigma \log^{-\gamma}(n+1) \varepsilon_n(\mathrm{Im}(K)) < \infty \tag{2.16}$$

for some $\sigma > 0$ and $\gamma \geq 0$, then the dyadic entropy numbers $e_n(T_{K,\nu})$ of the integral operator $T_{K,\nu}$ satisfy

$$\sup_{n \in \mathbb{N}} n^{\min\{1/2, 1/q\} + \sigma} \log^{-\gamma}(n+1) e_n(T_{K,\nu}) < \infty. \tag{2.17}$$

This follows from Theorem 2.4 with $\beta = \min\{1/2, 1/q\}$, because the dual of L_q is of type $\min\{2, q'\}$.

Example 2.6 Let $p, q \in (1, \infty)$, let (X, μ) and (Y, ν) be measure spaces and let K be an $(X \times Y, \mu \times \nu)$-measurable kernel satisfying the Hille–Tamarkin condition

$$\int_X \left(\int_Y |K(s,t)|^{q'} \, \mathrm{d}\nu(t) \right)^{p/q'} \mathrm{d}\mu(s) < \infty. \tag{2.18}$$

The integral operator $T_{K,\nu,\mu}$ given by

$$(T_{K,\nu,\mu} f)(s) = \int_Y K(s,t) f(t) \, \mathrm{d}\nu(t)$$

maps $L_q(Y, \nu)$ into $L_p(X, \mu)$, as an application of Hölder's inequality shows. Letting $K(s) := K(s, \cdot)$, we may consider K as a p-integrable, $L_{q'}$-valued kernel $K \in L_p((X, \mu), L_{q'}(Y, \nu))$. If in addition we assume that (X, μ) is a finite measure space and that the entropy numbers of the image of K,

$$\mathrm{Im}(K) = \{K(s, \cdot) \colon s \in X\} \subset L_{q'}(Y, \nu),$$

satisfy the condition

$$\sup_{n\in\mathbb{N}} n^{\sigma} \log^{-\gamma}(n+1)\varepsilon_n(\mathrm{Im}(K)) < \infty \tag{2.19}$$

for some $\sigma > 0$ and $\gamma \geq 0$, then the dyadic entropy numbers of the Hille–Tamarkin integral operator $T_{K,\nu,\mu}$ satisfy

$$\sup_{n\in\mathbb{N}} n^{\min\{1/2,1/q\}+\sigma} \log^{-\gamma}(n+1) e_n(T_{K,\nu,\mu}) < \infty. \tag{2.20}$$

To see this, note that as $\mathrm{Im}(K)$ is precompact we have

$$\sup_{s\in X} \|K(s,\cdot)\|_{L_{q'}} < \infty;$$

and as $\mu(X) < \infty$ we may, with the help of Hölder's inequality, factorise $T_{K,\nu,\mu}$ as follows:

$$\begin{array}{ccc} L_q(Y,\nu) & \overset{T_{K,\nu,\mu}}{\longrightarrow} & L_p(X,\mu) \\ T^0_{K,\nu,\mu}\searrow & & \nearrow I \\ & \ell_\infty(X) & \end{array}$$

Here I is the natural embedding and $T^0_{K,\nu,\mu}$ satisfies the conditions of Example 2.5. The multiplicity of the entropy numbers

$$e_n(T_{K,\nu,\mu}) \leq e_n(T^0_{K,\nu,\mu})\|I\| \leq e_n(T^0_{K,\nu,\mu})\mu^{1/p}(X),$$

and Example 2.5 now give (20).

As a consequence of the last example we obtain an improvement of the main theorem in [10], which is set in a Hilbert space context.

Example 2.7 Let (X,μ) and (Y,ν) be finite measure spaces and let $K \in L_2(X\times Y, \mu\times\nu)$ be a Hilbert–Schmidt kernel. Let

$$K_X := \{K(s,\cdot)\colon s\in X\} \subset L_2(Y,\nu)$$

and

$$K_Y := \{K(\cdot,t)\colon t\in Y\} \subset L_2(X,\mu),$$

and suppose that for some $\sigma > 0$ and $\gamma \geq 0$,

$$\sup_{n \in \mathbb{N}} n^{\sigma} \log^{-\gamma}(n+1)\varepsilon_n(K_X) < \infty \quad \text{or} \quad \sup_{n \in \mathbb{N}} n^{\sigma} \log^{-\gamma}(n+1)\varepsilon_n(K_Y) < \infty. \tag{2.21}$$

