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Bifurcation of Solutions of Nonlinear Sturm–Liouville Problems

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A global bifurcation theorem for the following nonlinear Sturm–Liouville problem is given

$$\begin{cases} u''(t) = -h(\lambda, t, u(t), u'(t)), & \text{a.e. on } (0,1) \\ u(0)\cos\eta - u'(0)\sin\eta = 0 & \\ u(1)\cos\zeta + u'(1)\sin\zeta = 0 & \text{with } \eta, \zeta \in [0, \frac{\pi}{2}]. \end{cases} \qquad (*)$$

Moreover we give various versions of existence theorems for boundary value problems

$$\begin{cases} u''(t) = -g(t, u(t), u'(t)), & \text{a.e. on } (0,1) \\ u(0)\cos\eta - u'(0)\sin\eta = 0 & \\ u(1)\cos\zeta + u'(1)\sin\zeta = 0. & \end{cases} \qquad (**)$$

The main idea of these proofs is studying properties of an unbounded connected subset of the set of all nontrivial solutions of the nonlinear spectral problem (*), associated with the boundary value problem (**), in such a way that $h(1,\cdot,\cdot,\cdot) = g$.

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In this paper we will study the nonlinear spectral Sturm–Liouville problem

$$\begin{cases} u''(t) = -h(\lambda, t, u(t), u'(t)), & \text{a.e. on } (0,1) \\ u \in \mathcal{S} & \end{cases} \qquad (*)$$

where

$$\mathcal{S} = \{u \in C^1[0,1] : u(0)\cos\eta - u'(0)\sin\eta \\ = 0 \wedge u(1)\cos\zeta + u'(1)\sin\zeta = 0\}$$

for $\eta, \zeta \in [0, (\pi/2)]$. Let us also denote

$$\mathcal{S}_0 = \{u \in C^1[0,1] : u(0)\cos\eta - u'(0)\sin\eta \\ = 0 \wedge u(1)\cos\zeta + u'(1)\sin\zeta = 0\}$$

where $\eta, \zeta \in (0, (\pi/2)]$. In Section 1 we give some sufficient conditions for the existence of an unbounded connected subset of the set of all nontrivial, nonnegative solutions of this problem.

In Section 2 we give some conditions for function $g : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, sufficient for existence of nonnegative solution of the following problem

$$\begin{cases} u''(t) = -g(t, u(t), u'(t)) & \text{for } t \in (0,1) \\ u \in \mathcal{S}. \end{cases} \qquad (**)$$

All we assume is a behaviour of $g(t,\cdot,\cdot) : \mathbf{R}^2 \to \mathbf{R}$ in the neighbourhood of the zero point $(0,0) \in \mathbf{R}^2$ uniformly with respect to $t \in [0,1]$ and for the large arguments $(s,y) \in \mathbf{R}^2$, and the Bernstein conditions (*cf.* [2]) need not be satisfied.

Mawhin and Omana showed in [4] that if a Caratheodory function $\hat{g} : [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfies the following conditions

$$\hat{g}(t,0) = 0; \liminf_{s \to +\infty} s^{-1}\hat{g}(t,s) > \mu_0 \quad \text{and} \quad \limsup_{s \to 0^+} s^{-1}\hat{g}(t,s) < \mu_0$$
$$\text{uniformly with respect to } t \in [0,1]$$

then there exists a nonnegative solution of the problem

$$\begin{cases} u''(t) + (p'(t)/p(t))u'(t) = -\hat{g}(t,u(t)) & \text{a.e. on } (0,1) \\ u(0) = 0, u(1) = 0 \end{cases} \qquad (\mathcal{M})$$

where μ_0 is the minimal eigenvalue of the linear problem

$$\begin{cases} u''(t) + (p'(t)/p(t))u'(t) = -\lambda u(t) & \text{a.e. on } (0,1) \\ u(0) = 0, u(1) = 0 \end{cases}.$$

and $p : [0,1] \to \mathbf{R}$ is continuous and such that $p|_{(0,1]} \in C^1(0,1]$, $p(0) = 0$, $p(t) > 0$ for $t \in (0,1]$ and $(1/p) \in L^1(0,1)$.

In this paper we prove a theorem which is a generalization of the above result for some class of Picard problems with a Caratheodory right hand side $\hat{g}$ depending on t, u, u'.

All proofs of the existence theorems are based on ideas differing from those used in papers [2] or [4]; we can see that *a priori* bounds and topological transversality theorems are not necessary here. The main idea of these proofs is studying properties of an unbounded connected subset of the set of all nontrivial solutions of a nonlinear spectral problem (*) associated with the boundary value problem (**), such that $h(1,\cdot,\cdot,\cdot)=g$. The existence of this subset can be established by the global bifurcation theorem (*cf.* [6, 3]).

1. GLOBAL BIFURCATION THEOREM FOR STURM-LIOUVILLE PROBLEM

In this paper we will need the following notations. Let $\langle\cdot,\cdot\rangle$ be a scalar product in $L^2(0,1)$. Let $\|\cdot\|_0$ be the supremum norm in $C[0,1]$ and $\|\cdot\|_1$ be the norm in $C^1[0,1]$ given by $\|u\|_1=\|u\|_0+\|u'\|_0$.

Let $F:(0,+\infty)\times C^1[0,1]\to C^1[0,1]$ be a completely continuous map such that $F(\cdot,0)=0$ and let $f:(0,+\infty)\times C^1[0,1]\to C^1[0,1]$ be a map given by $f(\lambda,u)=u-F(\lambda,u)$. The point $(\lambda_0,0)$ is a bifurcation point of the map $f:(0,+\infty)\times C^1[0,1]\to C^1[0,1]$ if for all open $U\subset(0,+\infty)\times C^1[0,1]$ satisfying $(\lambda_0,0)\in U$ there exists $(\lambda,u)\in U$, such that $u\neq 0$ and $f(\lambda,u)=0$.

If $(\lambda_0,0)$ is the bifurcation point that is an isolated one in the set of all bifurcation points of the map f then there exists such $\varepsilon_0>0$ that for any $\delta\in(0,\varepsilon_0)$ there exists positive $R>0$, such that

$$\begin{aligned}&[f(\lambda,\cdot)]^{-1}(0)\cap\overline{K(0,R)}=\{0\}\quad\text{for}\\&\lambda\in[\lambda_0-\varepsilon_0,\lambda_0-\delta]\cup[\lambda_0+\delta,\lambda_0+\varepsilon_0]\end{aligned}\tag{1.1}$$

Moreover, by additivity and homotopy properties of topological degree the number

$$\begin{aligned}s[f,\lambda_0]=\ &\deg(f(\lambda_0+\varepsilon_0,\cdot),K(0,R),0)-\\&-\deg(f(\lambda_0-\varepsilon_0,\cdot),K(0,R),0)\end{aligned}$$

is well defined.

