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Factoring Linear Differential Operators on Measure Chains

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We extend to measure chains Abel's theorem, the existence of a set of positive Wronskians, and the Polya and Trench factorizations of disconjugate linear Δ-differential operators.

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1 INTRODUCTION

In this paper we are concerned with the extensions to measure chains of some of the fundamental ideas in the theory of nth order linear differential equations on the real line. In particular our object is to show that the factorizations of disconjugate differential operators developed by Polya [10] and extended to difference equations by Hartman [4] is valid also in the measure chain setting. Similarly, we show that the canonical factorization due to Trench [11] and extended to difference equations by Krueger [7] is also valid for measure chains. Let $\mathbb{R}$ be the set

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of real numbers and let $\mathbb{T}$ be a closed subset of $\mathbb{R}$. Let $\mathbb{T}^{\kappa}$ denote $\mathbb{T}\backslash\sup\mathbb{T}$ and let $\mathbb{T}^{\kappa^n}$ be defined inductively by $\mathbb{T}^{\kappa^n} = (\mathbb{T}^{\kappa^{n-1}})^{\kappa}$. Define $\sigma : \mathbb{T}^{\kappa} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}$. σ is called the right jump operator. Similarly, the left jump operator ρ is defined by $\rho(t) = \sup\{s \in \mathbb{T}: s < t\}$. The so-called Δ-derivative of a function $f: \mathbb{T} \to \mathbb{R}$ is denoted by f^{Δ} and is defined by

$$f^{\Delta}(t) = \lim_{s\to t}\frac{f(\sigma(s)) - f(t)}{\sigma(s) - t},$$

or equivalent by

$$f^{\Delta}(t) = \lim_{s\to t}\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

See the papers by Erbe and Hilger [3] and Hilger [5] for details regarding measure chains and Δ-derivatives. We use the standard notation f^{σ} for the function $f: t \longmapsto f(\sigma(t))$ defined on $\mathbb{T}^{\kappa}$. Throughout this paper I will denote an interval of $\mathbb{T}$ which is just an interval of $\mathbb{R}$ intersected with $\mathbb{T}$.

The existence, uniqueness, and extension theorems for solutions of initial value problems as well as the continuous dependence of solutions on initial data, all familiar from differential equations on $\mathbb{R}$, are also valid for differential equations on measure chains. Again see [3] and [5] and the book [8] for details.

2 ABEL'S THEOREM

Consider the nth order Δ-differential equation

$$L_n[y] := y^{\Delta^n} + \sum_{i=1}^{n} a_i y^{\Delta^{n-i}}, \tag{1}$$

where y^{Δ^j} denotes the jth Δ-derivative of the real-valued function y and each a_i is a real-valued, right-dense continuous function on $\mathbb{T}$. Recall that a function is right-dense continuous on the interval I of $\mathbb{T}$ if it is continuous at right-dense points of I and the left-hand limit exist at left-dense points. Note that σ is right-dense continuous. Let I be an interval of $\mathbb{T}^{\kappa^n}$. Let $w_n(t) = w(x_1, x_2, \ldots, x_n)(t) = \det[x_j^{\Delta^{i-1}}]$. A typical term of this Wronskian determinant is $\epsilon(i)x_{i_1}x_{i_2}^{\Delta}x_{i_3}^{\Delta^2}\cdots x_{i_n}^{\Delta^{n-1}}$, where $\epsilon(i)$ is the sign of the permutation $i=(i_1,i_2,\ldots,i_n)$ of $(1,2,\ldots,n)$. Applying the product rule $(uv)^{\Delta}=u^{\Delta}v+u^{\sigma}v^{\Delta}$ for the Δ-derivative to this typical term of w_n gives the terms

$$\begin{aligned}&\epsilon(i)x_{i_1}^{\Delta}x_{i_2}^{\Delta}x_{i_3}^{\Delta^2}\cdots x_{i_n}^{\Delta^{n-1}}+\epsilon(i)x_{i_1}^{\sigma}x_{i_2}^{\Delta^2}x_{i_3}^{\Delta^2}\cdots x_{i_n}^{\Delta^{n-1}}\\&+\epsilon(i)x_{i_1}^{\Delta\sigma}x_{i_2}^{\Delta^2\sigma}x_{i_3}^{\Delta^3}x_{i_4}^{\Delta^3}\cdots x_{i_n}^{\Delta^{n-1}}+\cdots\\&+\epsilon(i)x_{i_1}^{\sigma}x_{i_2}^{\Delta\sigma}x_{i_3}^{\Delta^2\sigma}\cdots x_{i_{n-1}}^{\Delta^{n-2}\sigma}x_{i_n}^{\Delta^n}.\end{aligned}$$

All but the last term in this sum contains a pair of factors of the form $x_{i_j}^{\Delta^j}x_{i_{j+1}}^{\Delta^j}$. This corresponds to the determinant of a matrix with two equal rows which gives 0 for that determinant. Hence w_n^{Δ} is the sum over all terms of the form $\epsilon(i)x_{i_1}^{\sigma}x_{i_2}^{\Delta^{\sigma}}x_{i_3}^{\Delta^{2\sigma}}\cdots x_{i_{n-1}}^{\Delta^{n-2\sigma}}x_{i_n}^{\Delta^n}$. Since $u^{\sigma}=u+u^{\Delta}\mu$ where $\mu(t)=\sigma(t)-t$, then the typical term has the form

$$\epsilon(i)(x_{i_1}+x_{i_1}^{\Delta}\mu)(x_{i_2}^{\Delta}+x_{i_2}^{\Delta^2}\mu)\cdots(x_{i_{n-1}}^{\Delta^{n-2}}+x_{i_{n-1}}^{\Delta^{n-1}}\mu)x_{i_n}^{\Delta^n}.$$

When this product is expanded, many terms contain products of the form $x_{i_j}^{\Delta^j}x_{i_{j+1}}^{\Delta^j}$. These sum to zero since again they correspond to the determinant of a matrix with two equal rows. Hence there remains only the sum of the terms

$$\begin{aligned}&\epsilon(i)x_{i_1}x_{i_2}^{\Delta}\cdots x_{i_{n-1}}^{\Delta^{n-2}}x_n^{\Delta^n}+\epsilon(i)x_{i_1}x_{i_2}^{\Delta}\ldots x_{i_{n-2}}^{\Delta^{n-3}}\left(x_{i_{n-1}}^{\Delta^{n-2}}\mu\right)x_n^{\Delta^n}+\cdots\\&+\epsilon(i)x_{i_1}x_{i_2}^{\Delta^2}x_{i_3}^{\Delta^3}\cdots x_{i_{n-1}}^{\Delta^{n-1}}x_n^{\Delta^n}\mu^{n-2}\\&+\epsilon(i)x_{i_1}^{\Delta}x_{i_2}^{\Delta^2}x_{i_3}^{\Delta^3}\cdots x_{i_{n-1}}^{\Delta^{n-1}}x_n^{\Delta^n}\mu^{n-1}.\end{aligned}$$