Let $T_{K,\nu,\mu} : L_2(Y,\nu) \to L_2(X,\mu)$ be the integral operator given by

$$(T_{K,\nu,\mu} f)(s) = \int_Y K(s,t) f(t) \, \mathrm{d}\nu(t).$$

Then the dyadic entropy numbers of $T_{K,\nu,\mu}$ satisfy

$$\sup_{n \in \mathbb{N}} n^{1/2+\sigma} \log^{-\gamma}(n+1) e_n(T_{K,\nu,\mu}) < \infty. \tag{2.22}$$

Indeed, application of Example 2.6 with $p = q = 2$ to $T_{K,\nu,\mu}$ as well as to the dual operator $T'_{K,\nu,\mu}$ immediately gives (2.22) since $e_n(T_{K,\nu,\mu}) = e_n(T'_{K,\nu,\mu})$.

In particular, when (2.21) holds with $\gamma = 0$, we see that $T_{K,\nu,\mu}$ belongs to the weak Schatten class $\mathcal{L}_{s,\infty}$, with $1/s = (1/2) + \sigma$. The class $\mathcal{L}_{s,\infty}$ can be characterised by the dyadic entropy numbers as follows:

$$T \in \mathcal{L}_{s,\infty} \quad \text{if, and only if,} \ \sup_{n \in \mathbb{N}} n^{1/s} e_n(T) < \infty$$

(see [6, p. 27]). The case studied in [10] corresponds to $\sigma > 1/2$. In fact, since in a Hilbert space setting entropy ideals coincide with approximation ideals (see [2] and [6, p. 120]), which in turn are related to nuclear operators (see, for example, [13, p. 66, Proposition 1.d.12]), we see that when $\sigma > 1/2$, T is nuclear. This is the result in [10].

Finally, we turn to the eigenvalue distributions of integral operators generated by Hille–Tamarkin kernels satisfying certain entropy conditions.

THEOREM 2.8 *Let $1 < p < \infty$, let (X,μ) be a finite measure space, suppose that $K : X \times X \to \mathbb{C}$ is a $\mu \times \mu$-measurable kernel satisfying the Hille–Tamarkin condition*

$$\int_X \left(\int_X |K(s,t)|^{p'} \, \mathrm{d}\mu(t) \right)^{p/p'} \mathrm{d}\mu(s) < \infty, \tag{2.23}$$

and put

$$(T_{K,\mu} f)(s) = \int_X K(s,t) f(t)\, \mathrm{d}\mu(t), \quad f \in L_p(X,\mu), \ s \in X.$$

Then $T_{K,\mu}$ maps $L_p(X,\mu)$ into itself. Assume further that the entropy numbers $\varepsilon_n(\operatorname{Im} K)$ of the image

$$\operatorname{Im}(K) := \{K(s,\cdot) \colon s \in X\} \subset L_{p'}(X,\mu)$$

satisfy

$$\sup_{n \in \mathbb{N}} n^{\sigma} \varepsilon_n(\operatorname{Im}(K)) < \infty \tag{2.24}$$

for some $\sigma > 0$. Then the eigenvalues $\lambda_n(T_{K,\mu})$ of $T_{K,\mu}$ satisfy

$$\sup_{n \in \mathbb{N}} n^{\min\{1/2, 1/p\}+\sigma} |\lambda_n(T_{K,\mu})| < \infty. \tag{2.25}$$

Here the eigenvalues are counted according to their algebraic multiplicities and ordered by decreasing modulus; if $T_{K,\mu}$ has only m eigenvalues, we put $\lambda_n(T_{K,\mu}) = 0$ for $n > m$.

Proof The assertion follows immediately from Example 2.6 and the well-known inequality between eigenvalues and entropy numbers (see [2] and [6, p. 146])

$$|\lambda_n(T_{K,\mu})| \leq \sqrt{2} e_n(T_{K,\mu}).$$

A typical application of the last theorem is the following:

COROLLARY 2.9 *Let $1 < p < \infty$, let (X,μ) be a finite measure space, let (X,d) be a compact metric space satisfying the condition*

$$\sup_{n \in \mathbb{N}} n^{\sigma} \varepsilon_n(X) < \infty$$

for some $\sigma > 0$, and let $K \colon X \times X \to \mathbb{C}$ be a $\mu \times \mu$-measurable Hille–Tamarkin kernel:

$$\int_X \left(\int_X |K(s,t)|^{p'}\, \mathrm{d}\mu(t) \right)^{p/p'} \mathrm{d}\mu(s) < \infty.$$