On the other hand if $\lambda_0 \in (0, +\infty)$ satisfies condition (1.1) and $s[f, \lambda_0] \neq 0$ then $(\lambda_0, 0)$ is the isolated bifurcation point of f.

The next theorem is a corollary from the global bifurcation theorem (*cf.* [6, 3]) and will be the main tool used in this paper.

THEOREM A *If* $(\lambda_0, 0) \in (0, +\infty) \times C^1[0, 1]$ *is the unique bifurcation point of* f *and* $s[f, \lambda_0] \neq 0$ *then there exists a connected component* C *of the set*

$$\mathcal{R}_f = \overline{\{(\lambda, u) \in \boldsymbol{R} \times C^1[0,1] : f(\lambda, u) = 0 \wedge u \neq 0\}}$$

such that C *is not compact and* $(\lambda_0, 0) \in C$. ∎

Let us remind that $h : [0, +\infty) \times [0, 1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function, if $h(\cdot, t, \cdot, \cdot) : [0, +\infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous for $t \in [0, 1]$, $h(\lambda, \cdot, s, y) : [0, 1] \to \mathbf{R}$ is measurable for $(\lambda, s, y) \in [0, +\infty) \times \mathbf{R} \times \mathbf{R}$ and

$$\forall_{R>0} \exists_{m_R \in L^1(0,1)} \forall_{(\lambda,s,y) \in [0,+\infty) \times \mathbf{R}^2} \forall_{t \in [0,1]} |\lambda| + |s| + |y| \\ \leq R \Rightarrow |h(\lambda, t, s, y)| \leq m_R(t).$$

Assume that

$$h(\cdot, \cdot, 0, \cdot) = 0 \tag{1.2}$$

$$h(0, \cdot, \cdot, \cdot) = 0 \tag{1.3}$$

and

$$\exists_{m>0} \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{s \geq 0} \forall_{y \in \mathbf{R}} \forall_{\lambda \geq 0} \forall_{t \in [0,1]} |s| + |y| \\ \leq \delta \Rightarrow |h(\lambda, t, s, y) - m\lambda s| \leq \varepsilon \lambda |s| \tag{1.4}$$

THEOREM 1 *Assume* $h : [0, +\infty) \times [0, 1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ *is the Caratheodory function satisfying conditions* (1.2)–(1.4). *Then there exists unbounded connected subset* $C \subset (0, +\infty) \times C^1[0, 1]$ *of the set of all nontrivial solutions of the problem*

$$\begin{cases} u''(t) + \mu u(t) + h(\lambda, t, u(t), u'(t)) = 0 & a.e. \text{ on } (0,1) \\ u \in \mathcal{S} \end{cases} \tag{1.5}$$

such that $\{(0,((\mu_0-\mu)/m))\}\in\bar{C}$ *and for every* $(\lambda,u)\in C$ *we have* $u\geq 0$, *where* $\mu<\mu_0$ *is any nonpositive number and* μ_0 *is the minimal eigenvalue of the linear problem*

$$\begin{cases} u''(t)+\lambda u(t)=0 & t\in(0,1) \\ u\in\mathcal{S}. \end{cases} \tag{$\mathcal{L}_\lambda$}$$

Proof First let us remind some properties of linear problem $(\mathcal{L}_\lambda)$. It is well known (see [5]) that there exists the minimal eigenvalue $\mu_0\geq 0$ of the problem $\mathcal{L}_\lambda$, such that the space of its eigenvectors is generated by a function $u_0\in C^2[0,1]$ satisfying $u_0(t)>0$ for $t\in(0,1)$. For every λ which is not the eigenvalue of $\mathcal{L}_\lambda$ there exists a continuous linear map $T_\lambda: L^1(0,1)\to C^1[0,1]$, such that

$$T_\lambda v=u \Leftrightarrow \begin{cases} u''(t)+\lambda u(t)+v(t)=0 & \text{a.e. on } (0,1) \\ u\in\mathcal{S}. \end{cases} \tag{1.6}$$

and $T_\lambda: L^2(0,1)\to L^2(0,1)$ is self-adjoint, and $T_\lambda: C[0,1]\to C[0,1]$ is completely continuous (see [1]).

We can see (*cf.* [5]), that for $\lambda\leq 0$ and $v\geq 0$ we have $T_\lambda v\geq 0$.

Condition (1.4) implies that

$$\exists_{\beta>0}\exists_{r_0>0}\forall_{s\geq 0}\forall_{y\in\mathbf{R}}\forall_{\lambda\geq 0}\forall_{t\in[0,1]}|s|+|y|\leq r_0\Rightarrow h(\lambda,t,s,y)\geq\lambda\beta s \tag{1.7}$$

Let us define the Caratheodory function $\tilde{h}:[0,+\infty)\times[0,1]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ by

$$\tilde{h}(\lambda,t,s,y)=\begin{cases} h(\lambda,t,s,y) & \text{for } s\geq 0 \\ \lambda\beta|s| & \text{for } s<0 \end{cases}$$

Let us choose $\mu<\min\{\mu_0,0\}$. We will consider the map $f:[0,+\infty)\times C^1[0,1]\to C^1[0,1]$ given by $f(\lambda,u)=u-T_\mu G(\lambda,u)$, where $G:[0,+\infty)\times C^1[0,1]\to L^1(0,1)$ is Niemytskii operator for function $\tilde{h}$, given by $G(\lambda,u)(t)=\tilde{h}(\lambda,t,u(t),u'(t))$.

We can see that for $u\geq 0$ we have

$$f(\lambda,u)=0$$
$$\Leftrightarrow\begin{cases} u''(t)+\mu u(t)+h(\lambda,t,u(t),u'(t))=0 & \text{a.e. on } (0,1) \\ u\in\mathcal{S} \end{cases}.$$

First we are going to show the map f is a completely continuous vector field. We will prove that $T_\mu \circ G : [0, +\infty) \times C^1[0,1] \to C^1[0,1]$ is completely continuous. Let us then take a sequence $\{(\lambda_n, u_n)\} \subset [0, +\infty) \times C^1[0,1]$ such that $|\lambda_n| \leq (R/2)$ and $\|u_n\|_1 \leq (R/2)$ for some positive $R > 0$. Then $|G(\lambda_n, u_n)(t)| \leq m_R(t)$. We will prove that if $v_n = T_\mu G(\lambda_n, u_n)$ then sequences $\{v_n\}$ and $\{v_n'\}$ are uniformly bounded and $\{v_n'\}$ is equicontinuous.

