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Existence Theory for Nonlinear Volterra Integral and Differential Equations

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In this paper we prove the existence theorems for the integrodifferential equation

$$y'(t) = f\left(t, y(t), \int_0^t k(t, s, y(s))\,\mathrm{d}s\right), \quad t \in I = [0, T],$$
$$y(0) = y_0,$$

where in first part f, k, y are functions with values in a Banach space E and the integral is taken in the sense of Bochner. In second part f, k are weakly–weakly sequentially continuous functions and the integral is the Pettis integral. Additionaly, the functions f and k satisfy some boundary conditions and conditions expressed in terms of measure of noncompactness or measure of weak noncompactness.

Keywords: Integral equations; Existence theorem; Pseudo-solutions; Measures of noncompactness

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1 INTRODUCTION

In this paper we establish some existence principles for integrodifferential operator equations and present existence result for integro-differential and integral equations.

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The paper is divided into two main sections.

In Section 1 we prove some existence theorems for the problem

$$
\begin{aligned}
&y'(t) = f\left(t, y(t), \int_0^t k(t,s,y(s))\,\mathrm{d}s\right),\\
&y(0) = y_0,
\end{aligned}
\tag{1}
$$

where $I=[0,T]$, E is a Banach space with the norm $\|\cdot\|$, f, k, y are functions with values in a Banach space E and the integral is the Bochner integral.

In Section 2 we prove some existence theorem for the problem (1), where f, k, y are functions with values in a Banach space E, f, k are functions weakly-weakly sequentially continuous and the integral is the Pettis integral [1]. The results of this paper extends existence theorems from Krzyśka [12], Cichoń [6], Meehan and O'Regan [13], O'Regan [16,17], Cramer *et al.* [7].

In this paper we use the measure of noncompactness developed by Kuratowski [11], and the measure of weak noncompactnes developed by de Blasi [4].

Let A be a bounded nonvoid subset of E. *The Kuratowski measure of noncompactness* $\alpha(A)$ is defined by

$$\alpha(A) = \inf\{\varepsilon > 0\text{: there exists } C \in \mathcal{K} \text{ such that } A \subset C + \varepsilon B_0\},$$

where $\mathcal{K}$ is the set of compact subsets of E and B_0 is the norm unit ball.

The de Blasi measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0\text{: there exists } C \in \mathcal{K}^w \text{ such that } A \subset C + tB_0\},$$

where $\mathcal{K}^w$ is the set of weakly compact subsets of E and B_0 is the norm unit ball.

The properties of measure of noncompactness $\alpha(A)$ are:

(1^0) if $A \subset B$ then $\alpha(A) \le \alpha(B)$;
(2^0) $\alpha(A) = \alpha(\bar{A})$, where $\bar{A}$ denotes the closure of A;
(3^0) $\alpha(A) = 0$ if and only if A is relatively compact;
(4^0) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;

(5^0) $\alpha(\lambda A) = |\lambda|\alpha(A)$ $(\lambda \in \mathbb{R})$;
(6^0) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
(7^0) $\alpha(\operatorname{conv} A) = \alpha(A)$.

The properties of weak measure of noncompactness β are analogous to the properties of measure of noncompactness, see [2–5,14]. Moreover, we can construct many other measures with the above properties, by using a scheme from [5]. We now gather some well-known definitions and results from the literature, which we will use throughout this paper.

DEFINITION 1 *A function $f: I \times E \times E \to E$ is L^1-Carathéodory, if the following conditions hold*:

(i) *the map $t \mapsto f(t, x, y)$ is measurable for all $(x, y) \in E^2$*;
(ii) *the map $(x, y) \mapsto f(t, x, y)$ is continuous for almost all $t \in I$.*

DEFINITION 2 *A function $k: I \times I \times B \to E$ is L^1-Carathéodory, if the following conditions hold*:

(i) *the map $(t, s) \to f(t, s, y)$ is measurable for all $y \in B$*;
(ii) *the map $y \to f(t, s, y)$ is continuous for almost all $(t, s) \in I^2$.*

In the proof of the main theorem in Section 1 we will apply the following fixed point theorem.