Suppose that for some $\alpha \in (0,1]$ and some $\rho > 0$ the kernel K satisfies the following integral Hölder condition for all $s_0, s_1 \in X$:

$$\left(\int_X |K(s_0,t) - K(s,t)|^{p'} \,\mathrm{d}\mu(t)\right)^{1/p'} \leq \rho d^{\alpha}(s_0,s_1). \tag{2.26}$$

Then the induced integral operator $T_{K,\mu}$,

$$(T_{K,\mu}f)(s) = \int_X K(s,t)f(t)\,\mathrm{d}\mu(t), \quad f \in L_p(X,\mu), \ s \in X,$$

maps $L_p(X,\mu)$ into itself. Its eigenvalues $\lambda_n(T_{K,\mu})$, defined according to the conventions of Theorem 2.8, satisfy

$$\sup_{n\in\mathbb{N}} n^{\min\{1/2,1/p\}+\alpha\sigma} |\lambda_n(T_{K,\mu})| < \infty. \tag{2.27}$$

Proof As before we put $K(s) := K(s,\cdot)$ and consider K as a map from X to $L_{p'}(X,\mu)$. Because of the integral Hölder condition (2.26) we have for all $s_0, s_1 \in X$,

$$\|K(s_0) - K(s_1)\|_{L_{p'}} \leq \rho d^{\alpha}(s_0,s_1),$$

which implies that

$$\varepsilon_n(\mathrm{Im}(K)) \leq \rho\varepsilon_n^{\alpha}(X).$$

Since $\sup_{n\in\mathbb{N}} n^{\sigma}\varepsilon_n(X) < \infty$, it follows that

$$\sup_{n\in\mathbb{N}} n^{\alpha\sigma}\varepsilon_n(\mathrm{Im}\,K) < \infty,$$

and the desired assertion is now an immediate consequence of Theorem 2.8.

Example 2.10 If we put $X = [0,1]^N$ and take μ to be Lebesgue N-measure in Corollary 2.9, then since

$$\sup_{n\in\mathbb{N}} n^{1/N}\varepsilon_n([0,1]^N) < \infty,$$

we obtain

$$\sup_{n\in\mathbb{N}} n^{\min\{1/2,1/p\}+\alpha/N} |\lambda_n(T_{K,\mu})| < \infty.$$

This contains as a special case the example given in [10] to illustrate their Theorem 1.

3 ENTROPY OF LIPSCHITZ-CONTINUOUS OPERATORS ON $\ell_1(X)$

Here we study the situation 'dual' to that of Section 2. We consider operators $T: \ell_1(X) \to E$, where (X, d) is a precompact metric space and E is a Banach space. Such an operator T is said to belong to $\mathcal{L}\text{ip}_\alpha(\ell_1(X), E)$, $0 < \alpha \leq 1$, and is called Hölder-continuous of type α, if its dual T' belongs to $\mathcal{L}\text{ip}_\alpha(E', \ell_\infty(X))$. This is equivalent to the condition

$$\sup\{\|T(f_t) - T(f_s)\|/\text{d}^\alpha(s,t): s, t \in X, s \neq t\} < \infty,$$

where (f_t) is the canonical basis of $\ell_1(X)$. The quantity

$$\text{Lip}_\alpha(T) := \text{Lip}_\alpha(T') = \max\{\|T\|, |T'|_\alpha\}$$

is a norm on $\mathcal{L}\text{ip}_\alpha(\ell_1(X), E)$, and with this norm the space becomes a Banach space. When $\alpha = 1$ we suppress the index 1 and simply write $[\mathcal{L}\text{ip}(\ell_1(X), E), \text{Lip}]$, calling the operators Lipschitz-continuous. As in Section 2 we can transform Hölder-continuous operators into Lipschitz-continuous ones by a change of metric.

Now we can state a theorem corresponding to Theorem 2.1.