Now by (1), $x_k^{\Delta^n}=-\sum_{i=1}^{n}a_ix_k^{\Delta^{n-i}}$. Hence w_n^{Δ} is the sum of n determinants with typical columns

$$\begin{bmatrix}x_k\\x_k^{\Delta}\\\vdots\\-a_1x_k^{\Delta^{n-1}}\end{bmatrix},\begin{bmatrix}x_k\\x_k^{\Delta}\\\vdots\\x_k^{\Delta^{n-3}}\\\mu x_k^{n-1}\\-a_2x_k^{\Delta^{n-2}}\end{bmatrix},\begin{bmatrix}x_k\\x_k^{\Delta}\\\vdots\\x_k^{\Delta^{n-4}}\\\mu x_k^{n-2}\\\mu x_k^{\Delta^{n-1}}\\-a_3x_k^{\Delta^{n-3}}\end{bmatrix},\ldots,\begin{bmatrix}\mu x_k^{\Delta}\\\mu x_k^{\Delta^2}\\\vdots\\\mu x_k^{\Delta^{n-1}}\\\mu x_k^{n-1}\\-a_nx_k\end{bmatrix},$$

respectively. To evaluate these determinants first interchange rows to give

$$\begin{aligned} w_n^{\Delta} &= -a_1 w_n + a_2 \mu w_n - a_3 \mu^2 w_n + \cdots \\ &\quad + (-1)_i a_i \mu^i w_n + \cdots + (-1)^n \mu^{n-1} w_n \\ &= -P^*(-\mu) w_n, \end{aligned}$$

where $P^*(s) = \sum_{i=1}^{N} a_i s^{i-1}$. Thus we have proved

THEOREM 1 (Abel's Theorem) *Let $x_1, x_2, \ldots, x_n$ be solutions to (1). Let $w_n = w(x_1, x_2, \ldots, x_n)$ be the Wronskian of these solutions. Then*

$$w_n^{\Delta}(t) = -P_n^*(\mu(t)) w_n(t), \tag{2}$$

where $P^*(s) = \sum_{i=1}^{N} a_i s^{i-1}$ and $\mu(t) = \sigma(t) - t$ is the graininess.

Observe that if $\mathbb{T}$ is an interval of $\mathbb{R}$, then $\mu(t) = 0$ for all $t \in I \cap \mathbb{T}^{\kappa}$. Thus $-P_n^*(-\mu) = -a_1$. Hence (2) becomes the familiar $w_n' = -a_1 w_n$ for differential equations. For $\mathbb{T} = \mathbb{Z}$, the set of integers, (1) is the difference equation $\Delta^n x_t + \sum_{i=1}^{n} a_i \Delta^{n-i} x_t = 0$. When written as an n-term recurrence relation, the coefficient b_n of x_t is $b_n = (-1)^n + \sum_{i=1}^{n} a_i (-1)^{n+i}$, and thus $(-1)^n b_n = 1 + \sum_{i=1}^{n} a_i (-1)^i = 1 - \sum_{i=1}^{n} a_i (-1)^{i-1} = 1 - P_n^*(-1)$. In order for this recurrence relation form of the difference equation to be disconjugate, it is necessary that $(-1)^n b_n > 0$, i.e., $1 - P_n^*(-1) > 0$. But in this case $\mu(t) = 1$ for all t, so $-\mu(t) P_n^*(-\mu(t)) = -\sum_{i=1}^{n} a_i (-1)^{i-1}$. Hence the condition $(-1) b_n > 0$ can be written as $1 - P_n^*(-1) > 0$. In the case of an arbitrary measure chain $\mathbb{T}$, this condition would be

$$1 - \mu(t) P_n^*(-\mu(t)) > 0, \tag{3}$$

where $\mu(t) = \sigma(t) - t$. The general exponential function $e_h(\alpha, \tau)$, as defined by Hilger [5,6], is given by $e_h(\alpha, \tau)(t) = \exp(\int_{\tau}^{t} \xi_h(\alpha(s))\,\mathrm{d}s)$ where

$$\xi_h(z) = \begin{cases} \log(zh + 1), & h > 0, \\ z, & h = 0. \end{cases}$$

Thus the solution of (3) which equals 1 when $t = \tau$ is given by $\exp(\int_{\tau}^{t} \xi_{\mu(s)} (-P_n(\mu(s)))\,\mathrm{d}s)$. In this case, $zh + 1 = -\mu(s) P_n^*(\mu(s)) + 1$. Then (3) implies that $\log(zh + 1) = \log(1 - \mu(s) P_n^*(\mu(s)))$ is real-valued, and hence $e_h(\alpha, \tau) > 0$. Then $w_n(t) = w_n(\tau) e_{\mu(t)}(-\mu(t) P_n^*(-\mu(t))$ which is an extension of Abel's formula to measure chains under the assumption (3).

3 POLYA FACTORIZATION

In 1922 Georg Polya [10] showed that if an nth order linear differential equation

$$L[y] := y^{(n)} + \sum_{i=1}^{n} a_i y^{(n-i)} = 0$$

is disconjugate on an interval I or $\mathbb{R}$, then for a given point $a \in I$, L has the form

$$L[y] = \alpha_n(\alpha_{n-1}(\cdots \alpha_1(\alpha_0)' \cdots)')'$$

on $I \cap (a, \infty)$ where each $\alpha_i > 0$ on $I \cap (a, \infty)$. In this section we show that the analogous result holds on measure chains. We begin with the definition of disconjugacy on an interval I of a measure chain $\mathbb{T}$.