THEOREM 1 [15] *Let $\mathcal{D}$ be a closed convex subset of E, and let F be a continuous map from $\mathcal{D}$ into itself. If for some $x \in \mathcal{D}$ the implication*

$$\bar{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V)) \implies V \text{ is relatively compact,}$$

holds for every countable subset V of $\mathcal{D}$, *then F has a fixed point.*

In Section 2 we will apply the following theorem:

THEOREM 2 [10] *Let E be a metrizable locally convex topological vector space and let $\mathcal{D}$ be a closed convex subset of E, and let F be a weakly sequentially continuous map of $\mathcal{D}$ into itself. If for some $x \in \mathcal{D}$ the implication*

$$\bar{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V)) \implies V \text{ is relatively weakly compact,}$$

holds for every subset V of $\mathcal{D}$, then F has a fixed point.

2 AN EXISTENCE RESULT FOR INTEGRODIFFERENTIAL EQUATIONS

Observe that the problem (1) is equivalent to the integral equation

$$y(t) = y_0 + \int_0^t f\left(z, y(z), \int_0^z k(z, s, y(s))\,\mathrm{d}s\right)\mathrm{d}z, \quad \text{for } t \in I. \tag{1'}$$

Assume that

(1) a function $a \in L^1[0,T]$,
(2) $B = \{x\colon \|x\| \le b,\ b = \|y_0\| + \int_0^T a(t)\,\mathrm{d}t\}$,
(3) k is a L^1-Carathéodory function from $I^2 \times B$ into E,
(4) f is a L^1-Carathéodory function from $I \times B \times B$ into E,
(5) $\|f(t, y(t), \int_0^t k(t,s,y(s))\,\mathrm{d}s)\| \le a(t)$ almost everywhere on I for $y \in \tilde{B}$, where $\tilde{B} = \{y \in C[0,T]\colon \|y\| \le b,\ b = \|y_0\| + \int_0^T a(t)\,\mathrm{d}t\}$.

THEOREM 3 *Assume, that conditions* (1)–(5) *holds and in addition, that*

(6) *there exists a constant* c_1 *such that* $\alpha(f(t,A,C)) \le c_1 \max\{\alpha(A), \alpha(C)\}$, *for any subsets* A, C *of* B,
(7) *there exists an integrable function* $c_2 : I^2 \to R^+$ *such that for every* $t \in I$, $\varepsilon > 0$ *and for every bounded subset* X *of* B *there exists a closed subset* I_ε *of* I *such that* $\mathrm{mes}(I \backslash I_\varepsilon) < \varepsilon$ *and*

$$\alpha(k(t, T \times X)) \le \sup_{s\in T} c_2(t,s)\alpha(X) \text{ for any compact subset } T \text{ of } I_\varepsilon.$$

(8) *the zero function is the unique continuous solution of the inequality*:

$$p(t) \le c_1 T \sup_{z\in I} \int_0^T c_2(z,s)p(s)\,\mathrm{d}s \text{ on } I.$$

Then there exists at least one solution of problem (1).

Proof We define the operator $\mathbb{N} : C[0,T] \to C[0,T]$ by

$$\mathbb{N}y(t) = y_0 + \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\mathrm{d}z.$$

We require that $N : \tilde{B} \to \tilde{B}$ is continuous. Because

(i)

$$\begin{aligned}\|Ny(t)\| &= \left\| y_0 + \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\mathrm{d}z \right\| \\ &\le \|y_0\| + \left\| \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\mathrm{d}z \right\| \\ &\le \|y_0\| + \int_0^t \left\| f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\right\| \mathrm{d}z \\ &\le \|y_0\| + \int_0^T a(t)\,\mathrm{d}t = b\end{aligned}$$

so $Ny(t) \in B$, for $t \in I$.

Now we will show continuity of N.

