

Picone-type Inequalities for Nonlinear Elliptic Equations and their Applications

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Picone-type inequalities are derived for nonlinear elliptic equations, and Sturmian comparison theorems are established as applications. Oscillation theorems for forced superlinear elliptic equations and superlinear-sublinear elliptic equations are also obtained.

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1. INTRODUCTION

In 1909, in order to prove Sturmian comparison theorems for ordinary differential equations of the second order, Picone [9] established an identity which now bears his name. Since the pioneering work of Picone [9, 10], the Picone identity has been extended in various directions, and the extended identities of Picone type have played a significant role in the qualitative study of linear ordinary and partial differential equations. The reader is referred to Kreith [6, 7], Swanson [11, 12] and Yoshida [13] for Sturmian comparison results for even order linear elliptic partial differential equations.

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Very recently, an attempt has begun to generalize the Picone identity to the case of nonlinear differential operators for the purpose of developing the analogue of Sturmian theory for the associated nonlinear differential equations; see *e.g.*, the papers [2–4]. Particular mention is made of our previous paper [4] in which a Picone-type identity is derived for a class of quasilinear elliptic operators including the p -Laplacian and is effectively applied to demonstrate Sturm-type comparison and oscillation theorems for half-linear perturbations of the p -Laplace equation.

The objective of this paper is to establish Picone-type *inequalities* which connect the linear elliptic operator

$$\ell[u] = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u \quad (1)$$

with nonlinear elliptic operators of the types

$$L[v] = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial v}{\partial x_j} \right) + C(x)|v|^{\beta-1}v \quad (2)$$

and

$$\tilde{L}[v] = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial v}{\partial x_j} \right) + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v \quad (3)$$

where β and γ are positive constants with $\beta > 1$ and $0 < \gamma < 1$, and to utilize the inequalities thus obtained to deduce Sturmian comparison and oscillation theorems for the forced superlinear elliptic equation

$$L[v] = f(x) \quad (4)$$

as well as the unforced superlinear-sublinear elliptic equation

$$\tilde{L}[v] = 0. \quad (5)$$

Sections 2 and 3 concern the Eqs. (4) and (5), respectively; in each of them Sturmian comparison theorems are proved on the basis of the Picone-type inequalities derived for a pair of operators $\{\ell, L\}$ or $\{\ell, \tilde{L}\}$. Oscillation theorems for the Eqs. (4) and (5) are presented in Section 4.

2. PICONE-TYPE INEQUALITY WITH APPLICATION TO EQUATION (4)

In this section we will derive a Picone-type inequality relating the nonlinear elliptic operator (2) to the linear elliptic operator (1) and apply it to establish Sturmian comparison theorems for forced super-linear elliptic equations of the type (4).

All the operators and equations are defined in a bounded domain G in \mathbb{R}^n , $n \geq 2$, with piecewise smooth boundary ∂G and are assumed to satisfy the following conditions:

- (A₁) $a_{ij}(x) \in C(\bar{G}; \mathbb{R})$, $A_{ij}(x) \in C(\bar{G}; \mathbb{R})$ ($i, j = 1, 2, \dots, n$), and the matrices $(a_{ij}(x))$, $(A_{ij}(x))$ are symmetric and positive definite in G ;
- (A₂) $c(x) \in C(\bar{G}; \mathbb{R})$, $C(x) \in C(\bar{G}; [0, \infty))$;
- (A₃) $f(x) \in C(\bar{G}; \mathbb{R})$;
- (A₄) β and γ are constants such that $\beta > 1$ and $0 < \gamma < 1$.

The domain $\mathcal{D}_L(G)$ of L is defined to be the set of all functions v of class $C^1(\bar{G}; \mathbb{R})$ with the property that $A_{ij}(x)(\partial v / \partial x_j) \in C^1(G; \mathbb{R}) \cap C(\bar{G}; \mathbb{R})$, and the domain $\mathcal{D}_\ell(G)$ of ℓ is defined to be the set of all functions u of class $C^1(\bar{G}; \mathbb{R})$ with the property that $a_{ij}(x)(\partial u / \partial x_j) \in C^1(G; \mathbb{R})(\bar{G}; \mathbb{R})$.

Basic to the derivation of the desired Picone-type inequality (Theorem 2) is the differential inequality given in the following theorem.