THEOREM 3.1 *Let (X, d) be a precompact metric space such that for some $\sigma > 0$ and $\gamma \geq 0$,*

$$\sup_{n\in\mathbb{N}} n^\sigma \log^{-\gamma}(n+1)\varepsilon_n(X) < \infty. \tag{3.1}$$

Let E be a Banach space with the property that there is a constant $\beta > 0$ such that for each $\varepsilon > 0$ there is a constant $c(\varepsilon) \geq 1$ with

$$e_k(S) \leq c(\varepsilon)\|S\|k^{-\beta}(n/k)^\varepsilon \tag{3.2}$$

for all $n \in \mathbb{N}$, all $k \in \mathbb{N}$ with $1 \leq k \leq n$, and all bounded linear operators $S: \ell_1^n \to E$. Then for any $T \in \mathcal{L}\mathrm{ip}(\ell_1(X), E)$ we have for the dyadic entropy numbers $e_n(T)$ the estimate

$$\sup_{n \in \mathbb{N}} n^{\beta+\sigma} \log^{-\gamma}(n+1) e_n(T) < \infty. \tag{3.3}$$

In particular, if E is a Banach space of type $p > 1$, we have $\beta = 1 - 1/p$.

We omit the proof of this theorem as it proceeds along similar lines to that of Theorem 3.3 in [5], where $\gamma = 0$. Note that if E is an L_q space, a Sobolev space W_q^s or a Besov space B_q^s, we have $\beta = \min\{1/2, 1 - 1/q\}$ when $1 < q < \infty$.

Remark If $T \in \mathcal{L}\mathrm{ip}_\alpha(\ell_1(X), E)$ for some α, $0 < \alpha \leq 1$, in Theorem 3.1, then just as in Remark (i) following Theorem 2.1 the conclusion is changed to

$$\sup_{n \in \mathbb{N}} n^{\beta+\alpha\sigma} \log^{-\alpha\gamma}(n+1) e_n(T) < \infty.$$

As an immediate consequence of Theorem 3.1 we recover the main result in [3].

COROLLARY 3.2 *Let E be a Banach space of type $p > 1$ and let $S: \ell_1(\mathbb{N}) \to E$ have the property that for some $\sigma > 0$,*

$$\sup_{n \in \mathbb{N}} n^{\sigma} \|Sf_n\| < \infty, \tag{3.4}$$

where (f_n) is the canonical unit vector basis in $\ell_1(\mathbb{N})$. Then the dyadic entropy numbers of S satisfy

$$\sup_{n \in \mathbb{N}} n^{1-1/p+\sigma} e_n(S) < \infty. \tag{3.5}$$

Proof We consider

$$X := \{n^{-\sigma} f_n : n \in \mathbb{N}\} \subset \ell_1(\mathbb{N})$$

as a subset of $\ell_1(\mathbb{N})$ and equip it with the metric d defined by

$$d(s,t) = \|s - t\|_{\ell_1} \quad \text{for } s, t \in X.$$

We thus have a precompact metric space (X, d) with

$$\sup_{n \in \mathbb{N}} n^{\sigma} \varepsilon_n(X) < \infty.$$

Define $T : \ell_1(X) \to E$ by

$$T\hat{f}_t = Sf_n \quad \text{for } t = n^{-\sigma} f_n \in X;$$

here $\hat{f}_t$ denotes an element of the canonical unit vector basis in $\ell_1(X)$. Then the dyadic entropy numbers of T and S coincide:

$$e_n(T) = e_n(S).$$

Moreover, given any $s, t \in X$ with $s \neq t$, so that $s = m^{-\sigma} f_m$ and $t = n^{-\sigma} f_n$, say, we have

$$\begin{aligned} \|T\hat{f}_t - T\hat{f}_s\| &= \|Sf_n - Sf_m\| \leq \|Sf_n\| + \|Sf_m\| \\ &\leq (n^{-\sigma} + m^{-\sigma}) \sup_{k \in \mathbb{N}} k^{\sigma} \|Sf_k\| \\ &= \|n^{-\sigma} f_n - m^{-\sigma} f_m\| \sup_{k \in \mathbb{N}} k^{\sigma} \|Sf_k\| \\ &= d(t, s) \sup_{k \in \mathbb{N}} k^{\sigma} \|Sf_k\|. \end{aligned}$$

Hence $T \in \mathcal{L}\text{ip}(\ell_1(X), E)$. Application of Theorem 3.1 with $\beta = 1 - 1/p$, $\gamma = 0$ and $\sigma > 0$ now gives

$$\sup_{n \in \mathbb{N}} n^{1-1/p+\sigma} e_n(S) = \sup_{n \in \mathbb{N}} n^{1-1/p+\sigma} e_n(T) < \infty,$$

as required.