(ii) Let $y_n \to y$ in $C[0, T]$. Then

$$\begin{aligned}\|Ny_n - Ny\| &= \sup_{t\in[0,T]} \left\| \int_0^t f\left(z, y_n(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s\right)\mathrm{d}z \right. \\ &\qquad \left. - \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\mathrm{d}z \right\| \\ &\le \sup_{t\in[0,T]} \left\| \int_0^t \left[f\left(z, y_n(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s\right) \right. \right. \\ &\qquad \left.\left. - f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\right] \mathrm{d}z \right\| \\ &\le \sup_{t\in[0,T]} \int_0^t \left\| f\left(z, y_n(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s\right) \right. \\ &\qquad \left. - f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\right\| \mathrm{d}z \\ &\le \sup_{t\in[0,T]} \int_0^t \left\| f\left(z, y_n(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s\right) \right. \\ &\qquad \left. - f\left(z, y(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s\right)\right\| \mathrm{d}z \\ &\quad + \sup_{t\in[0,T]} \int_0^t \left\| f\left(z, y(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s\right) \right. \\ &\qquad \left. - f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\right\| \mathrm{d}z.\end{aligned}$$

Because f and k are L^1-Carathéodory functions and $\|y_n - y\| \to 0$ so $\|Ny_n - Ny\| \to 0$.

From (i) and (ii) follows that $N : \tilde{B} \to \tilde{B}$ is continuous.

Now we will show that the set $N(\tilde{B})$ is equicontinuous subset. This follows from inequality:

$$\begin{aligned}\|Ny(t) - Ny(\tau)\| &= \sup_{t\in[0,T]} \left\| \int_\tau^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right) \mathrm{d}z \right\| \\ &\le \sup_{t\in[0,T]} \int_\tau^t \left\| f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right) \right\| \mathrm{d}z \\ &\le \int_\tau^t a(z)\,\mathrm{d}z \quad \text{for every } y \in B.\end{aligned}$$

Observe that the fixed point of the operator N is the solution of the problems (1) and (1′). Now we prove that fixed point of the operator N exists using fixed point Theorem 1.

Let $V \subset \tilde{B}$ be a countable set and $\bar{V} = \overline{\mathrm{conv}}(\mathbb{N}(V) \cup \{x\})$. Because V is an equicontinuous then $t \mapsto v(t) = \alpha(V(t))$ is continuous on I. Let $t \in I$ and $\varepsilon > 0$. Using the Lusin's theorem, there exists a compact subset I_ε of I such that $\mathrm{mes}(I \setminus I_\varepsilon) < \varepsilon$ and a function $s \to c_2(t,s)$ is continuous on I_ε. We divide on interval $I = [0, T]$: $0 = t_0 < t_1 < \cdots < t_n = T$, like this

$$\|c_2(t,s)v(r) - c_2(t,u)v(z)\| < \varepsilon \quad \text{for } s,r,u,z \in T_i = \mathcal{D}_i \cap I_\varepsilon,$$

where $\mathcal{D}_i = [t_{i-1}, t_i]$, $i = 1,2,\ldots,n$. Let $V_i = \{u(s)\colon u \in V,\ s \in \mathcal{D}_i\}$.

We notice

$$\begin{aligned}\alpha\left(\int_I k(t,s,V(s))\,\mathrm{d}s\right) &\le \alpha\left(\int_{I_\varepsilon} k(t,s,V(s))\,\mathrm{d}s + \int_{I\setminus I_\varepsilon} k(t,s,V(s))\,\mathrm{d}s\right) \\ &\le \alpha\left(\int_{I_\varepsilon} k(t,s,V(s))\,\mathrm{d}s\right) + \varepsilon_1,\end{aligned}$$

where $\varepsilon_1 \to 0$ if $\varepsilon \to 0$

and

$$\int_I k(z,s,V(s))\,ds \subset \sum_{i=1}^{n} \int_{T_i} k(z,s,V(s))\,ds$$
$$\subset \sum_{i=1}^{n} \text{mes } T_i \overline{\text{conv}}\, k(z, T_i \times V_i).$$

Using the properties of measure of noncompactness α we have

$$\alpha\left(\int_I k(z,s,V(s))\,ds\right) \leq \sum_{i=1}^{n} \text{mes } T_i \alpha(k(z, T_i \times V_i))$$
$$\leq \sum_{i=1}^{n} \text{mes } T_i \sup_{s\in T_i} c_2(z,s)\alpha(V_i)$$
$$= \sum_{i=1}^{n} \text{mes } T_i c_2(z,q_i)v(s_i),$$

where $q_i \in T_i$, $s_i \in \mathcal{D}_i$.