DEFINITION *We say that the function $x : I \to \mathbb{R}$, where I is an interval of the measure chain $\mathbb{T}$, has a* generalized zero *at $\tau \in I$ in case either* (i) $x(\tau) = 0$, *or* (ii) $\tau > \rho(\tau) \in I$ *and* $x(\rho(\tau))x(\tau) < 0$. *We say that x has a* generalized zero of multiplicity k *at $\tau \in I$ in case either* (i) $x^{\Delta^j}(\tau) = 0$ *for* $0 \le j \le k-1$ *and* $x^{\Delta^k}(\tau) \neq 0$, *or* (ii) $\tau > \rho(\tau) \in I$, $x^{\Delta^j}(\tau) = 0$ *for* $0 \le j \le k-2$, *and*

$$(-1)^k x(\rho(\tau)) x^{\Delta^{k-1}}(\tau) > 0. \tag{4}$$

We say that the operator L_n is disconjugate *on the interval I of $\mathbb{T}$ in case each nontrivial solution y* of *$L_n[y] = 0$ has fewer that n generalized zeros counting multiplicity in I.*

The first step in the Polya factorization is to get a collection of positive Wronskians as we do next. In Theorem 4 we show that with the fundamental set of solutions $x_1, \ldots, x_n$ that will be chosen presently, $w_k(t)$ can neither equal zero nor change sign for $t > \sigma^{n-2}(a)$ when (1) is disconjugate. Then by successively replacing x_i by $-x_i$ as needed, we could inductively choose solutions $x_1, x_2, \ldots, x_n$ so that each $w_k(t)$ would be positive. This requires that $t > \sigma^{n-2}(a)$. To extend this positivity of the $w_k(t)$ to the entire interval in the case of a compact interval, we must show, as is evident in the proof of Theorem 5, that the fundamental set of solutions $x_1, x_2, \ldots, x_n$ defined next gives the positivity of the Wronskians $w_k(t)$. Throughout the remainder of this section, $x_1, \ldots, x_n$ will be the fundamental set of solutions of (1) satisfying the initial conditions

$$x_i^{\Delta^{j-1}}(a) = \begin{cases} (-1)^{i-1}, & j = n+1-i, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Also, w_k will denote the Wronskian determinant of $x_1, \ldots, x_k$. When first considering Lemma 2, we assumed the result would be found in the literature. However, our search produced no reference to this result, so we have included the statement and proof here.

LEMMA 2 *Let a be some point in $I \cap \mathbb{T}^{\kappa^n}$ and let $x_1, x_2, \ldots, x_n$ be n solutions of (1) satisfying (5). Let w_k denote the Wronskian determinant of $x_1, x_2, \ldots, x_k$. If $\sigma(a) = a$, then w_k has a zero of multiplicity $k(n-k)$ at a and, for small $\epsilon > 0$, $w_k(t) > 0$ for $0 < t - a < \epsilon$.*

Proof Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be integers with $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_k \le n-1$, and let $D(\lambda_1, \ldots, \lambda_k)$ denote the determinant of the $k \times k$ matrix whose ith row is the row of derivatives $x_1^{(\lambda_i)}, \ldots, x_k^{(\lambda_i)}$. Also let $D^{(j)}(\lambda_1, \ldots, \lambda_k)$ denote the jth derivative of $D(\lambda_1, \ldots, \lambda_k)$ with respect to t. For $1 \le j \le n-k$, $w_k^{(j)}$ is the sum of determinants of the form $D(\lambda_1, \ldots, \lambda_k)$ where $\Lambda := \lambda_1 + \lambda_2 + \cdots + \lambda_k = j + k(k-1)/2$ since each differentiation increases Λ by 1. In order to have a determinant which is not zero at $t = a$, we must have the term $D(n-k, n-k+1, \ldots, n-2) = 1$. This determinant arises if we avoid differentiating the kth row of $D(\lambda_1, \ldots, \lambda_{k-1}, n-1)$ and will first occur after $k(n-k)$ differentiations. Note that for $j = n-k-1$, $w_k^{(j)}$ involves

$$D(\lambda_1, \ldots, \lambda_{k-1}, n) = -\sum_{i=1}^{n-1} a_{n-i} D((\lambda_1, \ldots, \lambda_{k-1}, i).$$

$D(\lambda_1, \ldots, \lambda_{k-1}, n)$ need not be differentiable, but it is the sum of terms each being the product of the right-dense continuous coefficient $-a_{n-i}$ and the differentiable function $D(\lambda_1, \ldots, \lambda_{k-1}, i)$. The order of the zero at a of these determinants must be larger than the order of the zero at a of the other determinants $D(\mu_1, \ldots, \mu_k)$ in the sum in (3) since $\mu_1 + \mu_2 + \cdots + \mu_k = \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + n > \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + i$. Since each application of the derivative, or derivative of each $D(\lambda_1, \ldots, \lambda_{k-1}, i)$

in the sum in (3), increases by 1 the index of the derivative in one row of the determinant, we arrive at the first nonzero determinant $D(n-k, n-k+1, \dots, n-2)$ after each of the k rows has been moved down $n-k$ positions, i.e., $k(n-k)$ differentiation operations have been performed. The coefficient of $D(n-k,\ n-k+1, \dots, n-2)$ in this final sum is a positive integer, so $w_k(t)$ must be positive for $0 < t-a < \epsilon$ when ϵ is small enough.

LEMMA 3 *Let a be some point in $I \cap \mathbb{T}^{\kappa^n}$ and let $x_1, \dots, x_n$ be the solutions to* (1) *satisfying* (5). *Suppose that $a < \sigma(a) < \cdots < \sigma^r(a) = \sigma^{r+1}(a)$ for some integer r, $1 \le r \le n-1$.*

(i) *If $n-j > r$, then w_{n-j} has a zero of order j at a and $w_{n-j}(\sigma^j(a)) > 0$.*
(ii) *If $n-j \le r$, then w_{n-j} has a zero of order $(n-j)(j-r)$ at $\sigma^r(a)$ and $w_{n-j}(t) > 0$ for $0 < t-\sigma^r(a) < \epsilon$ for some small ϵ.*

Proof Suppose that $a<\sigma(a)<\cdots<\sigma^r(a)=\sigma^{r+1}(a)$ where $1 \le r \le n$. Recall that for an $n \times n$ matrix $A=[a_{ij}]$, the antidiagonal is the set of entries a_{ij} with $i+j=n+1$. We say the matrix A is lower anti-triangular in case $a_{ij}=0$ if $i+j<n+1$, i.e., if all entries above the antidiagonal are 0. Also we say that such a matrix A is alternately signed if

$$(-1)^j a_{ij} \begin{cases} = 0, & \text{for } i < n+1-j, \\ > 0, & \text{for } i = n+1-j, \\ \ge 0, & \text{for } i > n+1-j. \end{cases}$$