THEOREM 1 *If $v \in \mathcal{D}_L(G)$, $v \neq 0$ in G and $v \cdot f(x) \leq 0$ in G , then the following inequality holds for any $u \in C^1(G; \mathbb{R})$:*

$$\begin{aligned} & \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) \\ & \quad + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ & \leq \sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \beta(\beta - 1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} \\ & \quad \times |f(x)|^{((\beta-1)/\beta)} u^2 + \frac{u^2}{v} (L[v] - f(x)). \end{aligned} \tag{6}$$

Proof A simple computation yields

$$\begin{aligned} & \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ &= \sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{u^2}{v} \left(\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \right) \end{aligned} \quad (7)$$

which, combined with

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial v}{\partial x_j} \right) = -C(x)|v|^{\beta-1}v + L[v],$$

gives

$$\begin{aligned} & \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ &= \sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \left(C(x)|v|^{\beta-1} - \frac{f(x)}{v} \right) u^2 + \frac{u^2}{v} (L[v] - f(x)). \end{aligned} \quad (8)$$

Since $v \cdot f(x) \leq 0$, we have

$$C(x)|v|^{\beta-1} - \frac{f(x)}{v} = C(x)|v|^{\beta-1} + \frac{|f(x)|}{|v|}.$$

Applying Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \left(a \geq 0, b \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1 \right) \quad (9)$$

to the case where

$$\begin{aligned} p &= \beta (> 1), \\ q &= \frac{\beta}{\beta - 1}, \\ a &= C(x)^{(1/\beta)} |v|^{((\beta-1)/\beta)}, \\ b &= \left| \frac{f(x)}{(\beta - 1)v} \right|^{((\beta-1)/\beta)}, \end{aligned}$$

we see that

$$(\beta - 1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} |f(x)|^{((\beta-1)/\beta)} \leq \frac{1}{\beta} \left(C(x)|v|^{\beta-1} + \left| \frac{f(x)}{v} \right| \right)$$

or

$$C(x)|v|^{\beta-1} + \frac{|f(x)|}{|v|} \geq \beta(\beta - 1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} |f(x)|^{((\beta-1)/\beta)}. \tag{10}$$

The desired inequality (4) then follows from (8) and (10).

THEOREM 2 (Picone-type inequality) *Assume that $u \in \mathcal{D}_\ell(G)$, $v \in \mathcal{D}_L(G)$, $v \neq 0$ in G and $v \cdot f(x) \leq 0$ in G . Then we have the following Picone-type inequality*

$$\begin{aligned} & \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(u a_{ij}(x) \frac{\partial u}{\partial x_j} - \frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ & \geq \sum_{i,j=1}^n (a_{ij}(x) - A_{ij}(x)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \\ & \quad + (\beta(\beta - 1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} |f(x)|^{((\beta-1)/\beta)} - c(x)) u^2 \\ & \quad + \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) \\ & \quad + \frac{u}{v} (v\ell[u] - u(L[v] - f(x))). \end{aligned} \tag{11}$$

Proof To prove this theorem it suffices to combine the inequality (6) with the identity

$$u\ell[u] = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(u a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x)u^2. \tag{12}$$

We now formulate Sturmian comparison theorems for the nonlinear elliptic equation (4). Such a theorem provides a principle which guarantees the existence of zeros of the solutions of (4) in a nodal domain of a solution of the comparison equation

$$\ell[u] = 0. \tag{13}$$

THEOREM 3 *If there is a nontrivial function $u \in C^1(\bar{G}; \mathbb{R})$ such that $u = 0$ on ∂G and*

$$M[u] \equiv \int_G \left[\sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \beta(\beta - 1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} |f(x)|^{((\beta-1)/\beta)} u^2 \right] dx \leq 0, \quad (14)$$

then every solution $v \in \mathcal{D}_L(G)$ of (4) satisfying $v \cdot f(x) \leq 0$ in G vanishes at some point of \bar{G} . Furthermore, if $\partial G \in C^1$, then every solution $v \in \mathcal{D}_L(G)$ of (4) satisfying $v \cdot f(x) \leq 0$ in G has one of the following properties:

- (1) *v has a zero in G , or*
- (2) *v is a constant multiple of u .*

Proof (The first statement) Suppose to the contrary that there exists a solution $v \in \mathcal{D}_L(G)$ of (4) which satisfies $v \cdot f(x) \leq 0$ in G and $v \neq 0$ on \bar{G} . Then, the inequality (6) of Theorem 1 holds. Integrating (6) over G and then using the divergence theorem, we obtain

$$M[u] \geq \int_G \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) dx. \quad (15)$$

Since $u = 0$ on ∂G and $v \neq 0$ on \bar{G} , we observe that u is not a constant multiple of v , and hence $\nabla(u/v) \neq 0$. Therefore, we see that

$$\int_G \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) dx > 0,$$

which, together with (15), implies that $M[u] > 0$. This contradicts the hypothesis $M[u] \leq 0$. The proof of the first statement is complete.

(The second statement) Next we consider the case where $\partial G \in C^1$. Let $v \in \mathcal{D}_L(G)$ be a solution of (4) satisfying $v \cdot f(x) \leq 0$ in G and $v \neq 0$ in G . Since $\partial G \in C^1$, $u \in C^1(\bar{G}; \mathbb{R})$ and $u = 0$ on ∂G , we find that u belongs to the Sobolev space $H_1(G)$ which is the closure in the norm

$$\|u\| = \|u\|_1 = \left(\int_G \sum_{|\alpha| \leq 1} |D^\alpha u|^2 dx \right)^{1/2} \quad (16)$$

of the class $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G (see, e.g., Agmon [1, p.131]). Let $\{u_k\}$ be a sequence of functions in $C_0^\infty(G)$ converging to u in the norm (16). Then we easily see that the inequality (6) with $u = u_k$ holds. Since (15) holds for $u = u_k$, we find that $M[u_k] \geq 0$. Since $A_{ij}(x)$ ($i, j = 1, 2, \dots, n$) and $\beta(\beta - 1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} |f(x)|^{((\beta-1)/\beta)}$ are uniformly bounded in G , there is a constant $K > 0$ such that

$$|M[u_k] - M[u]| \leq K \int_G \left| \sum_{i,j=1}^n \left(\frac{\partial u_k}{\partial x_i} \frac{\partial(u_k - u)}{\partial x_j} + \frac{\partial(u_k - u)}{\partial x_i} \frac{\partial u}{\partial x_j} \right) \right| dx + K \int_G |u_k(u_k - u) + (u_k - u)u| dx.$$

Applying the Schwarz inequality, we obtain

$$|M[u_k] - M[u]| \leq K(n^2 + 1)(\|u_k\| + \|u\|)\|u_k - u\|. \tag{17}$$

Since $\lim_{k \rightarrow \infty} \|u_k - u\| = 0$, we observe that $\lim_{k \rightarrow \infty} M[u_k] = M[u] \geq 0$, and therefore $M[u] = 0$ in view of (14). Let B denote an arbitrary ball with $\bar{B} \subset G$ and define

$$H_B[u] \equiv \int_B \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) dx$$

for $u \in C^1(G; \mathbb{R})$. It is easy to see that

$$0 \leq H_B[u_k] \leq M[u_k],$$

and the estimate

$$|H_B[u_k] - H_B[u]| \leq \tilde{K}(\|w_k\|_{1,B} + \|w\|_{1,B})\|w_k - w\|_{1,B}$$

holds, where $w_k = u_k/v$, $w = u/v$, \tilde{K} is a positive constant and the subscript B indicates the integrals involved in the norm (16) are to be taken over B only. Since $v \neq 0$ on \bar{B} , we find that $\|w_k - w\|_{1,B} \rightarrow 0$ as $\|u_k - u\| \rightarrow 0$, and hence $H_B[u_k] \rightarrow H_B[u]$ ($k \rightarrow \infty$). Since $\lim_{k \rightarrow \infty} M[u_k] = M[u] = 0$, it follows that $H_B[u] = 0$, and consequently $\nabla(u/v) \equiv 0$ in B . Since B is arbitrary, we conclude that $u/v = K_0$ in G , and hence on \bar{G} by continuity, for some nonzero constant K_0 . This completes the proof of the second statement.