We know (see [1]) that there exists continuous Green function $\hat{G} : [0,1]^2 \to \mathbf{R}$ such that $(T_\mu u)(t) = \int_0^1 \hat{G}(t,s)u(s)ds$ and $(\partial\hat{G}/\partial t)(t,s)$ exists for $(t,s) \in \{(t,s) \in [0,1]^2 : t \neq s\}$. We can also see that $(\partial\hat{G}/\partial t)(t,s)$ is uniformly continuous on triangles $\{(t,s) \in [0,1]^2 : t < s\}$ and $\{(t,s) \in [0,1]^2 : s < t\}$, hence is bounded function.

That is why we have $\|v_n\|_0 \leq \sup_{(t,s) \in [0,1]^2} |\hat{G}(t,s)| \int_0^1 m_R(t)dt$ and $v_n'(t) = \int_0^1 (\partial\hat{G}/\partial t)(t,s)u_n(s)$. We can see then that $\|v_n'\|_0 \leq \|(\partial\hat{G}/\partial t)\|_{L^\infty([0,1]^2)} \int_0^1 m_R(t)dt$. Now we will show that $\{v_n'\}$ are equicontinuous. Let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$; then we have

$$\begin{aligned}
|v_n'(t_1) - v_n'(t_2)| \leq \int_0^1 \left|\frac{\partial\hat{G}}{\partial t}(t_1,s) - \frac{\partial\hat{G}}{\partial t}(t_2,s)\right| m_R(s)ds = \\
= \int_0^{t_1} \left|\frac{\partial\hat{G}}{\partial t}(t_1,s) - \frac{\partial\hat{G}}{\partial t}(t_2,s)\right| m_R(s)ds + \\
+ \int_{t_2}^1 \left|\frac{\partial\hat{G}}{\partial t}(t_1,s) - \frac{\partial\hat{G}}{\partial t}(t_2,s)\right| m_R(s)ds + \\
+ \int_{t_1}^{t_2} \left|\frac{\partial\hat{G}}{\partial t}(t_1,s) - \frac{\partial\hat{G}}{\partial t}(t_2,s)\right| m_R(s)ds
\end{aligned}$$

We can see that because $(\partial\hat{G}/\partial t)$ is uniformly continuous we can choose $\delta > 0$ such that for $|t_1 - t_2| \leq \delta$ the first and second terms of the right hand side of the above inequality are less then $(\varepsilon/3)$. The third term can be bounded by $|t_2 - t_1| \|(\partial\hat{G}/\partial t)\|_{L^\infty([0,1]^2)} \int_0^1 m_R(t)dt$ so we can see that for any $\varepsilon > 0$ we can find $\delta > 0$ such that for every $n \in \mathbf{N}$ and $t_1, t_2 \in [0,1]$ such that $|t_1 - t_2| < \delta$ there is $|v_n'(t_1) - v_n'(t_2)| < \varepsilon$. Hence by Arzela–Ascoli theorem there exists a subsequence of $\{v_n\}$ convergent in $C^1[0,1]$.

The rest of the proof will be divided into 3 Steps.

Step 1 First we will prove that if $f(\lambda, u) = 0$ then $u \geq 0$. Let us observe that if $t_0 \in (0,1)$ is a negative minimum of u then for t from the neighbourhood $(t_0-\delta,\ t_0+\delta)$ there is $u(t)<0$ and for almost every $t\in(t_0-\delta,\ t_0+\delta)$ we have

$$u''(t)+\mu u(t)+\tilde{h}(\lambda,t,u(t),u'(t))=0$$
$$u''(t)+\mu u(t)-\lambda\beta u(t)=0$$

hence

$$u''(t)<0$$

which is impossible in the neighbourhood of local minimum. So we can see that the negative minimum of u is achieved on the boundary of the interval $[0,1]$, but it is impossible for $u\in\mathcal{S}$ (see [5]). That is why for every solution (λ,u) of $f(\lambda,u)=0$ there is $u\geq 0$.

Step 2 Now we are going to show that if $(\lambda,0)$ is a bifurcation point of f then $\lambda=((\mu_0-\mu)/m)$. To prove the above statement we will observe that for any sequence $\{(\lambda_n,u_n)\}$ such that $f(\lambda_n,u_n)=0$, $\|u_n\|_1\neq 0$, $\lambda_n\rightarrow\bar{\lambda}$ and $\|u_n\|_1\rightarrow 0$ there is $\bar{\lambda}=((\mu_0-\mu)/m)$. Let

$$u_n=T_\mu G(\lambda_n,u_n).$$

Labeling $v_n=(u_n/\|u_n\|_0)$ we have

$$v_n=T_\mu(m\lambda_n v_n)+T_\mu\left(\frac{G(\lambda_n,u_n)-m\lambda_n u_n}{\|u_n\|_0}\right)$$

we can see then that $\lim_{n\rightarrow+\infty}((\|G(\lambda_n,u_n)-m\lambda_n u_n\|_0)/\|u_n\|_0)=0$ so the sequence $\{v_n\}$ must have a subsequence convergent in $C[0,1]$. Labeling this subsequence as $\{v_n\}$ and letting $n\rightarrow+\infty$ we have

$$v_0=m\bar{\lambda}T_\mu v_0$$

where $\lim_{n\rightarrow+\infty}v_n=v_0$. Because $\|v_0\|_0=1$ and $v_n\geq 0$ we can see that v_0 is nonnegative eigenvector of the problem $u=\lambda T_\mu u$, and $m\bar{\lambda}$ is an eigenvalue associated with it, hence $m\bar{\lambda}=\mu_0-\mu$, which is our claim.