Let M_a denote the $n \times n$ matrix of initial values at a, i.e., $M=[m_{ij}]$, where $m_{ij}=(-1)^{j-1}\delta_{j,n+1-i}$. We now use the equation $x^\sigma(t)=x(t)+x^\Delta(t)\mu(t)$ to get the $(n-1)\times(n-1)$ matrix $M_{\sigma(a)}$ of initial values for $x_i^{\Delta^j}(\sigma(a))$, $0 \le j \le n-2$. Note that this is again an alternately signed lower anti-triangular matrix (ASLAT). If we continue using $x^\sigma(t)=x(t)+x^\Delta(t)\mu(t)$ for $t=\sigma(a), t=\sigma^2(a), \dots, t=\sigma^{r-1}(a)$, we will generate the r initial value matrices $M_{\sigma^i(a)}$, $1 \le i \le r$. Each of these is again ASLAT. Now $w_{n-j}(\sigma^j(a)) = M_{\sigma^i(a)} > 0$ for $1 \le j \le r$. Thus, if $r=n-1$, each w_j is positive at some point. Suppose that $r \le n-2$. Since $\sigma^r(a)$ is right-dense and since the matrix of initial values $M_{\sigma^r(a)}$ is ASLAT, the argument of the previous lemma, with n there replaced by $n-r$ and k replaced by $n-j$, applies to give a point $t > \sigma^r(a)$, where $w_{n-j}(t) > 0$ for $j \ge n-r$. The order of the zero of $w_{n-j}(t)$ at $\sigma^r(a)$ is $(n-j)(n-r-(n-j)) = (n-j)(j-k)$ by Lemma 2.

THEOREM 4 *Suppose* (1) *is disconjugate on the interval* $I \subseteq \mathbb{T}$ *and* (3) *holds. Let* $a \in I$. *Then there are solutions* $x_1, x_2, \dots, x_n$ *of* (1) *so that the Wronskians* $w_i := w(x_1, x_2, \dots, x_i)$ *satisfy*

$$w_i(t) > 0 \quad \textit{for } t \in (\sigma^{n-i-1}(a), \infty) \cap \mathbb{T}^{\kappa^i} \cap I, \; 1 \le i \le n-1. \tag{6}$$

Proof Let $x_1, x_2, \dots, x_n$ be the collection of solutions of (1) defined in (5). First observe that $w_1 = x_1$ has a zero of multiplicity $n-1$ at a, so that x_1 can neither equal zero nor change sign for $t > \sigma^{n-1}(a)$ since otherwise x_1 would have n generalized zeros. But $x_1^{\Delta^{n-1}}(a) = 1$, so $x(t) > 0$ for $t > \sigma(a)$. Next consider w_k for $1 < k < n$. If $w_k(t_0) = 0$ for some $t_0 > \sigma^{n-k-1}(a)$, then there are constant $c_1, c_2, \dots, c_k$, not all zero, so that $y := c_1x_1 + c_2x_2 + \cdots + c_kx_k$ has a zero of multiplicity k at t_0. But y has a zero of multiplicity $n-k$ at a giving y a total of at least n zeros. Since y is nontrivial, that contradicts the disconjugacy of L_n. Hence $w_k(t) \neq 0$ for $t > \sigma^{n-k}(a)$.

Next suppose there are points $t_1 < t_2$ in $I \cap (\sigma^{n-k-1}(a), \infty) \cap \mathbb{T}^{\kappa^k}$ so that $w_k(t_1)w_k(t_2) < 0$. Let $t_0 = \sup\{t \in I\colon w_k(t_1)w_k(s) > 0 \text{ for } t_1 \le s \le t\}$. Then $t_0 \le t_2$. Since $w_k(t_0) \neq 0$, $w_k(t_1)w_k(t_0) > 0$. If t_0 were right-dense, then we could find a sequence $s_m \to t_0^+$ so that $w_k(t_1)w_k(t_m) < 0$. But $w_k(t_1)w_k(s_m) \to w_k(t_1)w_k(t_0) > 0$, and that is impossible. Hence t_0 must be right-scattered. Next let x be the solution to the BVP $L_n[x] = 0$, $x^{\Delta^i}(a) = 0$ for $0 \le i \le n-k-1$, $x(t_0) = 1$, and $x^{\Delta^i}(\sigma(t_0)) = 0$ for $0 \le i \le k-1$. Then $x = c_1x_1 + c_2x_2 + \cdots + c_kx_k$ since x has a zero of multiplicity $n-k$ at a. Let W be the $(k+1) \times (k+1)$ matrix with first column $[x(t_0), x(\sigma(t_0)), \dots, x^{\Delta^{k-1}}(\sigma(t_0))]^T = [0, 0, \dots, 0, x^{\Delta^{k-1}}(\sigma(t_0))]$ and column $j+1$ equal to $[x_j(t_0), x_j(\sigma(t_0)), x_j^{\Delta}(\sigma(t_0)), \dots, x_j^{\Delta^{k-1}}(\sigma(t_0))]$ for $1 \le j \le k$. Then $0 = \det W = w_k(\sigma(t_0)) + (-1)^k x^{\Delta^{k-1}}(\sigma(t_0)) \det W_\sigma$ where W_σ is the $k \times k$ matrix with jth column $[x_j(t_0), x_j(\sigma(t_0)), x_j^{\Delta}(\sigma(t_0)), \dots, x_j^{\Delta^{k-2}}(\sigma(t_0))]^T$ for $1 \le j \le k$. For function y on $\mathbb{T}$ which is Δ-differentiable at t we have $y^\sigma(t) = y(t) + y^\Delta(t)\mu(t)$. Using this identity on the columns of W_σ we get that the jth column of W_σ has the form $[x_j(t_0), x_j(t_0) + x_j^{\Delta}(t_0)\mu(t_0), \dots, x_j^{\Delta^{k-2}} + x_j^{k-1}(t_0)\mu(t_0)]^T$ for $1 \le j \le k$. But $\mu(t_0) > 0$, so elementary row operations applied to W_σ give $\det W_\sigma = (\mu(t_0))^{k-1}w_k(t_0)$. Then $0 = \det W = w(\sigma(t_0)) + (-1)^k x^{\Delta^{k-1}}(\sigma(t_0)) \times (\mu(t_0))^{k-1} 2_\kappa(t_0)$. Now $w_k(t_0)w_k(\sigma(t_0)) < 0$, so $(-1)^k x^{\Delta^{k-1}}(\sigma(t_0))x(t_0) > 0$ since $x(t_0) = 1$. Then (4) holds with $\tau = \sigma(t_0)$, and x has a generalized zero of multiplicity k at t_0. This contradicts the disconjugacy of L_n on I. We conclude that $w_k(t)$ does not have a change of sign for $t \in (\sigma^{n-k-1}(a), \infty) \cap I \cap \mathbb{T}^{\kappa^k}$. It remains only to show that $w_k(t) > 0$ for some t. But this follows from Lemmas 2 and 3.

THEOREM 5 *Suppose* (1) *is disconjugate on* $I = [a, \sigma^n(b)]$, *a compact interval of the measure chain* $\mathbb{T}$. *Then there are n solutions* $y_1, y_2, \dots, y_n$ *of* (1) *so that the Wronskians* $w_k(t) := w(y_1, y_2, \dots, y_k)$ *are positive on* $[a, b]$.