COROLLARY 1 *Assume that $f(x) \geq 0$ [or $f(x) \leq 0$] in G . If there is a nontrivial function $u \in C^1(\bar{G}; \mathbb{R})$ such that $u = 0$ on ∂G and $M[u] \leq 0$, then (4) has no negative [or positive] solution on \bar{G} .*

Proof Let $v \in \mathcal{D}_L(G)$ be a solution of (4) which is negative [or positive] on \bar{G} . Then, it is clear that $v \cdot f(x) \leq 0$ in G , and therefore it follows from Theorem 3 that v must vanish at some point of \bar{G} . This is a contradiction, and the proof is complete.

THEOREM 4 *If there is a nontrivial solution $u \in \mathcal{D}_\ell(G)$ of (13) such that $u = 0$ on ∂G and*

$$V[u] \equiv \int_G \left[\sum_{i,j=1}^n (a_{ij}(x) - A_{ij}(x)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + (\beta(\beta - 1))^{((1-\beta)/\beta)} C(x)^{(1/\beta)} \times |f(x)|^{((\beta-1)/\beta)} - c(x)u^2 \right] dx \geq 0, \tag{18}$$

then every solution $v \in \mathcal{D}_L(G)$ of (4) satisfying $v \cdot f(x) \leq 0$ in G vanishes at some point of \bar{G} . Furthermore, if $\partial G \in C^1$, then every solution $v \in \mathcal{D}_L(G)$ of (4) satisfying $v \cdot f(x) \leq 0$ in G has one of the following properties:

- (1) v has a zero in G , or
- (2) v is a constant multiple of u .

Proof It suffices to start from the inequality (11) and apply the same argument as that used in the proof of Theorem 3. The details are left to the reader.

An alternative proof will be presented here. By the definitions of $V[u]$ and $M[u]$ we have

$$M[u] = -V[u] + \int_G \left[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - c(x)u^2 \right] dx.$$

Since the last integral over G vanishes for any solution u of (13) such that $u = 0$ on ∂G (*cf.* (12)), it follows that $M[u] = -V[u]$, which, in view of (18), implies that $M[u] \leq 0$. The conclusion of the theorem then follows from Theorem 3.

COROLLARY 2 Assume that $f(x) \geq 0$ [or $f(x) \leq 0$] in G . If there is a nontrivial solution $u \in \mathcal{D}_\ell(G)$ of (13) such that $u = 0$ on ∂G and $V[u] \geq 0$, then (4) has no negative [or positive] solution on \bar{G} .

The following variants of Theorems 3 and 4 will be useful in the study of the oscillatory behavior of the Eq. (4) in unbounded domains.

THEOREM 5 Suppose that G is divided into two subdomains G_1 and G_2 by an $(n-1)$ -dimensional piecewise smooth hypersurface in such a way that

$$f(x) \geq 0 \text{ in } G_1 \quad \text{and} \quad f(x) \leq 0 \text{ in } G_2. \tag{19}$$

If there are nontrivial functions $u_k \in C^1(\bar{G}_k; \mathbb{R})$ such that $u_k = 0$ on ∂G_k and

$$M_k[u_k] = \int_{G_k} \left[\sum_{i,j=1}^n A_{ij}(x) \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} - \beta(\beta-1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} \right. \\ \left. \times |f(x)|^{((\beta-1)/\beta)} u_k^2 \right] dx \leq 0 \quad (k = 1, 2), \tag{20}$$

then every solution $v \in \mathcal{D}_L(G)$ of (4) has a zero on \bar{G} .

THEOREM 6 Suppose that G is divided into two adjacent subdomains G_1 and G_2 as mentioned in Theorem 5. If there are two nontrivial solutions $u_k \in \mathcal{D}_\ell(G_k)$ of (13) such that $u_k = 0$ on ∂G_k and

$$V_k[u_k] = \int_{G_k} \left[\sum_{i,j=1}^n (a_{ij}(x) - A_{ij}(x)) \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right. \\ \left. + (\beta(\beta-1)^{((1-\beta)/\beta)} C(x)^{(1/\beta)} |f(x)|^{((\beta-1)/\beta)} - c(x)) u_k^2 \right] dx \geq 0, \quad (k = 1, 2), \tag{21}$$

then every solution $v \in \mathcal{D}_L(G)$ of (4) has a zero on \bar{G} .