Moreover, because $\|c_2(t,s)v(s) - c_2(t,q_i)v(s_i)\| < \varepsilon$ for $s \in T_i$ we have

$$\sum_{i=1}^{n} \text{mes } T_i c_2(t,q_i)v(s_i)$$
$$\leq \sum_{i=1}^{n} \text{mes } T_i \|c_2(t,q_i)v(s_i) - c_2(t,s_i)v(s_i)\| + \sum_{i=1}^{n} \text{mes} T_i c_2(t,s_i)v(s_i)$$
$$\leq \varepsilon_2 + \sum_{i=1}^{n} \text{mes } T_i c_2(t,s_i)v(s_i),$$

where $\varepsilon_2 \to 0$ if $\varepsilon \to 0$. So

$$\alpha\left(\int_I k(z,s,y(s))\,ds\right) \leq \int_{I_\varepsilon} c_2(z,s)v(s)\,ds + \varepsilon_2$$

then, because $\varepsilon_2 \to 0$ if $\varepsilon \to 0$ so

$$\alpha\left(\int_I k(z,s,y(s))\,ds\right) \leq \int_I c_2(z,s)v(s)\,ds.$$

Because $\bar{V} = \overline{\text{conv}}(N(V) \cup \{x\})$, then by the property of measure of noncompactness we have

$$
\begin{aligned}
\alpha(V(t)) &= \alpha(\overline{\text{conv}}(N(V(t)) \cup \{x\})) \leq \alpha(N(V(t))) \\
&\leq \alpha\left(\int_0^t f(z, V(z)), \int_0^z k(z, s, V(s))\,\mathrm{d}s\right)\mathrm{d}z \\
&\leq \int_0^t \alpha\left(f(z, V(z)), \int_0^z k(z, s, V(s))\,\mathrm{d}s\right)\mathrm{d}z \\
&\leq \int_0^t c_1 \cdot \max(\alpha(V(z))), \alpha\left(\int_0^z k(z, s, V(s))\,\mathrm{d}s\right)\mathrm{d}z \\
&\leq c_1 \cdot T \cdot \sup_{z \in I} \alpha\left(\int_0^z k(z, s, V(s))\,\mathrm{d}s\right) \\
&\leq c_1 \cdot T \cdot \sup_{z \in I} \int_I c_2(z, s) v(s)\,\mathrm{d}s.
\end{aligned}
$$

So

$$v(t) \leq c_1 \cdot T \sup_{z \in I} \int_0^T c_2(z, s) v(s)\,\mathrm{d}s.$$

By (8) we have that $v(t) = \alpha(V(t)) = 0$. Using Arzelá–Ascoli's theorem we obtain that V is relatively compact. By Theorem 1 the operator N has a fixed point. This means that there exists a solution of problem (1).

Remark Theorem 1 extends the existence theorem from Meehan and O'Regan [13] and O'Regan [17].

3 AN EXISTENCE RESULT FOR INTEGRODIFFERENTIAL EQUATIONS IN WEAK SENSE

In this part we prove a theorem for the existence of pseudo-solutions to the Cauchy problem

$$
\begin{aligned}
y'(t) &= f\left(t, y(t), \int_0^t k(t, s, y(s))\,\mathrm{d}s\right), \\
y(0) &= y_0
\end{aligned}
\tag{2}
$$

in Banach spaces. Functions f and k will be assumed Pettis integrable but this assumption is not sufficient for the existence of solutions. We impose a weak compactness type condition expressed in terms of measures of weak noncompactness. Throughout this part $(E, \|\cdot\|)$ will denote a real Banach space, E^* the dual space. Unless otherwise stated, we assume that "$\int$" denotes the Pettis integral.

A function $g: E \to E$ is said to be *weakly–weakly sequentially continuous* if for each weakly convergent sequence $(x_n) \subset E$, a sequence $(g(x_n))$ is weakly convergent in E.