The next result can be reduced to the previous corollary by using B-spline techniques.

COROLLARY 3.3 *Let E be a Banach space of type $p > 1$, let $s > 0$, let $B_1^s((0,1))$ be the usual Besov space and let $T : B_1^s((0,1)) \to E$ be an* *operator which can be factorised as follows*

$$\begin{array}{ccccc} B_1^s((0,1)) & & \xrightarrow{T} & & E \\ & I\searrow & & \nearrow S & \\ & & B_1^t((0,1)) & & \end{array}$$

Here $s > t \geq 0$, I is the natural embedding and S is a bounded linear operator from $B_1^t((0,1))$ to E. Then the dyadic entropy numbers of T satisfy

$$\sup_{n\in\mathbb{N}} n^{1-1/p+s-t} e_n(T) < \infty. \tag{3.6}$$

We shall not give the details of the proof as the result is contained in the following Proposition, for which we provide a direct proof using B-spline techniques.

PROPOSITION 3.4 *Let $s \geq 0$, let $n \in \mathbb{N}$, put $Q = (0,1)^N$ and let F_1^s stand for the Sobolev space $W_1^s(Q)$ or the Besov space $B_1^s(Q)$ if $s > 0$, and $L_1(Q)$ if $s = 0$. Suppose that E is a Banach space of type $p > 1$ and that $T : F_1^s \to E$ is an operator which can be factorised as follows*

$$\begin{array}{ccccc} F_1^s & & \xrightarrow{T} & & E \\ & I\searrow & & \nearrow S & \\ & & F_1^t & & \end{array}$$

Here $s > t \geq 0$, I is the natural embedding and S is a bounded operator. Then the dyadic entropy numbers of T satisfy

$$\sup_{n\in\mathbb{N}} n^{1-1/p+(s-t)/N} e_n(T) < \infty. \tag{3.7}$$

Proof We need some basic facts about spline functions (see [1,8,12] and also [9, Chap. 2]). Divide the cube Q into cubes of side 2^{-k}. The corresponding space of smooth splines of degree ℓ is denoted by

$$S_k := S_k(N, \ell) \subset C^{\ell-1}(\bar{Q})$$

and has dimension $d_k := d_k(N, \ell)$ with

$$c_0 2^{kN} \leq d_k \leq c_1 2^{kN},$$

where c_0 and c_1 are positive constants independent of k. The space S_k is spanned by tensor products of one-dimensional B-splines $M_j := M_j(k, \ell, N)$, normalised by

$$\sum_{j=1}^{d_k} M_j(x) = 1 \quad \text{for all } x \in \bar{Q}.$$

We identify $a = \sum_{j=1}^{d_k} a_j M_j \in S_k$ with the sequence of coefficients a_j and denote by $A_k : S_k \to \ell_1^{d_k}$ the corresponding isomorphism. Then

$$\|A_k : L_1 \to \ell_1^{d_k}\| \leq c2^{kN}, \quad \text{and} \quad \|A_k^{-1} : \ell_1^{d_k} \to L_1\| \leq c2^{-kN}. \tag{3.8}$$

In what follows we choose ℓ so large that $S_k \subset F_1^s$. Then we have the following well-known inequalities of Berstein and Jackson type respectively for F_1^s (see [8,12] and [9, Chapter 2]):

$$\|a\|_{F_1^s} \leq c2^{ks}\|a\|_{L_1}, \quad a \in S_k \tag{3.9}$$

and

$$\|I - P_k : F_1^s \to L_1\| \leq 2^{-ks} \quad \text{for some } P_k : F_1^s \to S_k, \tag{3.10}$$

where I is the embedding map from F_1^s to L_1.