Proof Let $y_{i,\epsilon}$, $1 \le i \le n$, be the solution of the BVP

$$y_{i,\epsilon}^{\Delta^j}(a) = \begin{cases} (-1)^{i-1} \epsilon^{n-i-j}/(n-i-j)!, & 0 \le j \le n-i, \\ 0, & n-i < j \le n-1, \end{cases}$$

where $\epsilon > 0$. Then $w_{k,\epsilon}(t) := w(y_{1,\epsilon}(t), y_{2,\epsilon}(t), \dots, y_{k,\epsilon}(t)) > 0$ for $t \ge \sigma^{n-k-1}(a)$ since $w_{k,\epsilon}(t) \to w_k(t)$ as $\epsilon \to 0^+$. Here w_k and x^k are as defined in Theorem 4. We look first at the case $k = 2$ which gives insight into the general case. Here we must show that $w_{2,\epsilon}$ for $a \le t \le \sigma^{n-3}(a)$. Note first that $w_{2,\epsilon}(a) = (\epsilon^{2n-4})/(n-1)!(n-2)! > 0$. For convenience of notation we denote

$$w_{2,\epsilon(t)} = \begin{vmatrix} y_{1,\epsilon}(t) & y_{2,\epsilon}(t) \\ y_{1,\epsilon}^{\Delta}(t) & y_{2,\epsilon}^{\Delta}(t) \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} y(t) \\ y^{\Delta}(t) \end{vmatrix}.$$

Then

$$\begin{aligned} w_{2,\epsilon}(\sigma^2(a)) &= \begin{vmatrix} y(\sigma(a)t) \\ y^{\Delta}(\sigma(a)) \end{vmatrix} = \begin{vmatrix} y(a) \ + y^{\Delta}(a)\mu(a) \\ y^{\Delta}(a) + y^{\Delta^2}(a)\mu(a) \end{vmatrix} \\ &= \begin{vmatrix} y(a) \\ y^{\Delta}(a) \end{vmatrix} + \mu(a) \begin{vmatrix} y(a) \\ y^{\Delta^2}(a) \end{vmatrix} + \mu^2(a) \begin{vmatrix} y^{\Delta}(a) \\ y^{\Delta^2}(a) \end{vmatrix} \\ &> 0 \end{aligned}$$

since each of the determinants is positive.

$$w_{2,\epsilon}(\sigma^2(a)) = w_{2,\epsilon}(\sigma(a)) + \mu(\sigma(a)) \begin{vmatrix} y(\sigma(a)) \\ y^{\Delta}(\sigma(a)) \end{vmatrix} + \mu^2(a) \begin{vmatrix} y^{\Delta}(\sigma(a)) \\ y^{\Delta^2}(\sigma(a)) \end{vmatrix},$$

which can be written as a linear combination, with coefficients products and powers of $\mu(a)$ and $\mu(\sigma(a))$, of the determinants of 2×2 submatrices of the matrix E whose ith column is the column of initial values of $y_{i,\epsilon}$ at a. All these determinants and indeed the determinant of each of $k \times k$ submatrices obtained by deleting $n-k$ rows and the last $n-k$ columns of E is positive. Hence we get that $w_{2,\epsilon}(\sigma^j(a)) > 0$ for $0 \le j \le n-4$. In the case of a general k, $w_k(\sigma^j(a))$ is a linear combination of positive determinants of $k \times k$ submatrices of E with coefficients $\mu(\sigma^i(a))$, $0 \le i \le j-1$, and products and powers of these nonnegative numbers. Hence $w_{k,\epsilon}(\sigma^j(a)) > 0$ for $0 \le j \le n-k-2$, and thus w_k is positive on $[a, b]$. To show that the matrix E has this positive submatrix determinant property, recall that the polynomials $1, t, t^2/2!, t^3/3!, \ldots, t^{n-1}/(n-1)!$ form a Descartes system on $(0, \infty)$ for the disconjugate differential equation $\mathrm{d}^n y/\mathrm{d}t^n = 0$. See Coppel [2]. That means that if k columns $1 \le j_1 < j_2 < \cdots < j_k \le n$ are selected from the Wronskian matrix W of this set of solutions, and the bottom $n-k$ rows are deleted, the resulting submatrix has a positive determinant. Now consider the transpose W^T which has the property that if the k rows $1 \le i_1 < i_2 < \cdots < i_k \le n$ are selected and the last $n-k$ columns are deleted, the resulting matrix has a positive determinant. Next reverse the order of the rows placing row $n-i-1$ as the ith row for $1 \le i \le n$ to obtain a new matrix $\hat{W}$. Next multiply the even numbered columns of $\hat{W}$ by -1 to give the matrix E with $\epsilon = t$. Consider the submatrix M formed by deleting $n-k$ rows and the last $n-k$ columns of E. The determinant of the corresponding submatrix in W^T has undergone $k(k-1)/2$ sign changes in the transformation from W^T to $\hat{W}$ and $\lfloor k/2 \rfloor$ changes in going from $\hat{W}$ to M. But $(k(k-1)/2) + \lfloor k/2 \rfloor$ is even, so $\det S > 0$ as claimed.

THEOREM 6 (Polya Factorization) *Let $L_n x = x^{\Delta^n} + a_1 x^{\Delta^{n-1}} + \cdots + a_{n-1} x + a_n$ where each a_i is right-dense continuous on the interval I of the measure chain $\mathbb{T}$. Suppose $L_n x = 0$ is disconjugate on I. Then L_n is given by*

$$L_n x = \alpha_n(\alpha_{n-1}(\alpha_{n-2}(\cdots \alpha_2(\alpha_1(\alpha_0 x)^{\Delta})^{\Delta} \cdots)^{\Delta})^{\Delta})^{\Delta}, \tag{7}$$

where each $\alpha_i(t)$ is right-dense continuous on (a, b) and positive for $t \in (\sigma^{n-1}(a), b)$ if $\rho^n(b) \in I$.