Fix $x^* \in E^*$, and consider the equation

(9)

$$(x^*x)'(t) = x^*f\left(t, x(t), \int_0^t k(t, s, x(s))\, ds\right), \quad t \in I.$$

Now, we can introduce the following definition:

DEFINITION 3 [6,8] *A function $x: I \to E$ is said to be a pseudo-solution of the Cauchy problem* (2) *if it satisfies the following conditions*:

(i) *$x(\cdot)$ is absolutely continuous,*
(ii) $x(0) = x_0$,
(iii) *for each $x^* \in E^*$ there exists a negligible set $A(x^*)$ (i.e.* mes $A(x^*) = 0$), *such that for each $t \notin A(x^*)$*:

$$(x^*x)'(t) = x^*\left(f\left(t, x(t), \int_0^t k(t, s, y(s))\, ds\right)\right).$$

In other words by a pseudo-solution of (2) *we will understand an absolutely continuous function such that $x(0) = x_0$, and $x(\cdot)$ satisfies* (2) *a.e., for each $x^* \in E^*$.*

In this part we use a weak measure of noncompactness of de Blasi's β. It is necessary to remark that the following lemma is true:

LEMMA 1 [9,14] *Let $\mathcal{H} \subset C_w(I, E)$ be a family of strongly equicontinuous functions. Then the function $t \mapsto v(t) = \beta(\mathcal{H}(t))$ is continuous and $\beta(\mathcal{H}(I)) = \sup\{\beta(\mathcal{H}(t)) : t \in I\}$.*

Assume that in addition to (1), (2), (5) and (6),

(10) k is a Carathéodory's weakly–weakly sequentially continuous function $I^2 \times B$ into E;
(11) f is Carathéodory's weakly–weakly sequentially continuous function from $I \times B \times B$ into E;
(12) for any continuous function $y: I \to E$, functions $k(\cdot, \cdot, y(\cdot))$ and $f\left(\cdot, y(\cdot), \int_0^{(\cdot)} k(\cdot, s, y(s))\, ds\right)$ are Pettis integrable.

THEOREM 4 *Assume, in addition to* (1), (2), (5) *and* (10–12) *that*

(13) *there exists a constant* c_3 *such that for every interval* $J \subset I$ *and for any subsets* A, C *of* B

$$\beta(f(J, A, C) \leq c_3 \max\{\beta(A), \beta(C)\},$$

(14) *there exists an integrable function* $c_4 : I \to R^+$ *such that for every* $t \in I$, $\varepsilon > 0$ *and for every bounded subset* X *of* B *there exists a closed subset* I_ε *of* I *such that* mes $(I \backslash I_\varepsilon) < \varepsilon$ *and*

$$\beta(k(J, J \times X)) \leq \sup_{s \in J} c_4(s)\beta(X), \quad \textit{for any } J \subset I.$$

Then there exists at least one pseudo-solution of the problem (2).

Proof We define the operator $G : C[0, T] \to C[0, T]$ by

$$Gy(t) = y_0 + \int_0^t f(z, y(z), \int_0^z k(z, s, y(s)\, \mathrm{d}s)\, \mathrm{d}z.$$

We require that $G : \tilde{B} \to \tilde{B}$ is weakly sequentially continuous, where

$$\tilde{B} = \left\{ y \in C[0, T]: \|y\| \leq b,\ b = \|y_0\| + \int_0^T a(t)\, \mathrm{d}t \right\}.$$

Because

(i) For any $y^* \in E^*$ such that $\|y^*\| \leq 1$ and for any $y \in$ B,

$$\begin{aligned} &\left| y^* \left[f\left(z, y(z), \int_0^z k(z, s, y(s))\, \mathrm{d}s \right) \right] \right| \\ &\quad \leq \|y^*\| \left\| f\left(z, y(z), \int_0^z k(z, s, y(s))\, \mathrm{d}s \right) \right\| \\ &\quad \leq \left\| f\left(z, y(z), \int_0^z k(z, s, y(s))\, \mathrm{d}s \right) \right\| \leq a(z) \end{aligned}$$

so

$$\begin{aligned} |y^* Gy(t)| &\leq |y^* y_0| + \int_0^t \left| y^* \left[f\left(z, y(z), \int_0^z k(z, s, y(s))\, \mathrm{d}s \right) \right] \right| \mathrm{d}z \\ &\leq \|y_0\| + \int_0^t a(t)\, \mathrm{d}t \leq \|y_0\| + \int_0^T a(t)\, \mathrm{d}t = b. \end{aligned}$$

From here

$$\sup\{|y^*Gy(t)|\colon y^* \in E^*,\ \|y^*\| \leq 1\} \leq b \text{ and } \|Gy(t)\| \leq b$$

so $Gy(t) \in B$.