The above reasoning allows us to observe that for every compact interval $[a,b]\subset[0,+\infty)\setminus\{((\mu_0-\mu)/m)\}$ there exists $r>0$ such that

$$\forall_{(\lambda,u)\in[a,b]\times\overline{K(0,r)}}f(\lambda,u)=0\Rightarrow u=0.$$

Step 3 Now we are going to show that the point $(\lambda_0, 0)$ satisfies condition (1.1) and $s[f, \lambda_0] = -1$ for $\lambda_0 = ((\mu_0 - \mu)/m)$. Let us choose any $0 < \lambda_1 < ((\mu_0 - \mu)/m)$. There exists $r > 0$, such that

$$\forall_{(\lambda,u) \in [0,\lambda_1] \times \overline{K(0,r)}} f(\lambda, u) = 0 \Rightarrow u = 0$$

Hence the homotopy $h_1 : [0,1] \times \overline{K(0,r)} \to C^1[0,1]$ given by $h_1(t,u) = f(\lambda_1 t, u)$ is well defined and $d^- = \deg(f(\lambda_1, 0), K(0,r), 0) = \deg(f(0,\cdot), K(0,r), 0)$. Because $f(0,u) = u - T_\mu G(0,u) = u$ we have $d^- = \deg(I, K(0,r), 0) = 1$.

Let $\lambda_2 > ((\mu_0 - \mu)/m)$ be fixed. Just like before we can observe that for any $\lambda_3 > \lambda_2$ there exists $r \in (0, r_0)$, such that $f(\lambda_2, \cdot)$ may be joined by homotopy with $f(\lambda_3, \cdot)$ on $\overline{K(0,r)}$. Let $\lambda_3 > \lambda_2$ be such that $(\beta\lambda_3/(\mu_0 - \mu)) > 1$. Consider the homotopy $h_2 : [0,1] \times \overline{K(0,r)} \to C^1[0,1]$ given by $h_2(t,u) = f(\lambda_3, u) - tu_0$. We will show that $h_2(t,u) \neq 0$ for $t \in (0,1]$ and $u \in \overline{K(0,r)}$. On the contrary, assume that $h_2(t,u) = 0$. First we should notice that for $u \in \overline{K(0,r)}$ there is $G(\lambda_3, u) \geq 0$ and if $u = \lambda_3 \alpha T_\mu G(\lambda_3, u) + tu_0$ then $u \geq 0$. So we have

$$\begin{aligned} 0 &= \langle u, u_0 \rangle - \langle T_\mu G(\lambda_3, u), u_0 \rangle - t\|u_0\|^2 \leq \\ &\leq \langle u, u_0 \rangle - \langle T_\mu \beta\lambda_3 u, u_0 \rangle - t\|u_0\|^2 = \\ &= \langle u, u_0 \rangle - \beta\lambda_3 \langle u, T_\mu u_0 \rangle - t\|u_0\|^2 = \\ &= \left(1 - \frac{\beta\lambda_3}{\mu_0 - \mu}\right)\langle u, u_0 \rangle - t\|u_0\|^2 < 0 \end{aligned} \tag{1.8}$$

for T_μ is self-adjoint and $T_\mu u_0 = (1/(\mu_0 - \mu))u_0$, a contradiction. That is why $d^+ = \deg(f(\lambda_2, \cdot), K(0,r), 0) = 0$.

Because $s[f, ((\mu_0 - \mu)/m)] = -1$ and $(((\mu_0 - \mu)/m), 0)$ is the unique bifurcation point of f by Theorem A there exists an unbounded connected component C of the set of nontrivial solutions of $f(\lambda, u) = 0$ such that $(((\mu_0 - \mu)/m), 0) \in \bar{C}$. Because for all $(\lambda, u) \in C$ we have $u \geq 0$ and (λ, u) is a solution of the problem (1.5). So C is a connected and unbounded subset of the set of all nontrivial solutions of the problem (1.5). ■

We can see that the Niemytskii operator $G : [0, +\infty) \times C^1[0,1] \to L^1(0,1)$ for Caratheodory function $\tilde{h} : [0, +\infty) \times [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$

given by $G(\lambda,u)(t)=\tilde{h}(\lambda,t,u(t),u'(t))$ satisfies following conditions

$$\begin{aligned}\exists_{m>0}\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{u\in C^1[0,1],u\geq 0}\forall_{\lambda\geq 0}\forall_{t\in[0,1]}\|u\|_1\\ \leq\delta\Rightarrow|G(\lambda,u)(t)-m\lambda u(t)|\leq\varepsilon\lambda|u(t)|.\end{aligned} \tag{1.9}$$

and

$$\forall_{t\in[0,1]}\forall_{u\in C^1[0,1]}\forall_{\lambda\geq 0}u(t)<0\Rightarrow G(\lambda,u)(t)>0. \tag{1.10}$$

We call the map $G:[0,+\infty)\times C^1[0,1]\to L^1(0,1)$ an integrably bounded when

$$\begin{aligned}\forall_{R>0}\exists_{m_R\in L^1(0,1)}\forall_{(\lambda,u)\in[0,+\infty)\times C^1[0,1]}\forall_{t\in[0,1]}|\lambda|\\ +\|u\|_1\leq R\Rightarrow|G(\lambda,u)(t)|\leq m_R(t).\end{aligned}$$

We can see that the proof of the above theorem remains unchanged if we assume $G:[0,+\infty)\times C^1[0,1]\to L^1(0,1)$ is an integrably bounded map satisfying (1.9)–(1.10), not necessarily a Niemytskii operator for a Caratheodory function. So we have the following lemma.

LEMMA 1.11 *Assume $G:[0,+\infty)\times C^1[0,1]\to L^1(0,1)$ is a continuous and an integrably bounded map satisfying conditions* (1.9)–(1.10) *and f: $[0,+\infty)\times C^1[0,1]\to C^1[0,1]$ is given by $f(\lambda,u)=u-T_\mu G(\lambda,u)$, where $T_\mu: L^1(0,1)\to C^1[0,1]$ is given by* (1.6) *for any nonpositive number $\mu<\mu_0$ and μ_0 is the minimal eigenvalue of the problem $(\mathcal{L}_\lambda)$. Then there exists unbounded connected subset $C\subset(0,+\infty)\times C^1[0,1]$ of the set of all nontrivial solutions of the equation $f(\lambda,u)=0$ such that $(((\mu_0-\mu)/m),0)\in\bar{C}$ and for $(\lambda,u)\in C$ we have $u\geq 0$.* ■