Proof of Theorem 5 Suppose that (4) has a solution $v \in \mathcal{D}_L(G)$ with no zero on \bar{G} . Then, either $v < 0$ on \bar{G} or $v > 0$ on \bar{G} . If $v < 0$ on \bar{G} then $v < 0$ on \bar{G}_1 , so that $v \cdot f(x) \leq 0$ in G_1 . Applying Corollary 1, we see that

no solution of (4) can be negative on \bar{G}_1 . This contradiction shows that it is impossible that $v < 0$ on \bar{G} . Likewise it cannot occur that $v > 0$ on \bar{G} , and the conclusion follows.

Theorem 6 can be proved similarly.

3. PICONE-TYPE INEQUALITY WITH APPLICATION TO EQUATION (5)

Our aim in this section is to establish a Picone-type inequality for the superlinear-sublinear elliptic operator (3) in relation to the linear elliptic operator (1) and use it to prove Sturmian comparison theorems for the Eq. (5) adopting the linear Eq. (13) as a comparison equation.

We suppose as before that G is a bounded domain in \mathbb{R}^n with piecewise smooth boundary and retain the hypotheses (A_1) , (A_2) and (A_4) made at the beginning of Section 2. With regard to the function $D(x)$ in (3) we assume that:

$$(A_5) \quad D(x) \in C(\bar{G}; [0, \infty)).$$

THEOREM 7 *If $v \in \mathcal{D}_{\tilde{L}}(G)$ and $v \neq 0$ in G , then the following inequality holds for any $u \in C^1(G; \mathbb{R})$:*

$$\begin{aligned} & \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) \\ & + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ & \leq \sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} \\ & \quad \times C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} u^2 + \frac{u^2}{v} \tilde{L}[v]. \end{aligned} \tag{22}$$

Proof We start with the identity (7). Combining (7) with (3) written as

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial v}{\partial x_j} \right) = -C(x)|v|^{\beta-1}v - D(x)|v|^{\gamma-1}v + \tilde{L}[v]$$

yields

$$\begin{aligned} & \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ &= \sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - (C(x)|v|^{\beta-1} + D(x)|v|^{\gamma-1})u^2 + \frac{u^2}{v} \tilde{L}[v]. \end{aligned} \tag{23}$$

In Young’s inequality (9), letting

$$\begin{aligned} p &= \frac{\beta - \gamma}{1 - \gamma} (> 1), \\ q &= \frac{\beta - \gamma}{\beta - 1}, \\ a &= C(x)^{(1-\gamma)/(\beta-\gamma)} |v|^{(1-\gamma)(\beta-1)/(\beta-\gamma)}, \\ b &= \left(\frac{D(x)}{(((\beta - 1)/(1 - \gamma))^{(1/(1-\gamma))} |v|)^{1-\gamma}} \right)^{(\beta-1)/(\beta-\gamma)}, \end{aligned}$$

we have

$$\begin{aligned} & \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} \\ & \leq \frac{1 - \gamma}{\beta - \gamma} (C(x)|v|^{\beta-1} + D(x)|v|^{\gamma-1}), \end{aligned}$$

or equivalently

$$\begin{aligned} C(x)|v|^{\beta-1} + D(x)|v|^{\gamma-1} & \geq \frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} \\ & \times C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)}. \end{aligned} \tag{24}$$

Using (24) in (23), we obtain the desired inequality (22).

THEOREM 8 (Picone-type inequality) *Assume that $u \in \mathcal{D}_\ell(G)$, $v \in \mathcal{D}_{\tilde{L}}(G)$ and $v \neq 0$ in G . Then we have the following inequality*

$$\begin{aligned} & \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(u a_{ij}(x) \frac{\partial u}{\partial x_j} - \frac{u^2}{v} A_{ij}(x) \frac{\partial v}{\partial x_j} \right) \\ & \geq \sum_{i,j=1}^n (a_{ij}(x) - A_{ij}(x)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} \right. \\
 & \quad \left. \times C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} - c(x) \right) u^2 \\
 & + \sum_{i,j=1}^n A_{ij}(x) \left(v \frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) \right) \left(v \frac{\partial}{\partial x_j} \left(\frac{u}{v} \right) \right) + \frac{u}{v} (v\ell[u] - u\tilde{L}[v]). \quad (25)
 \end{aligned}$$

Proof As in the proof of Theorem 2 the conclusion follows from (22) combined with (12).

Sturmian comparison theorems for the superlinear-sublinear elliptic Eq. (5) will be given below.