(ii) Now we will show that set $G(\tilde{B})$ is strongly equicontinuous subset. This follows from the inequality

$$\begin{aligned}
&|y^*[Gy(t) - Gy(\tau)]| \\
&= \left| y^* \left[\int_\tau^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s \right) \mathrm{d}z \right] \right| \\
&\leq \int_\tau^t \left| y^* f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s \right) \right| \mathrm{d}z \leq \int_\tau^t a(z)\,\mathrm{d}z.
\end{aligned}$$

(iii) Now we will show weakly sequentially continuity of G. Let $y_n \to y$ in $(C[0,T], \omega)$. Then

$$\begin{aligned}
|y^*[Gy_n(t) - Gy(t)]| &= \left| y^* \left[\int_0^t f\left(z, y_n(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s \right) \mathrm{d}z \right.\right. \\
&\quad \left.\left. - \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s \right) \mathrm{d}z \right] \right| \\
&\leq \int_0^t \left| y^* \left[f\left(z, y_n(z), \int_0^z k(z,s,y_n(s)\,\mathrm{d}s) \right) \right.\right. \\
&\quad \left.\left. - f\left(z, y(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s \right) \right] \right| \mathrm{d}z \\
&\quad + \int_0^t \left| y^* \left[f\left(z, y(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s \right) \right.\right. \\
&\quad \left.\left. - f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s \right) \right] \right| \mathrm{d}z \\
&\leq \int_0^T \left| y^* \left[f\left(z, y_n(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s \right) \right.\right. \\
&\quad \left.\left. - f\left(z, y(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s \right) \right] \right| \mathrm{d}z \\
&\quad + \int_0^T \left| y^* \left[f\left(z, y(z), \int_0^z k(z,s,y_n(s))\,\mathrm{d}s \right) \right.\right. \\
&\quad \left.\left. - f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s \right) \right] \right| \mathrm{d}z.
\end{aligned}$$

Because f and k are L^1-Carathéodory functions and $y_n \to y$ in $(C[0,T],\omega)$ so

$$|y^*[Gy_n(t) - Gy(t)]| \quad \to \quad 0.$$

From here

$$\sup\{y^*[Gy_n(t) - Gy(t)]:\, y^* \in E^*,\ \|y^*\| \leq 1\} \quad \to \quad 0.$$

From (i) and (iii), follows that $G : \tilde{B} \to \tilde{B}$ is weakly–weakly sequentially continuous.

Observe that the fixed point of the operator G is the pseudo-solution of the problem

$$y(t) = y_0 + \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\mathrm{d}z. \tag{2'}$$

Now we prove that fixed point of the operator G exists using fixed point Theorem 2.

Let $V \subset \tilde{B}$ be a countable set and $\bar{V} = \overline{\mathrm{conv}}(G(V) \cup \{0\})$. Because V is equicontinuous then $t \to v(t) = \beta(V(t))$ is continuous on I (by Lemma 1).

Let $t \in I$ and $\varepsilon > 0$. Using the Luzin's theorem, there exists a compact subset I_ε of I such that $\mathrm{mes}\,(I \backslash I_\varepsilon) < \varepsilon$ and a function $s \to c_4(s)$ is continuous. We divide an interval $I = [0,T]$: $0 = t_0 < t_1 < \cdots < t_n = T$, like this $\|c_4(s)v(r) - c_4(u)v(z)\| < \varepsilon$ for $s, r, u, z \in T = \mathcal{D}_i \cap I_\varepsilon$, where $\mathcal{D}_i = [t_{i-1}, t_i]$.