2. EXISTENCE THEOREMS

THEOREM 2 *Let μ_0 be the minimal eigenvalue of the linear problem $(\mathcal{L}_\lambda)$ and $\alpha,\beta\in\boldsymbol{R}$ be constants such that $\alpha<\mu_0<\beta$, let $p,q_1,q_2\in L^1(0,1)$ and let $g:[0,1]\times\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ be a Caratheodory function satisfying the following conditions*

$$\exists_{r_0>0}\forall_{s\geq 0}\forall_{y\in\boldsymbol{R}}s+|y|\leq r_0\Rightarrow g(t,s,y)\leq\alpha s,\quad \textit{a.e. in } t\in[0,1]; \tag{2.1}$$

$$\forall_{y\in\boldsymbol{R}}g(t,0,y)\geq 0,\quad \textit{a.e. in } t\in[0,1] \tag{2.2}$$

and

$$\exists_{R_0>0}\forall_{s\geq 0}\forall_{y\in\mathbf{R}}\, s+|y|\geq R_0 \Rightarrow \beta s\leq g(t,s,y) \leq p(t)+q_1(t)s+q_2(t)|y|, \quad \text{a.e. in } t\in[0,1] \tag{2.3}$$

Then there exists nonnegative, nonzero solution of the problem

$$\begin{cases} u''(t)+g(t,u(t),u'(t))=0 & \text{a.e. on } (0,1)\\ u\in\mathcal{S}. \end{cases}$$

Proof Let us choose any $\mu<\min\{0,\alpha\}$ and denote $\bar\alpha=\alpha-\mu>0$ and $\bar\beta=\beta-\mu$.

Let us define the Caratheodory function $\tilde g:[0,1]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ by

$$\tilde g(t,s,y)=\begin{cases} g(t,s,y)-\mu s & \text{for } s\geq 0\\ \bar\alpha|s|+g(t,0,y) & \text{for } s<0 \end{cases}$$

Let us observe that for $s<0$ because of (2.2) we have $\tilde g(t,s,y)\geq \bar\alpha|s|>0$.

Let $U_1=K(0,r_0)$ and $U_2=C^1[0,1]\backslash\overline{K(0,(r_0/2))}$. These sets are the open cover of the space $C^1[0,1]$. Let $\{\phi_1,\phi_2\}$ be a continuous partition of unity on $C^1[0,1]$ such that $\operatorname{supp}\phi_i\subset U_i$ for $i=1,2$. Let $G_0:[0,+\infty)\times C^1[0,1]\to C^1[0,1]$ be given by $G_0(\lambda,u)=\lambda\phi_1(u)\bar\alpha|u|+\lambda\phi_2(u)\tilde G(u)$, where $\tilde G:C^1[0,1]\to C^1[0,1]$ is Niemytskii operator for function $\tilde g$. Let $G_1:[0,+\infty)\times C^1[0,1]\to C^1[0,1]$ be given by $G_1(\lambda,u)=\lambda\phi_1(u)\bar\alpha|u|+\lambda\phi_2(u)\bar\beta|u|$.

Let μ_1, μ_2 be positive numbers such that $((\mu_0-\mu)/\bar\alpha)<\mu_1<\mu_2$. Consider the continuous partition of unity ψ_1, ψ_2 assigned to an open cover $\{[0,\mu_2),(\mu_1,+\infty)\}$ of the interval $[0,+\infty)$. Let $G:[0,+\infty)\times C^1[0,1]\to C^1[0,1]$ be given by

$$G(\lambda,u)=\psi_1(\lambda)G_0(\lambda,u)+\psi_2(\lambda)G_1(\lambda,u). \tag{2.4}$$

Of course for $\lambda>\mu_2$ we have $G(\lambda,u)=G_1(\lambda,u)$, and for $\lambda<\mu_1$ there is $G(\lambda,u)=G_0(\lambda,u)$. We can also see that for $\|u\|_1\leq(r_0/2)$ and any $\lambda\geq 0$ the equality holds $G(\lambda,u)=\lambda\bar\alpha|u|$.

Consider the completely continuous vector field $f:[0,+\infty)\times C^1[0,1]\to C^1[0,1]$ given by $f(\lambda,u)=u-T_\mu G(\lambda,u)$. We can see that G is an integrably bounded map satisfying (1.9) and (1.10) so by Lemma

1.11 there exists unbounded connected subset C of the set of nontrivial solutions of the equation $u = T_\mu G(\lambda, u)$ such that $(((\mu_0 - \mu)/\bar{\alpha}), 0) \in \bar{C}$ and for $(\lambda, u) \in C$ we have $u \geq 0$. If we additionaly assume that $\|u\|_1 \geq r_0$ and $\lambda \in [0, \mu_1]$ then u is a solution of the problem

$$\begin{cases} u''(t) + \mu u(t) + \lambda(g(t, u(t), u'(t)) - \mu u(t)) = 0 \\ u \in \mathcal{S} \end{cases}.$$

Because $\bar{\alpha} < \mu_0 - \mu$, there is $1 < ((\mu_0 - \mu)/\bar{\alpha}) < \mu_1$ and for $\|u\|_1 \geq r_0$ we have

$$u = T_\mu G(1, u) \Leftrightarrow \begin{cases} u''(t) + g(t, u(t), u'(t)) = 0 \\ u \in \mathcal{S} \end{cases}.$$

We have just seen that for $(\lambda, u) \in C$ such that $\lambda > \mu_2$ we have $G(\lambda, u) \geq \lambda\bar{\alpha}|u|$, which means (*cf.* (1.8)), that $1 - (\bar{\alpha}\lambda/(\mu_0 - \mu)) \geq 0$, so $\lambda \leq ((\mu_0 - \mu)/\bar{\alpha}) < \mu_2$. That is why the component $C \subset [0, \mu_2] \times C^1[0, 1]$, and there must exist a sequence $\{(\lambda_n, u_n)\} \subset C$ satisfying $\lim_{n \to +\infty} \|u_n\|_1 = +\infty$ and $\lambda_n \to \bar{\lambda} \in [0, \mu_2]$. We are going to show that $\bar{\lambda} < 1$. Let us denote by $\Pi(C) \subset [0, +\infty)$ the projection of the component C on the first factor of the product $[0, +\infty) \times C^1[0, 1]$. Of course $((\mu_0 - \mu)/\bar{\alpha}) \in \Pi(C)$ and if $\bar{\lambda} < 1$ then there must exist $\tilde{\lambda} < 1$ such that $\tilde{\lambda} \in \Pi(C)$ so we have also $1 \in \Pi(C)$. Hence there must exists such $u \in C^1[0, 1]$, that $u = T_\mu G(1, u)$.

Let the sequence $\{(\lambda_n, u_n)\} \subset C$ be such that $\|u_n\|_1 \to +\infty$ and $\lambda_n \to \bar{\lambda}$. We can assume that $\|u_n\|_1 \geq R_0 > r_0$. Let $m_{R_0} \in L^1(0, 1)$ be an integrable function such that $|g(t, u(t), u'(t))| \leq m_{R_0}(t)$ for $|u(t)| + |u'(t)| \leq R_0$.