THEOREM 9 *If there is a nontrivial function $u \in C^1(\bar{G}; \mathbb{R})$ such that $u = 0$ on ∂G and*

$$\begin{aligned}
 \tilde{M}[u] \equiv \int_G \left[\sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} \right. \\
 \left. \times C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} u^2 \right] dx \leq 0, \quad (26)
 \end{aligned}$$

then every solution $v \in \mathcal{D}_{\tilde{L}}(G)$ of (5) vanishes at some point of \bar{G} . Furthermore, if $\partial G \in C^1$, then every solution $v \in \mathcal{D}_{\tilde{L}}(G)$ of (5) has one of the following properties:

- (i) v has a zero in G , or
- (ii) v is a constant multiple of u .

Proof Suppose that there exists a solution $v \in \mathcal{D}_{\tilde{L}}(G)$ of (5) such that $v \neq 0$ on \bar{G} . Then, the inequality (22) of Theorem 7 holds with $\tilde{L}[v] = 0$. Integrating (22) over G and arguing as in the proof of Theorem 3, we observe that $\tilde{M}[u] > 0$, which contradicts the hypothesis $\tilde{M}[u] \leq 0$. This completes the proof of the first statement. In the case where $\partial G \in C^1$, let $v \in \mathcal{D}_{\tilde{L}}(G)$ be a solution of (5) such that $v \neq 0$ in G . By the same arguments as were used in Theorem 3, we conclude that $\tilde{M}[u] = 0$, which implies that v is a constant multiple of u . The proof of the second statement is complete.

THEOREM 10 *If there is a nontrivial solution $u \in \mathcal{D}_\ell(G)$ of (13) such that $u=0$ on ∂G and*

$$\begin{aligned} \tilde{V}[u] \equiv \int_G \left[\sum_{i,j=1}^n (a_{ij}(x) - A_{ij}(x)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \left(\frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} \right. \right. \\ \left. \left. \times C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} - c(x) \right) u^2 \right] dx \geq 0, \end{aligned} \tag{27}$$

then every solution $v \in \mathcal{D}_{\bar{L}}(G)$ of (5) vanishes at some point of \bar{G} . Furthermore, if $\partial G \in C^1$, then every solution $v \in \mathcal{D}_{\bar{L}}(G)$ of (5) has one of the following properties:

- (i) v has a zero in G , or
- (ii) v is a constant multiple of u .

Proof Using the fact that $\ell[u]=0$ in G and $u=0$ on ∂G , we have $\tilde{V}[u] = -\tilde{M}[u]$, so that $\tilde{M}[u] \leq 0$ by (27). The conclusion of the theorem is an immediate consequence of Theorem 9.

Our proof of Theorem 9 can be stated as a result on Wirtinger-type inequalities as follows.

THEOREM 11 (Wirtinger-type inequality) *Let $\partial G \in C^1$. If there exists a solution $v \in \mathcal{D}_{\bar{L}}(G)$ of (5) such that $v \neq 0$ in G , then the inequality*

$$\begin{aligned} \int_G \left[\sum_{i,j=1}^n A_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} \right. \\ \left. \times C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} u^2 \right] dx \geq 0 \end{aligned} \tag{28}$$

holds for any nontrivial function $u \in C^1(\bar{G}; \mathbb{R})$ such that $u=0$ on ∂G , where u is a constant multiple of v if equality holds.

4. OSCILLATION CRITERIA FOR EQUATIONS (4) AND (5)

The purpose of this section is to show that Sturmian comparison theorems of Sections 2 and 3 can be applied to establish oscillation

criteria for the nonlinear elliptic Eqs. (4) and (5) defined in an unbounded domain Ω in \mathbb{R}^n .

4.1. Oscillation Criterion for the Equation (4)

Let Ω be an unbounded domain in \mathbb{R}^n and assume that:

- (B₁) $A_{ij}(x) \in C(\Omega; \mathbb{R})$ ($i, j = 1, 2, \dots, n$), and the matrix $(A_{ij}(x))$ is symmetric and positive definite in Ω ;
- (B₂) $C(x) \in C(\Omega; [0, \infty))$, $f(x) \in C(\Omega; \mathbb{R})$;
- (B₃) β is a constant with $\beta > 1$.