From now on $c, c_0, c_1, \ldots$ will always denote positive constants which may depend upon s, t, p and N but not on k, m and n. Let $X_0 := P_0$, $X_k := P_k - P_{k-1}$, where the $P_k : F_1^s \to S_k$ are so chosen that the Jackson inequality (3.10) is satisfied. The embedding map

$$I : F_1^s \to F_1^t, \quad s > t \geq 0,$$

has a representation

$$I = \sum_{k=0}^{\infty} Y_k, \quad Y_k : F_1^s \to F_1^t, \tag{3.11}$$

with the Y_k defined by the diagram below:

$$Y_k : F_1^s \xrightarrow{X_k} L_1 \cap S_k \xrightarrow{A_k} \ell_1^{d_k} \xrightarrow{A_k^{-1}} L_1 \cap S_k \xrightarrow{I_k} F_1^t. \tag{3.12}$$

Here the A_k are defined as above and the I_k denote embedding maps. The following norm estimates hold for these operators:

$$\begin{aligned} &\|X_k\| \leq \|I - P_k\| + \|I - P_{k-1}\| \leq c_0 2^{-ks}, \\ &\|A_k\| \leq c_1 2^{kN}, \quad \|A_k^{-1}\| \leq c_2 2^{-kN}, \\ &\|I_k\| \leq c_3 2^{-kt}. \end{aligned} \tag{3.13}$$

Let

$$L_\sigma(S) := \sup_{n\in\mathbb{N}} n^{1/\sigma} e_n(S), \quad \sigma > 0;$$

this gives a quasi-norm on the entropy classes

$$\mathcal{L}_\sigma = \{S \in \mathcal{L}\colon \sup_{n\in\mathbb{N}} n^{1/\sigma} e_n(S) < \infty\},$$

where $\mathcal{L}$ denotes the ideal of all bounded linear operators between arbitrary Banach spaces. Then

$$L_\sigma(TY_k) \leq \|A_k X_k\| L_\sigma(TI_k A_k^{-1}) \leq c_4 2^{-ks+kN} L_\sigma(TI_k A_k^{-1}).$$

Choose σ so that $1/\sigma > 1 - 1/p$. Then

$$L_\sigma(TI_k A_k^{-1}) \leq c_5 2^{kN(1/\sigma+1/p-1)} \|TI_k A_k^{-1}\| \leq c_6 2^{kN(1/s+1/p-1)+k(t-N)} \|T\|.$$

This follows from the fact that for the dyadic entropy numbers of arbitrary operators S from ℓ_1^n into a Banach space E of type p we have the estimate (cf. [4])

$$e_k(S\colon \ell_1^n \to E) \leq c_7 \|S\| \left(\frac{\log(n/k+1)}{k}\right)^{1-1/p}, \quad 1 \leq k \leq n.$$

Combining the previous two estimates we have

$$L_\sigma(TY_k) \le c_8 2^{k(-s+t)+N(1/\sigma+1/p-1)k} \|T\|.$$

Write

$$TI = \sum_{k=0}^{m-1} TY_k + \sum_{k=m}^{\infty} TY_k.$$

Since L_σ is equivalent to an $r = r(\sigma)$ norm ([16, (6.2.5)]) we find that for $(1/\sigma) > (s-t)/N + 1 - 1/p$,

$$\begin{aligned} L_\sigma\left(\sum_{k=0}^{m-1} TY_k\right) &\le c_9\left(\sum_{k=0}^{m-1} L_\sigma^r(TY_k)\right)^{1/r} \\ &\le c_{10}\|T\|\left(\sum_{k=0}^{m-1} 2^{rk(-s+t+N(1/\sigma+1/p-1))}\right)^{1/r} \\ &\le c_{11}\|T\|2^{m(-s+t+N(1/\sigma+1/p-1))}. \end{aligned}$$

Hence

$$e_{2^{mN-1}}\left(\sum_{k=0}^{m-1} TY_k\right) \le c_{12}\|T\|2^{-mN(1-1/p+(s-t)/N)}.$$

To estimate the remainder $\sum_{k=m}^{\infty} TY_k$ we now choose σ so that $1 - 1/p < 1/\sigma < (s-t)/N + 1 - 1/p$. Then

$$\begin{aligned} L_\sigma\left(\sum_{k=m}^{\infty} TY_k\right) &\le c_{13}\|T\|\left(\sum_{k=m}^{\infty} 2^{rk(-s+t+N(1/\sigma+1/p-1))}\right)^{1/r} \\ &\le c_{14}\|T\|2^{-mN((s-t)/N-1/\sigma+1-1/p)}. \end{aligned}$$

Consequently

$$e_{2^{mN-1}}\left(\sum_{k=m}^{\infty} TY_k\right) \le c_{15}\|T\|2^{-mN((s-t)/N+1-1/p)}.$$

The additivity of the dyadic entropy numbers gives us

$$e_{2^{mN}}(TI) \le e_{2^{mN-1}}\left(\sum_{k=0}^{m-1} TY_k\right) + e_{2^{mN-1}}\left(\sum_{k=m}^{\infty} TY_k\right)$$
$$\le c_{16}\|T\|2^{-mN((s-t)/N+1-1/p)}.$$