Then we have $G(\lambda_n, u_n) \geq \lambda_n\bar{\beta}u_n - \lambda_n m_{R_0} - \lambda_n\bar{\beta}R_0$ and

$$\frac{|g(t, u_n(t), u_n'(t))|}{\|u_n\|_1} \leq \frac{m_{R_0}(t)}{\|u_n\|_1} \leq \frac{m_{R_0}(t)}{R_0}$$
$$\text{for } t \text{ such that } |u_n(t)| + |u_n'(t)| \leq R_0$$

$$\frac{|g(t, u_n(t), u_n'(t))|}{\|u_n\|_1} \leq |q_1(t)| + |q_2(t)| + \frac{|p(t)|}{R_0}$$
$$\text{for } t \text{ such that } |u_n(t)| + |u_n'(t)| > R_0.$$

We can see then that the sequence $(G(\lambda_n, u_n)/\|u_n\|_1) \subset L^1(0, 1)$ is uniformly bounded by an integrable function, so the sequence

$T_\mu(G(\lambda_n, u_n)/\|u_n\|_1)$ has a subsequence convergent in $C^1[0,1]$. Let us denote $v_n = (u_n/\|u_n\|_1)$. Because $v_n = \lambda_n T_\mu(G(\lambda_n, u_n)/\|u_n\|_1)$ we can assume that $v_n \to v_0$ in $C^1[0,1]$.

Since T_μ is a self-adjoint operator and $(\mu_0 - \mu)T_\mu u_0 = u_0$ we can observe that,

$$\langle v_n, u_0\rangle = \left\langle T_\mu \frac{G(\lambda_n, u_n)}{\|u_n\|_1}, u_0 \right\rangle = \frac{1}{\mu_0 - \mu}\left\langle \frac{G(\lambda_n, u_n)}{\|u_n\|_1}, u_0 \right\rangle.$$

By (2.3) and because $u_n \geq 0$ we have

$$\langle v_n, u_0\rangle \geq \frac{\lambda_n}{\mu_0 - \mu}\left\langle \bar{\beta} v_n - \bar{\beta}\frac{m_{R_0} - R_0}{\|u_n\|_1}, u_0 \right\rangle.$$

Letting $n \to +\infty$ we have

$$\langle v_0, u_0\rangle \geq \frac{\bar{\lambda}}{\mu_0 - \mu}\langle \bar{\beta} v_0, u_0\rangle$$

and

$$(\mu_0 - \mu - \bar{\lambda}\bar{\beta})\langle v_0, u_0\rangle \geq 0$$

which because u_0, $v_0 \in C^1\ [0,1]$ are nonzero and nonnegative gives

$$\bar{\lambda} \leq \frac{\mu_0 - \mu}{\bar{\beta}} < 1$$

which is the desired conclusion.

We will show that if $(1, u) \in C$ then $\|u\|_1 \geq r_0$. To obtain a contradiction, suppose that $\|u\|_1 \leq r_0$ and $u = T_\mu G(1, u)$. Then $u \geq 0$ and

$$G(1,u)(t) \leq \phi_1(u)\bar{\alpha}u(t) + \phi_2(u)\bar{\alpha}u(t) = \bar{\alpha}u(t)$$

So we can write

$$0 = \langle u, u_0\rangle - \langle T_\mu G(1,u), u_0\rangle = \langle u, u_0\rangle - \frac{1}{\mu_0 - \mu}\langle G(1,u), u_0\rangle \geq$$
$$\geq \langle u, u_0\rangle - \frac{\bar{\alpha}}{\mu_0 - \mu}\langle u, u_0\rangle = \left(1 - \frac{\bar{\alpha}}{\mu_0 - \mu}\right)\langle u, u_0\rangle.$$

This means that $1 - (\bar{\alpha}/(\mu_0 - \mu)) \leq 0$ and $\bar{\alpha} \geq \mu_0 - \mu$ which contradicts (2.1).

So we proved the existence of $u \in C^1[0,1]$, such that $u \geq 0$, $\|u\|_1 \geq r_0$, and $u = T_\mu G(1, u)$ which completes the proof. ■

As far as problems with $(\mathcal{S}_0)$ boundary conditions are concerned the condition (2.3) may be replaced by weaker ones. Let $R_0 > 0$ be a positive number, such that Caratheodory function $g: [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfies (2.1)–(2.2) and

$$\forall_{s \geq 0} \forall_{y \in \mathbf{R}} |y| \geq R_0 \Rightarrow g(t,s,y) \geq 0, \quad \text{a.e. in } t \in [0,1] \tag{2.5}$$

$$\forall_{s \geq 0} \forall_{y \in \mathbf{R}} s \geq R_0 \Rightarrow g(t,s,y) \geq \beta s, \quad \text{a.e. in } t \in [0,1] \tag{2.6}$$

THEOREM 3 *Suppose $g: [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ is a Caratheodory function satisfying conditions* (2.1), (2.2), (2.5), (2.6). *Then there exists nonzero, nonnegative solution of the problem*

$$\begin{cases} u''(t) + g(t, u(t), u'(t)) = 0, & \text{a.e. on } (0,1) \\ u \in \mathcal{S}_0. \end{cases}$$

Proof As in the proof of Theorem 2 we define the compact vector field $f: [0, +\infty) \times C^1[0,1] \to C^1[0,1]$ given by $f(\lambda, u) = u - T_\mu G(\lambda, u)$, where $G: [0, +\infty) \times C^1[0,1] \to L^1(0,1)$ is given by (2.4), such that there exists the connected component C of the set of nontrivial solutions of the equation $f(\lambda, u) = 0$ bifurcating from $(((\mu_0 - \mu)/\bar{\alpha}), 0)$ and $C \subset [0, \mu_2] \times C^1[0,1]$. It is sufficient to show that there exists $(\lambda, u) \in C$, such that $\lambda \leq 1$.