The domain $\mathcal{D}_L(\Omega)$ of L is defined to be the set of all functions $v \in C^1(\Omega; \mathbb{R})$ with the property that $A_{ij}(x)(\partial v / \partial x_j) \in C^1(\Omega; \mathbb{R})$ for $i, j = 1, 2, \dots, n$.

DEFINITION 1 A function $v : \Omega \rightarrow \mathbb{R}$ is said to be *oscillatory* in Ω if v has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in \mathbb{R}^n; |x| > r\}.$$

THEOREM 12 Assume that for any $r > 0$ there exists a bounded domain G in Ω_r with piecewise smooth boundary, which can be divided into two subdomains G_1 and G_2 by an $(n-1)$ -dimensional hypersurface in such a way that $f(x) \geq 0$ in G_1 and $f(x) \leq 0$ in G_2 . Assume furthermore that $C(x) \geq 0$ in G and that there are nontrivial functions $u_k \in C^1(\overline{G_k}; \mathbb{R})$ such that $u_k = 0$ on ∂G_k and $M_k[u_k] \leq 0$ ($k = 1, 2$), where M_k are defined by (20). Then every solution $v \in \mathcal{D}_L(\Omega)$ of (4) is oscillatory in Ω .

Proof We need only to apply Theorem 5 to make sure that v has a zero in any domain G as mentioned in the hypotheses of Theorem 12.

Example Consider the forced superlinear elliptic equation

$$\Delta v + K(\sin(x_1 - \pi)\sin x_2)|v|^{\beta-1}v = \cos x_1 \sin x_2, \quad (x_1, x_2) \in \Omega, \quad (29)$$

where Δ is the two-dimensional Laplacian, $K > 0$ is a constant and Ω is an unbounded domain in \mathbb{R}^2 containing a horizontal strip such that

$$[\pi, \infty) \times [0, \pi] \subset \Omega.$$

We will prove that every solution v of (29) is oscillatory in Ω provided $K > 0$ is sufficiently large. For any fixed $m \in \mathbb{N}$ consider the square $G = ((2m - 1)\pi, 2m\pi) \times (0, \pi)$, which is divided into two subdomains

$$G_1 = ((2m - 1)\pi, (2m - (1/2))\pi) \times (0, \pi)$$

and $G_2 = ((2m - (1/2))\pi, 2m\pi) \times (0, \pi)$

by the vertical line $x_1 = (2m - (1/2))\pi$. We see that $f(x) = \cos x_1 \sin x_2 \leq 0$ in G_1 and $f(x) \geq 0$ in G_2 . Let us put $u_k = \sin 2x_1 \sin x_2$ ($k = 1, 2$). Then, $u_k = 0$ on ∂G_k , and an elementary calculation yields

$$M_k[u_k] = \int_{G_k} \left[|\nabla u_k|^2 - \beta(\beta - 1)^{((1-\beta)/\beta)} (K \sin(x_1 - \pi) \sin x_2)^{(1/\beta)} \right. \\ \left. \times |\cos x_1 \sin x_2|^{((\beta-1)/\beta)} u_k^2 \right] dx_1 dx_2 \\ = \frac{5}{8} \pi^2 - \frac{8}{3} K^{(1/\beta)} \beta(\beta - 1)^{((1-\beta)/\beta)} B\left(\frac{3}{2} + \frac{1}{2\beta}, 2 - \frac{1}{2\beta}\right),$$

where $B(s, t)$ is the beta function ($k = 1, 2$). This implies that $M_k[u_k] \leq 0$ ($k = 1, 2$) if $K > 0$ is chosen so that

$$K \geq \left[\frac{15}{64} \pi^2 \cdot \left(\beta(\beta - 1)^{((1-\beta)/\beta)} B\left(\frac{3}{2} + \frac{1}{2\beta}, 2 - \frac{1}{2\beta}\right) \right)^{-1} \right]^\beta.$$

From Theorem 12 it follows that every solution $v \in C^2(\Omega; \mathbb{R})$ of (29) is oscillatory in Ω for all sufficiently large $K > 0$.

Remark Theorem 12 is a generalization of a result of Nasr [8, Corollary].

4.2. Oscillation Criteria for the Equation (5)

Our task here is to obtain oscillation criteria for the superlinear-sublinear elliptic Eq. (5) in an unbounded domain Ω by comparing it with a suitable linear elliptic Eq. (13) whose oscillatory behavior is already known.