Proof Suppose $x_1, x_2, \ldots, x_n$ form a linearly independent set of solutions to $L_n x = 0$. Note that $x_1, x_2, \ldots, x_n$ are solutions to $w(x_1, x_2, \ldots, x_n, x) = 0$

where w is the Wronskian determinant. Then both L_n and $w(x_1, x_2, \ldots, x_n, x)/w_n$, where $w_n = w(x_1, x_2, \ldots, x_n)$, are linear Δ-differential operators which annihilate $x_1, x_2, \ldots, x_n$ and have leading coefficient 1. Hence $L_n x = w(x_1, x_2, \ldots, x_n, x)/w_n$. Suppose now that $x_1, x_2, \ldots, x_n$ are solutions to $L_n x = 0$ so that $w_i := w(x_1, x_2, \ldots, x_i) > 0$ for $1 \le i \le n$. Consider the linear Δ-differential operator $M_k x = (w(x_1, x_2, \ldots, x_{k-1}, x)/w_k)^{\Delta}$. $M_k x_i = 0$ for $1 \le i \le k$, and the coefficient of x^{Δ^k} in $M_k x$ is $w_{k-1}^{\sigma}/w_k^{\sigma}$. Hence $(w_k^{\sigma}/w_{k-1}^{\sigma})M_k x = w(x_1, x_2, \ldots, x_k, x)/w_k$, and thus $w(x_1, x_2, \ldots, x_n, x) = w_k w_k{}^{\sigma}/w_{k-1}^{\sigma}(w(x_1, x_2, \ldots, x_{k-1}, x)/w_k)^{\Delta}$. Next apply the last result successively starting with $w(x_1, x_2, \ldots, x_n, x)$ to get

$$w(x_1, x_2, \ldots, x_n, x)$$
$$= \frac{w_n w_n^{\sigma}}{w_{n-1}^{\sigma}} \left(\frac{w_{n-1} w_{n-1}^{\sigma}}{w_n w_{n-2}^{\sigma}} \left(\cdots \left(\frac{w_2 w_2^{\sigma}}{w_3 w_1^{\sigma}} \left(\frac{w_1 w_1^{\sigma}}{w_2} \left(\frac{x}{w_1} \right)^{\Delta} \right)^{\Delta} \right)^{\Delta} \cdots \right)^{\Delta} \right)^{\Delta}.$$

Set $\alpha_n = (w_n w_n^{\sigma})/w_{n-1}^{\sigma}$ and $\alpha_k = (w_k w_k^{\sigma})/(w_{k+1} w_{k-1}^{\sigma})$ for $0 \le k \le n$, where $w_0 = w_{-1} = 1$. This gives each $\alpha_k > 0$ and shows the representation (6) for L_n.

4 TRENCH FACTORIZATION

We now turn our attention to a special form of an operator that has a Polya factorization. This form, the Trench factorization, is given by

$$L[y] = \frac{1}{\beta_n} \left(\frac{1}{\beta_{n-1}} \left(\cdots \frac{1}{\beta_1} \left(\frac{y}{\beta_0} \right)^{\Delta} \cdots \right)^{\Delta} \right)^{\Delta}, \tag{8}$$

where each $\beta_i > 0$ on I and $\int^b \beta_i(\tau)\Delta\tau = \infty$ for $1 \le i \le n$. We assume that b is a left-dense point in the closure of I in $\mathbb{T}$. If $\sup \mathbb{T} = \infty$, then $b = \infty$ is allowable. We denote by $\mathcal{P}(a, b)$ the collection of linear nth-order Δ-differential operators which have a Polya factorization (7) with each α_i right-dense continuous and positive in (a, b). Our goal in this section to show that the Trench factorization (cf. [11]) is valid as well on measure chains.

THEOREM 7 (Trench Factorization) *Let $L \in \mathcal{P}(a,b)$ with b a left-dense point in $\mathbb{T}$. Then L has a factorization* (8) *where each β_i is right-dense continuous and positive and $\int^b \beta_i(\tau)\Delta = \infty$, $1 \le i \le n-1$. This determines the β_i up to positive constants with product* 1.

This theorem shows that the Trench factorization serves a canonical form for the non-unique Polya factorization of $L \in \mathcal{P}(a,b)$. The non-uniqueness for the Polya factorization is evident from the proof of Lemma 9 below. Lemmas 8 and 9 serve as the first two steps in the induction process used to prove Theorem 7. The statements and proofs of the lemmas and Theorem 7 as well as the subsequent material on the Trench factorization are adaptations to measure chains of the corresponding material in Trench's original paper [11] where the analysis is for differential equations, i.e., for the case that the measure chain is $\mathbb{R}$ itself. Krueger [7] has recently shown the Trench factorization for difference operators written as recurrence relations.

LEMMA 8 *If $L \in \mathcal{P}(a,b)$ is given by $Lx = \{1/\alpha_2(1/\alpha_1(x/\alpha_0)^\Delta)^\Delta$ where $\int^\infty \alpha_1(\tau)\Delta\tau < \infty$, then L can be written as $Lx = 1/\beta_2(1/\beta_1(x/\beta_0)^\Delta)^\Delta$ where $\int^\infty \beta_1(\tau)\Delta\tau = \infty$.*

Proof An easy calculation shows that if $\beta_0(t) = \alpha_0(t)\int_t^b \alpha_1(\tau)\Delta\tau$, $\beta_2(t) = \alpha_2(t)\int_{\sigma(t)}^b \alpha_1(\tau)\Delta\tau$, and $\beta_1(t) = \alpha_1(t)/(\int_t^b \alpha_1(\tau)\Delta\tau \int_{\sigma(t)}^b \alpha_1(\tau)$. $\Delta\tau$, then the representation of L in terms of the $\beta_i's$ is correct. Note that $((\int_t^b \alpha_1(\tau)\Delta\tau)^{-1})^\Delta = \beta_1(t)$, so that for $a < c < b$, we have

$$\int_c^b \beta_1(\tau)\Delta\tau = \lim_{s\to b}\left(\frac{1}{\int_s^b \alpha_1(\tau)\Delta\tau} - \frac{1}{\int_c^b \alpha_1(\tau)\Delta\tau}\right) = \infty$$

as desired.

LEMMA 9 *If $L \in \mathcal{P}(a,b)$ is given by $Lx = 1/\alpha_3(1/\alpha_2(1/\alpha_1(x/\alpha_0)^\Delta)^\Delta)^\Delta$ where $\int^\infty \alpha_1(\tau)\Delta\tau = \infty$ and $\int^\infty \alpha_2(\tau)\Delta\tau < \infty$, then L can be written as $Lx = 1/\beta_3(1/\beta_2(1/\beta_1(x/\beta_0)^\Delta)^\Delta)^\Delta$ where each β_i is right-dense continuous and positive on (a,b) and $\int^\infty \beta_i(\tau)\Delta\tau = \infty$ for $i = 1, 2$.*