Finally, the monotonicity of the entropy numbers gives the desired result

$$e_n(S) \le c_{17}\|T\|n^{-((s-t)/N+1-1/p)}.$$

Remarks

(i) The representation (3.11) of the embedding map

$$I = \sum_{k=0}^{\infty} Y_k, \quad Y_k : F_1^s \to F_1^t,$$

by the Y_k and the factorisation (3.12) via $\ell_1^{d_k}$ indicate that the entropy numbers of $S = TI$ can be estimated by those of an operator $R : \ell_1(\mathbb{N}) \to E$ with

$$\sup_{n\in\mathbb{N}} n^{1-1/p+(s-t)/N} e_n(R) < \infty$$

(see Corollary 3.2).

(ii) We have formulated our result for the cube Q only. However, extension theorems for Sobolev and Besov spaces show that it also holds for bounded domains in $\mathbb{R}^n$ with sufficiently smooth boundary.

That the conclusion in Proposition 3.4 is asymptotically optimal is shown by the following example.

Example 3.5 Let $s > t \ge 0$, $1 \le p \le 2$ and $s - t > N(1 - 1/p)$; let $I : W_1^s(Q) \to W_1^t(Q)$ be the natural Sobolev embedding. Then the dyadic entropy numbers of I satisfy

$$e_n(I) \asymp n^{-(s-t)/N}.$$

The estimate from above follows easily from Proposition 3.4. Indeed, choose s_0 such that $s_0 - t = N(1 - 1/p)$. We may then factorise I as follows:

$$\begin{array}{ccccc} W_1^s(Q) & & \xrightarrow{I} & & W_p^t(Q) \\ & I_0 \searrow & & \nearrow I_1 & \\ & & W_1^{s_0}(Q) & & \end{array}$$

Since W_p^t is of type p we conclude from Proposition 3.4 that

$$\sup_{n \in \mathbb{N}} n^{(s-t)/N} e_n(I) = \sup_{n \in \mathbb{N}} n^{(s-s_0)/N+1-1/p} e_n(I) < \infty.$$

The estimate from below is also well-known: see [9, Chap. 3].

A typical application of Proposition 3.4 is the following result about the eigenvalue distribution for operators acting on $L_1(\Omega)$ with values in $W_1^s(\Omega)$.

COROLLARY 3.6 *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^∞ boundary, and let $S: L_1(\Omega) \to L_1(\Omega)$ be an operator such that $S(L_1(\Omega)) \subset W_q^s(\Omega)$, where $1 \leq q < \infty$ and $s > 0$. Let the eigenvalues of S be denoted by $\lambda_n(S)$, in accordance with the convention of Theorem* 2.8. *Then*

$$\sup_{n \in \mathbb{N}} n^{\min\{1/2, 1-1/q\}+s/N} |\lambda_n(S)| < \infty.$$

Proof By the closed graph theorem we may factorise S as follows:

$$\begin{array}{ccccc} L_1(\Omega) & & \xrightarrow{S} & & L_1(\Omega) \\ & S_0 \searrow & & \nearrow I & \\ & & W_q^s(\Omega) & & \end{array}$$

Here I is the natural embedding, which is compact, and S_0 is bounded; note that S is compact, and so it makes sense to discuss the eigenvalues of S. The principle of related operators [16] shows that the non-zero eigenvalues of S coincide with those of the operator T which has the factorisation

$$\begin{array}{ccc} W_q^s(\Omega) & \xrightarrow{T} & W_q^s(\Omega) \\ I_0 \downarrow & \searrow I & \uparrow S_0 \\ W_1^s(\Omega) & \xrightarrow{I_1} & L_1(\Omega) \end{array}$$

Since $W_q^s(\Omega)$ is of type $\min\{2, q\}$, Proposition 3.4 shows that

$$\sup_{n\in\mathbb{N}} n^{\min\{1/2,1-1/q\}+s/N} e_n(S_0 I) < \infty,$$

and hence the same estimate holds for $e_n(T)$. The desired result now follows from the well-known inequality [2]:

$$|\lambda_n(S)| = |\lambda_n(T)| \leq \sqrt{2} e_n(T), \quad n \in \mathbb{N}.$$

Remark The eigenvalue estimate is optimal. This may be seen by considering operators constructed in a similar fashion to the corresponding ones in [3].