As before we can observe that there exists a sequence $\{(\lambda_n, u_n)\} \subset C$ such that $\|u_n\|_1 \to +\infty$ and $\lambda_n \to \bar{\lambda}$. Assume that $\|u_n\|_1 \geq R_0$. Suppose, contrary to our claim, that $\bar{\lambda} \geq 1 > ((\mu_0 - \mu)/\beta)$. Then we have

$$\langle u_n, u_0 \rangle \geq \frac{\lambda_n \bar{\beta}}{\mu_0 - \mu} \langle u_n - m_{R_0} - R_0, u_0 \rangle$$

hence

$$\mu_2 \bar{\beta} \langle m_{R_0}, u_0 \rangle \geq (\lambda_n \bar{\beta} - (\mu_0 - \mu)) \langle u_n, u_0 \rangle$$

We can assume that $\lambda_n\bar{\beta} - (\mu_0 - \mu) \geq \varepsilon > 0$ for some $\varepsilon > 0$. Then

$$0 < \langle u_n, u_0 \rangle \leq \frac{\mu_2 \bar{\beta}}{\varepsilon} \langle m_{R_0}, u_0 \rangle.$$

Now we are going to show that the sequence $\{\langle u_n, u_0 \rangle\}$ is not bounded which contradicts the previous inequality and ends the proof.

We know that $\|u_n\|_1 = \|u_n\|_0 + \|u_n'\|_0 \to +\infty$. If there exists a constant $M > 0$ such that $\|u_n'\|_0 \leq M$ then we must have $\|u_n\|_0 \to +\infty$, and because

$$u_n(t) = u_n(0) + \int_0^t u_n'(s)ds$$

there is also

$$\forall_{t \in [0,1]} u_n(0) + M \geq u_n(t) \geq u_n(0) - M$$

and so $u_n(0) \to +\infty$ and $\inf_{t \in [0,1]} u_n(t) \to +\infty$. This means that $\lim_{n \to +\infty} \langle u_n, u_0 \rangle = +\infty$.

Let us assume then $\|u_n'\|_0 \to +\infty$. For η, $\zeta \in (0, (\pi/2)]$ there we have inequalities

$$u_n'(0) = u_n(0) \text{ ctg } \eta \geq 0. \tag{2.7}$$

$$u_n'(1) = u_n(1) \text{ ctg } \zeta \leq 0. \tag{2.8}$$

Assume $\|u_n'\|_0 > R_0$. First suppose there exists $t_0 \in (0,1)$ such that $\|u_n'\|_0 = u_n'(t_0)$. Then there exists $\delta > 0$, such that $u_n'(t) \geq R_0$ for $t \in [t_0 - \delta,\ t_0 + \delta]$ and because of (2.5) and $\lambda_n \geq 1$ there is

$$u_n''(t) = -\mu u(t)(1 - \lambda_n) - \lambda_n g(t, u(t), u'(t)) \leq 0$$

for a.e. $t \in [t_0 - \delta,\ t_0 + \delta]$. Hence $u_n'(t_0) \leq u_n'(t_0 - \delta)$. We can see then that there must be $u_n'(0) \geq u_n'(t_0)$. Similarly we show, that if $-\|u_n'\|_0 = u_n'(t_0)$ then there is $u_n'(1) \leq u_n'(t_0)$. That is why if $\|u_n'\|_0 \geq R_0$ then $\|u_n'\|_0 = \max\{u_n'(0), -u_n'(1)\}$.

We should consider four cases:

(A) If $u_n'(0) = u_n'(1) = 0$ then there must be $\|u_n'\|_0 \leq R_0$ which, as we have shown before, implies $\langle u_n, u_0\rangle \to +\infty$.

(B) Assume $u_n'(1) = 0$ and $u_n'(0) = \|u_n'\|_0 \geq R_0$. Because $\eta \neq (\pi/2)$ then, and $u_n'(t) \geq -R_0$ we have

$$u_n(t) \geq u_n(0) - R_0 = u_n'(0) \text{ tg } \eta - R_0 = \|u_n'\|_0 \text{ tg } \eta - R_0 \to \infty$$

for $t \in [0, 1]$. Of course, then we have $\langle u_n, u_0\rangle \to +\infty$.

(C) If $u_n'(0) = 0$ and $u_n'(1) = -\|u_n'\|_0 \leq -R_0$ then as in (B) we can see that $u_n'(t) \leq R_0$ and

$$u_n(t) \geq u_n(1) - R_0 = -u_n'(1) \text{ tg } \zeta - R_0 = \|u_n'\|_0 \text{ tg } \zeta - R_0 \to \infty$$

(D) Suppose $u_n'(1) < 0$ and $u_n'(0) > 0$. Then of course $\eta, \zeta \in (0, (\pi/2))$. We need only consider the case of $u_n'(0) \geq R_0$ and $u_n'(1) \leq -R_0$. Then we have $u_n'(0) \geq u_n'(t) \geq u_n'(1)$. Because

$$u_n(1) = u_n(0) + \int_0^1 u_n'(s)ds \leq u_n(0) + u_n'(0)$$

there is also

$$u_n(0) \geq \frac{u_n(1)}{1 + \text{ctg } \eta}.$$

Similarly we can get

$$u_n(1) \geq \frac{u_n(0)}{1 + \text{ctg } \zeta}.$$

We conclude from the above inequalities, (2.7) and (2.8) that

$$u_n'(0) \to +\infty \Leftrightarrow -u_n'(1) \to +\infty \Leftrightarrow u_n(0) \to +\infty \Leftrightarrow u_n(1) \to +\infty.$$

Since $\max\{u_n'(0), -u_n'(1)\} \to +\infty$ each part of the above equivalence must be true.

Let $\delta_1, \delta_2 \in (0, 1)$ be the numbers such that $u_n'(t) \geq R_0$ for $t \in [0, \delta_1]$ and $u_n'(t) \leq -R_0$ for $t \in [\delta_2, 1]$. Then $u_n(t) \geq u_n(0)$ for $t \in [0, \delta_1]$ and

$u_n(t) \geq u_n(1)$ for $t \in [\delta_2, 1]$. Then we have

$$u_n(t) = u_n(\delta_1) - \int_{\delta_1}^{t} u_n'(s)ds \geq u_n(\delta_1) - R_0 \geq u_n(0) - R_0,$$

for $t \in [\delta_1, \delta_2]$.

Hence $u_n(t) \geq \min\{u_n(0),\ u_n(1),\ u_n(0) - R_0\} \to +\infty$ which ends the proof. ∎

Now we are going to deal with the situation symmetrical to that considered in Theorems 2 and 3.