Proof An application of Lemma 8 to $1/\alpha_3(1/\alpha_2(\cdot/\alpha_1)^\Delta)^\Delta$. gives $L = 1/\gamma_3(1/\gamma_2(1/\gamma_1(\cdot/\gamma_0)^\Delta)^\Delta)^\Delta$ where $\gamma_2(t) = \alpha_2(t)(\int_t^b \alpha_2(\tau)\Delta\tau \times \int_{\sigma(t)}^b \alpha_2(\tau)\Delta\tau)^{-1}$, $\gamma_0 = \alpha_0$, $\gamma_1(t) = \alpha_1(t)\int_t^b \alpha_2(\tau)\Delta\tau$, and $\gamma_3(t) = \alpha_3(t)$ $\int_{\sigma(t)}^b \alpha_2(\tau)\Delta\tau$. By Lemma 9 we know that $\int^b \gamma_2(\tau)\Delta\tau = \infty$. Hence, if

$\int^b \gamma_1(\tau)\Delta\tau = \infty$, we have the desired representation of L. If $\int^b \gamma_1(\tau)\Delta\tau < \infty$, apply Lemma 8 to the operator $1/\gamma_2(1/\gamma_1(\cdot/\gamma_0)^\Delta)^\Delta$ to get $L = 1/\beta_3(1/\beta_2(1/\beta_1(\cdot/\beta_0)^\Delta)^\Delta)^\Delta$ where $\beta_0(t) = \gamma_0(t)\int_t^b \gamma_1(\tau)\Delta\tau$, $\beta_2(t) = \gamma_2(t)\int_{\sigma(t)}^b \gamma_1(\tau)\Delta\tau$, $\beta_3 = \gamma_3$, and $\beta_1(t) = \gamma_1(t)(\int_t^b \gamma_1(\tau)\Delta\tau \times \int_{\sigma(t)}^b \gamma_1(\tau)\Delta\tau)^{-1}$. $\int^b \beta_1(\tau)\Delta\tau = \infty$ clearly holds. To show that $\int^b \beta_2(\tau) \times \Delta\tau = \infty$, let $c \in (a, b)$ and let $\{b_n\}$ be a sequence from (c, b) with $b_n \nearrow b$ as $n \to \infty$. Then $\int_c^{b_n} \beta_2(\tau)\Delta\tau = \int_c^{b_n} \gamma_2(s)(\int_{\sigma(s)}^b \gamma_1(\tau)\Delta\tau)\Delta s = \int_c^{b_n}(1/\int_s^b \alpha_2(\tau)\Delta\tau)^\Delta \int_{\sigma(s)}^b \alpha_2(\tau)\Delta\tau\Delta s$. Applying the integration by parts formula

$$\int_c^{b_n} u^\sigma(\tau) v\Delta(\tau)\Delta\tau = u(s)v(s)|_c^{b_n} - \int_c^{b_n} v^\sigma(\tau)u^\Delta(\tau)\Delta\tau$$

gives

$$\begin{aligned}\int_c^{b_n} \beta_2(\tau)\Delta\tau &= \frac{1}{\int_s^{b_n} \alpha_2(\tau)\Delta\tau} \int_s^{b_n} \gamma_1 \tau\Delta\tau|_c^{b_n} \\ &\geq -\left(\int_c^{b_n} \alpha_2(\tau)\Delta\tau\right)^{-1} \int_c^{b_n} \gamma_1(\tau)\Delta\tau \\ &\quad + \int_c^{b_n} \alpha_1(\tau)\Delta\tau \to \infty \quad \text{as } n \to \infty.\end{aligned}$$

We now turn to the proof of the existence part of Theorem 7 by showing that a representation as in (8), with $\int^b \beta_i \Delta\tau = \infty$, $1 \leq i \leq n-1$, is always possible provided (7) holds. We delay the essential uniqueness until further terminology and techniques have been introduced. Our proof mimics the original proof of Trench [11] for differential operators when $\mathbb{T} = \mathbb{R}$. Some of the details are included here for clarity.

Proof (Existence in Theorem 7) As noted above, the preceding lemmas show the existence of representation (8) for $n = 2$ and $n = 3$. Suppose this representation exists for every Polya operator of order $n-1 \geq 3$. Let $L \in \mathcal{P}(a, b)$. Then

$$L[y] = \frac{1}{\beta_n}\left(\frac{1}{\beta_{n-1}}\left(\cdots\frac{1}{\beta_1}\left(\frac{y}{\beta_0}\right)^\Delta\cdots\right)^\Delta\right)^\Delta,$$

where each $\beta_i \geq 0$. By the induction assumption applied to the operator $1/\beta_{n-1}(1/\beta_{n-2}(\cdots 1/\beta_1(y/\beta_0)^{\Delta}\cdots)^{\Delta})^{\Delta}$ we may assume that $\int^b \beta_i = \infty$ for $1 \leq i \leq n-2$. If $\int^b \beta_{n-1} = \infty$, we have the desired representation. If $\int^b \beta_{n-1} < \infty$, consider the sequence of representations of L given by

$$L[y] = \frac{1}{\beta_{n,i}}\left(\frac{1}{\beta_{n-1,i}}\left(\cdots\frac{1}{\beta_{1,i}}\left(\frac{y}{\beta_{0,i}}\right)^{\Delta}\cdots\right)^{\Delta}\right)^{\Delta},$$

where

$$\begin{aligned} \beta_{j,0} &= \beta_{j,0} \leq j \leq n, \quad \text{and} \quad \text{for } i > 0, \\ \beta_{j,i} &= \beta_{j,i-1} \quad \text{for } j \notin \{n-i+1, n-1, n-i-1\}, \\ \beta_{n+1-j,i} &= \beta_{n+j-1,i-1} \int_{\sigma(t)}^{b} \beta_{n-j,i-1}\Delta\tau, \\ \beta_{n-i,i} &= \beta_{n-i,i-1}. \end{aligned}$$

For $i=1$, the divergence of $\int^b \beta_{n-2,0}\Delta\tau$ is transferred to $\beta_{n-1,1}$, as in Lemma 8, leaving $\int^b \beta_{n-2,1}\Delta\tau$ either finite or infinite. If it is infinite, we have the desired representation. If it is finite, the transformation from $\beta_{n-j,1}$ to $\beta_{n-1,2}$, which is the transformation in Lemma 9, gives $\int^b \beta_{n-1,2}\Delta\tau = \int^b \beta_{n-2,2}\Delta\tau = \infty$, but $\int^b \beta_{n-3,2}\Delta\tau$ could be finite. If it is infinite, we again have the desired representation for L. If it is finite, the next step in the sequence of representation gives $\int^b \beta_{n-3,3}\Delta\tau = \infty$ and $\int^b \beta_{n-3,2}\Delta\tau = \infty$ as in Lemma 9. We continue in this fashion applying the second half of the proof of Lemma 9. Since $\int^b \beta_{n-j,i}\Delta\tau < \infty$ can only hold for $j \in \{0, n, i+1\}$, if this process does not terminate prior to $i = n-1$, it does terminate for $i = n-1$ since then $\int^b \beta_{n-j,n-1}\Delta\tau = \infty$ except for $j=0$ and $j=n$. Hence every Polya operator has a Trench factorization.