4 ENTROPY OF CONVEX HULLS IN BANACH SPACES OF TYPE p

In the final section we show how the rate of decay of entropy numbers $\varepsilon_n(X)$ of a precompact subset X of a Banach space E of type p influences the rate of decay of the dyadic entropy numbers $e_n(co(X))$ of the (symmetric) absolutely convex hull $co(X)$ of X. We obtain optimal results which complement work of [5] and refine a Hilbert space theorem of [7].

PROPOSITION 4.1 *Let E be a Banach space of type $p > 1$, and let X be a precompact subset of E such that*

$$\sup_{n\in\mathbb{N}} n^\sigma \log^{-\gamma}(n+1)\varepsilon_n(X) < \infty \tag{4.1}$$

for some $\sigma > 0$ and $\gamma \geq 0$. Then the dyadic entropy numbers $e_n(co(X))$ of the absolutely convex hull of X satisfy

$$\sup_{n\in\mathbb{N}} n^{1-1/p+\sigma} \log^{-\gamma}(n+1) e_n(co(X)) < \infty. \tag{4.2}$$

Equivalently, if the covering number $N(X,\varepsilon)$ of X satisfies

$$N(X,\varepsilon) \leq c_0\varepsilon^{-1/\sigma}\log^{\gamma/\sigma}(1/\varepsilon) \quad \textit{as } \varepsilon \downarrow 0, \tag{4.3}$$

then

$$\log N(co(X),\varepsilon) \leq c_1\varepsilon^{-1/(1-1/p+\sigma)}\log^{\gamma}(1/\varepsilon). \tag{4.4}$$

Proof Define $T\colon \ell_1(X) \to E$ by

$$Tf_t = t \quad \textit{for } t \in X,$$

where (f_t) is the canonical unit basis of $\ell_1(X)$. Then

$$(T'a)(t) = \langle t, a\rangle \quad \text{for } t \in X,\ a \in E'.$$

Since

$$|(T'a)(t) - (T'a)(s)| = |\langle t-s, a\rangle| \leq \|t-s\|\,\|a\|,$$

we see that the modulus of continuity of T' satisfies

$$\omega(T',\delta) = \sup_{\|a\|\leq 1}\sup\{|(T'a)(t) - (T'a)(s)|\colon s,t \in X,\ \|s-t\| \leq \delta\} \leq \delta.$$

Hence

$$\mathrm{Lip}(T) = \max\left\{\|T\|, \sup_{\delta>0}\omega(T',\delta)/\delta\right\} \leq \max\left\{\sup_{t\in X}\|t\|, 1\right\} < \infty.$$

This means that $T \in \mathcal{L}\mathrm{ip}(\ell_1(X), E)$. Since E is a Banach space of type p and $e_n(co(X)) \leq e_n(T)$, the desired estimate (4.2) follows immediately from Theorem 3.1.

We observe that this result is optimal. To see this, let $1 < p \leq 2$, let (f_n) be the canonical basis for ℓ_p and take

$$X = \{n^{-\sigma}\log^{\gamma}(n+1)f_n\colon n \in \mathbb{N}\} \subset \ell_p.$$

Then

$$\sup_{n\in\mathbb{N}} n^{\sigma}\log^{-\gamma}(n+1)\varepsilon_n(X) < \infty,$$

and just as in [3] we have

$$e_n(co(X)) \asymp n^{-(1-1/p)-\sigma} \log^{\gamma}(n+1), \quad n \in \mathbb{N}.$$

Remark Proposition 4.1 provides a refinement of the Hilbert space result of Dudley [7]. For if we take $p = 2$, which includes the Hilbert space case, and set $\gamma = 0$, then the proposition tells us that if

$$\sup_{n \in \mathbb{N}} n^{\sigma} \varepsilon_n(X) < \infty,$$

then

$$\sup_{n \in \mathbb{N}} n^{1/2+\sigma} e_n(co(X)) < \infty;$$

or in terms of the covering number,

$$N(X, \varepsilon) \leq c_0 \varepsilon^{-1/\sigma} \quad \text{as } \varepsilon \to 0+$$

implies that

$$\log N(co(X), \varepsilon) \leq c_1 \varepsilon^{-1/(1/2+\sigma)}.$$

This is what Dudley proved, for Hilbert spaces.

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