We notice

$$\begin{aligned}
&\beta\left(\int_I f\left(z, V(z), \int_0^z k(t,s,V(s))\,\mathrm{d}s\right)\mathrm{d}z\right) \\
&\quad \leq \beta\left(\int_{I_\varepsilon} f\left(z, V(z), \int_0^z k(t,s,V(s))\,\mathrm{d}s\right)\mathrm{d}z\right) \\
&\qquad + \beta\left(\int_{I\backslash I_\varepsilon} f\left(z, V(z), \int_0^z k(t,s,V(s))\,\mathrm{d}s\right)\mathrm{d}z\right) \\
&\quad \leq \beta\left(\int_{I_\varepsilon} f\left(z, V(z), \int_0^z k(t,s,V(s))\,\mathrm{d}s\right)\mathrm{d}z\right) + \varepsilon'.
\end{aligned}$$

Using the properties of weak measure of noncompactness β we have

$$\begin{aligned}
&\beta\left(\int_{I_\varepsilon} f\left(z, V(z), \int_0^z k(t,s,V(s))\,\mathrm{d}s\right)\mathrm{d}z\right)\\
&\le \beta\left(\sum_{i=1}^n \operatorname{mes} T_i \overline{\operatorname{conv}} f\left(T_i, V(T_i), \sum_{i=1}^n \operatorname{mes} T_i \overline{\operatorname{conv}}\, k(t_i, T_i, V_i)\right)\right)\\
&\le \sum_{i=1}^n \operatorname{mes} T_i \beta\left(f\left(T_i, V(T_i), \sum_{i=a}^n \operatorname{mes} T_i \overline{\operatorname{conv}}\, k(T_i, T_i, V_i)\right)\right)\\
&\le \sum_{i=1}^n \operatorname{mes} T_i c_3 \cdot \max \beta(V(T_i)), \beta\left(\sum_{i=1}^n \operatorname{mes} T_i \overline{\operatorname{conv}}\, k(T_i, T_i, V_i)\right)\\
&\le \sum_{i=1}^n \operatorname{mes} T_i c_3 \sum_{i=1}^n \operatorname{mes} T_i \beta(k(T_i, T_i, V_i))\\
&\le Tc_3 \sum_{i=1}^n \operatorname{mes} T_i \sup_{s\in T_i} c_4(s)\beta(V_i)\\
&= Tc_3 \sum_{i=1}^n \operatorname{mes} T_i c_4(s_i)\beta(V(T_i))\\
&= Tc_3 \left[\sum_{i=1}^n \operatorname{mes} T_i c_4(t_i)\beta(V(t_i))\right.\\
&\quad \left. + \sum_{i=1}^n \operatorname{mes} T_i \left[c_4(s_i)\beta(V(t_i)) - c_4(t_i)\beta(V(t_i))\right]\right]
\end{aligned}$$

From here

$$\begin{aligned}
&\beta\left(\int_I f\left(z, V(z), \int_0^z k(z,s,V(s))\,\mathrm{d}s\right)\mathrm{d}z\right)\\
&\le Tc_3 \int_0^t c_4(s)\beta(V(s))\,\mathrm{d}s + \varepsilon_2,
\end{aligned}$$

Because $\varepsilon_2 \to 0$ if $\varepsilon \to 0$ we have

$$\begin{aligned}
\beta(V(t)) &\le \beta(G(V(t)))\\
&\le \beta\left(\int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,\mathrm{d}s\right)\mathrm{d}z\right)\\
&\le Tc_3 \int_0^t c_4(s)v(s)\,\mathrm{d}s.
\end{aligned}$$

So

$$v(t) \leq Tc_3 \int_0^t c_4(s)\beta(V(s))\,ds.$$

By Gronwall's inequality we have that $v(t) = \beta(V(t)) = 0$.

Using Arzelá–Ascoli's theorem we obtain that V is weakly relatively compact.

By Theorem 2 the operator G has a fixed point. This means that there exists a pseudo-solution of problem (2).

Remark Theorem 4 extends the existence theorems from Krzyśka [12], Cichoń [6], O'Regan [16] and others.

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