Before showing the essential uniqueness of the Trench factorization (8), let us consider why it is important. Suppose that L has the factorization in (8). Let $c \in (a,b)$. Then $x_1 = \beta_0$ is a solution to (1) as are $x_2 = \beta_0(t)\int_c^t \beta_1(\tau_1)\Delta\tau_1$ and $x_3 = \beta_0(t)\int_c^t \beta_1(\tau_1)\int_c^{\tau_1}\beta_2(\tau_2)\Delta\tau_2\Delta\tau_1$. In general,

$$x_{k+1} = \beta_0(t)\int_c^t \beta_1(\tau_1)\int_c^{\tau_1}\beta_2(\tau_2)\cdots\int_c^{\tau_{k-1}}\beta(\tau_k)\delta\tau_k\Delta\tau_{k-1}\cdots\Delta\tau_1 \quad (9)$$

is a solution to (1) for $1 \leq k \leq n-1$. Now observe that $x_1/x_2 = 1/\int_c^t \beta_1 \Delta\tau_1 \to 0$ as $t \nearrow b$. Next $x_2/x_3 = \int_c^t \beta_1(\tau_1)/\int_c^t \beta_1(\tau_1)\int_c^{\tau_1} \beta_2(\tau_2) \times \delta\tau_2 \Delta\tau_1$. By L'Hôpital's Rule on measure chains, see [1], the last quantity has the same limit as $1/\int_c^t \beta_2(\tau_2)\Delta\tau_2$ which has limit 0 as $t \nearrow b$. By repeated application of L'Hôpital's Rule on measure chains, we get that

$$\lim_{t \nearrow b} \frac{x_k(t)}{x_{k+1}(t)} = 0. \tag{10}$$

A collection $\{x_1, \ldots, x_n\}$ of n solutions to $L_n x = 0$ which are positive near b and satisfy (10) is called a *principal system of solutions at b* for $L_n x = 0$. We state this result here formally as a corollary.

COROLLARY 10 *Let $b \leq \infty$ be the left-dense right endpoint of the interval I. Suppose that L_n has a representation in the form* (8). *Then* (1) *has a principal system of solutions at b.*

The next lemma was stated for differential equations by Levin [9]. It is the principal tool used in showing the essential uniqueness of the Trench factorization.

LEMMA 11 *Suppose that $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are principal systems of solutions at b for* (1) *where the coefficients of L_n are right-dense continuous on (a, b). Then $y_i = \sum_{j=1}^{i} a_{ij} x_j$, where each a_{ij} is constant and $a_{ii} > 0$ for $1 \leq i \leq n$,*

Proof Let S be the solution space of $L_n y = 0$. For $1 < k \leq n$, let $V_k = \{y \in S : y(t)/x_k(t) \to 0 \text{ as } t \nearrow b\}$. Clearly, $x_1, \ldots, x_{k-1} \in V_k$ and span V_k. Since $y_j \in V_k$ implies that $y_i \in V_k$ for $i < j$, then $y_1, \ldots, y_{k-1} \in V_k$. It follows that $y_i = \sum_{j=1}^{i} a_{ij} x_j$ for constants a_{ij}, $1 \leq j \leq i$. Then $y_i/x_i = \sum_{j=1}^{i} a_{ij}(x_j/x_i) \to a_{ii}$ as $t \nearrow b$. Since both y_i and x_i are positive for t near b, $a_{ii} > 0$.

We are now prepared to finish the proof of Theorem 7.

Proof (Theorem 7 – essential uniqueness) Suppose that L_n has a Trench factorization as in (8). Then $x_1, \ldots, x_n$, as defined by (9), form a principal system of solutions at b. Define $L_0 x = x/\beta_0$ and $L_j x = 1/\beta_j (L_{j-1} x)^{\Delta}$ for $1 \leq j \leq n$. Then $L_j x_{j+1} = 1$ for $0 \leq j \leq n$ and $L_j x_j = 0$ for

$1 \le j \le n$. Now suppose that

$$L_n y = \frac{1}{\alpha_n}\left(\frac{1}{\alpha_{n-1}}\left(\cdots\left(\frac{1}{\alpha_1}\left(\frac{y}{\alpha_0}\right)^{\Delta}\right)^{\Delta}\cdots\right)^{\Delta}\right)$$

is a second representation of L_n. Let $y_1, \ldots, y_n$ be the principal system of solutions of (1) at b defined by (9) with β replaced by α. By the preceding lemma, $y_i = \sum_{j=1}^{i} a_{ij} x_j$, and so $L_{i-1} y_i = a_{ii}$ for $1 \le i \le n$. Now $y_1 = \alpha_0 = a_{11} x_1 = a_{11}\beta_0$. Hence $a_{11} = \alpha_0(t)/\beta_0(t)$. Next, $a_{22} = L_1 y_2 = 1/\beta_1(t)\left(\alpha_0(t)/\beta_0(t)\int_c^t \alpha_1(\tau)\Delta\tau\right)^{\Delta} = a_{11}(\alpha_1(t)/\beta_1(t)$, so $a_{22}/a_{11} = \alpha_1(t)/\beta_1(t)$. Suppose that $a_{jj} = \alpha_{j-1}(t)/(\beta_{j-1}(t))a_{j-1,j-1}$ for $2 \le j \le k$. Then $a_{k+1,k+1} = L_k y_{k+1} = a_{kk}(\alpha_k/\beta_k$, so in general we have $\alpha_k(t)/\beta_k(t) = a_{k+1,k+1}/a_{kk}$ for $1 \le k \le n-1$ and $a_{11} = \alpha_0(t)/\beta_0(t)$. Now let x_{n+1} be given by (9) for $k = n$. Then $L_n x_{n+1} = 1$. But $L_n x_{n+1} = \beta_n(t)/(\alpha_n a_{nn}(t))$. Hence $\alpha_n(t)/\beta_n(t) = 1/a_{nn}$ and $\prod_{i=0}^{n-1}(\alpha_i(t)/\beta_i(t)) = a_{11}(\prod_{i=1}^{n-1} a_{k+1,k+1}/a_{k,k})/a_{nn} = 1$.

The interested reader can check that the necessary and sufficient condition for $L_n x = 0$ to have a fundamental principal system of solutions on (a, b), i.e., principal at both a and b, given in Theorem 2 of [11] also extends to the measure chain setting if a is right-dense.

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