The conditions we assume for (5) and (13) are as follows:

- (\tilde{B}_1) $A_{ij}(x) \in C(\Omega; \mathbb{R})$ ($i, j = 1, 2, \dots, n$), and the matrix $(A_{ij}(x))$ is symmetric and positive definite in Ω ; and the same is true of $a_{ij}(x)$ ($i, j = 1, 2, \dots, n$);
- (\tilde{B}_2) $C(x) \in C(\Omega; [0, \infty))$, $D(x) \in C(\Omega; [0, \infty))$;
- (\tilde{B}_3) β and γ are constants such that $\beta > 1$ and $0 < \gamma < 1$.

The domain $\mathcal{D}_{\tilde{L}}(\Omega)$ of \tilde{L} is defined to be the same as that of L , that is, $\mathcal{D}_{\tilde{L}}(\Omega) = \mathcal{D}_L(\Omega)$. The domain $\mathcal{D}_\ell(\Omega)$ of ℓ is defined similarly.

DEFINITION 2 A bounded domain G with $\bar{G} \subset \Omega$ is said to be a *nodal domain* for the Eq. (13) if there exists a nontrivial function $u \in \mathcal{D}_\ell(G)$ such that $\ell[u] = 0$ in G and $u = 0$ on ∂G . The Eq. (13) is called *nodally oscillatory* in Ω if (13) has a nodal domain contained in Ω_r for any $r > 0$.

THEOREM 13 *Assume that:*

$$(a_{ij}(x) - A_{ij}(x)) \text{ is positive semidefinite in } \Omega, \quad (30)$$

$$c(x) \leq \frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} \text{ in } \Omega. \quad (31)$$

If (13) is nodally oscillatory in Ω , then every solution $v \in \mathcal{D}_{\tilde{L}}(\Omega)$ of (5) is oscillatory in Ω .

Proof Since (13) is nodally oscillatory in Ω , there is a nodal domain $G \subset \Omega_r$ for any $r > 0$, and hence there exists a nontrivial function $u \in \mathcal{D}_\ell(G)$ such that $\ell[u] = 0$ in G and $u = 0$ on ∂G . The conditions (30) and (31) ensures that $\tilde{V}[u] \geq 0$ is satisfied. From Theorem 10 it follows that every solution $v \in \mathcal{D}_{\tilde{L}}(\Omega)$ of (5) vanishes at some point of \bar{G} , that is, v must have a zero in Ω_r for any $r > 0$. This implies that v is oscillatory in Ω .

COROLLARY 3 *If the linear elliptic equation*

$$\Delta u + \frac{\beta - \gamma}{1 - \gamma} \left(\frac{\beta - 1}{1 - \gamma} \right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} u = 0 \quad (32)$$

is nodally oscillatory in Ω , then every solution $v \in C^2(\Omega; \mathbb{R})$ of the superlinear-sublinear equation

$$\Delta v + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v = 0 \tag{33}$$

is oscillatory in Ω .

This is an immediate consequence of Theorem 13.

Various criteria for nodal oscillation of linear elliptic equations of the form

$$\Delta u + c(x)u = 0, \quad x \in \mathbb{R}^n, \tag{34}$$

$c(x)$ being a continuous function in \mathbb{R}^n , have been given by Kreith and Travis [7]. They have shown in particular that (34) is nodally oscillatory if

$$\int_{\mathbb{R}^2} c(x)dx = \infty \quad (n = 2),$$

$$\int_0^\infty S[c(x)](r)dr = \infty \quad (n \geq 3),$$

where $S[c(x)](r)$ denotes the spherical mean of $c(x)$ over the sphere $\{x \in \mathbb{R}^n; |x|=r\}$. Applying this result to the Eq. (32) we obtain from Corollary 3 the following concrete oscillation criterion for the Eq. (33).

COROLLARY 4 *Let $\Omega = \mathbb{R}^n$. If*

$$\int_{\mathbb{R}^2} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} dx = \infty \quad (n = 2), \tag{35}$$

$$\int_0^\infty S[C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)}](r)dr = \infty \quad (n \geq 3), \tag{36}$$

then every C^2 -solution v of (33) is oscillatory in \mathbb{R}^n .

Note that (35) and (36) trivially hold if $C(x)$ and $D(x)$ are bounded below by positive constants in \mathbb{R}^n .

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