THEOREM 4 *Let μ_0 be the minimal eigenvalue of the linear problem $(\mathcal{L}_\lambda)$. Suppose $g : [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ is a Caratheodory function such that there exist integrable functions b, p, q_1, $q_2 \in L^1(0,1)$ and real constants α, $\beta \in \boldsymbol{R}$ such that $\alpha < \mu_0 < \beta$ and following conditions are satisfied*

$$\exists_{r_0 > 0} \forall_{s \geq 0} \forall_{y \in \boldsymbol{R}}\, s + |y| \leq r_0 \Rightarrow g(t,s,y) \geq \beta s, \quad a.e.\ in\ t \in [0,1] \tag{2.9}$$

$$\forall_{y \in \boldsymbol{R}}\, g(t,0,y) \geq 0, \quad a.e.\ in\ t \in [0,1] \tag{2.10}$$

$$\begin{aligned} \exists_{R_0 > 0} \forall_{s \geq 0} \forall_{y \in \boldsymbol{R}}\, s + |y| > R_0 \Rightarrow p(t) + q_1(t)s + q_2(t)|y| \\ \leq g(t,s,y) \leq b(t) + \alpha s, \\ a.e.\ in\ t \in [0,1] \end{aligned} \tag{2.11}$$

Then there exists nonzero, nonnegative solution of the problem

$$\begin{cases} u''(t) + g(t,u(t),u'(t)) = 0, & a.e.\ on\ (0,1) \\ u \in \mathcal{S}. \end{cases} \tag{2.12}$$

Proof Let us choose $\mu < \min\{0, \alpha\}$ and denote $\bar{\beta} = \beta - \mu$. Of course $\mu < \mu_0$ and $\bar{\beta} > \mu_0 - \mu$. Define $\tilde{g} : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$\tilde{g}(t,s,y) = \begin{cases} g(t,s,y) - \mu s & \text{for } s \geq 0 \\ \bar{\beta}|s| + g(t,0,y) & \text{for } s < 0 \end{cases}$$

by (2.10) we have $\tilde{g}(t,s,y) \geq \bar{\beta}|s| > 0$ for $s < 0$.

Let $\{\phi_1, \phi_2\}$ be the continuous partition of unity associated with the open coverage $U_1 = K(0, r_0)$, $U_2 = C^1[0,1] \setminus \overline{K(0,(r_0/2))}$ of $C^1[0,1]$ defined as in the proof of the Theorem 2. Let $G : [0, +\infty) \times C^1[0,1] \to$

$C^1[0,1]$ be given by $G(\lambda,u) = \lambda\phi_1(u)\bar{\beta}|u| + \lambda\phi_2(u)\tilde{G}(u)$, where $\tilde{G}: C^1[0,1] \to L^1(0,1)$ is the Niemytskii operator for $\tilde{g}$ and $f(\lambda, u) = u - T_\mu G(\lambda, u)$. We can see that G is an integrably bounded map satisfying (1.9)–(1.10) so by Lemma 1.11 there exists an unbounded component C of the set of nontrivial solutions of $f(\lambda, u) = 0$ such that $(((\mu_0 - \mu)/\bar{\beta}), 0) \in \bar{C}$ and for $(\lambda, u) \in C$ we have $u \geq 0$.

We conclude from the definition of G that if $\|u\|_1 \leq r_0$ and $u \geq 0$ then

$$G(1,u)(t) \geq \phi_1(u)\bar{\beta}u(t) + \phi_2(u)\bar{\beta}u(t) = \bar{\beta}u(t).$$

Hence, as in the proof of Theorem 2 we can see that for u satisfying $u = T_\mu G(1, u)$ we have $\|u\|_1 \geq r_0$.

That is why

$$u = T_\mu G(1,u) \Leftrightarrow \begin{cases} u''(t) + g(t,u(t),u'(t)) = 0 \\ u \in S \end{cases}$$

Because the component C is unbounded the set $C \cap ((0,+\infty) \times (C^1[0,1] \backslash \overline{K(0,r_0)}))$ is unbounded, too. We will show that there exists $u \in C^1[0,1]$ such that $(1,u) \in C$. By $\Pi(C)$ we denote the projection of C to the first factor of the product $(0, +\infty) \times C^1[0,1]$.

There are two possible situations:

(A) $1 \in \Pi(C)$. Then there exists such $u \in C^1[0,1]$, that is nonnegative solution of problem (2.12).

(B) $\Pi(C) \subset (0,1)$.

If (B) is satisfied then because the component C is unbounded there must exist the sequence $\{(\lambda_n, u_n)\} \subset C$ such that $\|u_n\|_1 \to +\infty$ and $\lambda_n \to \bar{\lambda} \in [0,1]$. As in the proof of Theorem 2 we can assume that $(u_n/\|u_n\|_1) = v_n \to v_0$ in $C^1[0,1]$.

Then we have

$$\langle v_n, u_0 \rangle = \lambda_n \left\langle T_\mu \frac{G(u_n)}{\|u_n\|_1}, u_0 \right\rangle = \frac{\lambda_n}{\mu_0 - \mu} \left\langle \frac{G(u_n)}{\|u_n\|_1}, u_0 \right\rangle.$$

Letting $n \to +\infty$ we have

$$(\mu_0 - \mu + \mu - \alpha)\langle v_0, u_0 \rangle \leq 0$$

by (2.9) and because $u_n \geq 0$. That is why $(\mu_0 - \alpha) \leq 0$, a contradiction.

That proves that (B) is impossible, which completes the proof. ∎

Assume $p\colon [0,1]\to\mathbf{R}$ is an absolutely continuous function such that $p(t)>0$ for $t\in(0,1]$ and $(1/p)\in L^1(0,1)$. Let $g\colon[0,1]\times\mathbf{R}\to\mathbf{R}$ be a Caratheodory function, and $g_0 : \mathbf{R}\to\mathbf{R}$ be continuous. We follow Mawhin and Omana [4] in studying the problem

$$\begin{cases} u''(t) + (p'(t)/p(t))u'(t) + g(t,u(t))g_0(u'(t)) = 0, & \text{a.e. on } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{2.13}$$

Let $\mu_0 > 0$ be the minimal eigenvalue of the linear problem (*cf.* [1,4])

$$\begin{cases} u''(t) + (p'(t)/p(t))u'(t) + \lambda u(t) = 0, & \text{a.e. on } (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Assume, as before, that there exist constants α, $\beta\in\mathbf{R}$ such that $0<\alpha<\mu_0<\beta$. Additionally let a_1, $a_2>0$. Suppose g_0 satisfies

$$\forall_{y\in\mathbf{R}} a_1 \le g_0(y) \le a_2 \tag{2.14}$$

Let g satisfies following conditions

$$\exists_{r_0>0}\forall_{s\ge 0} s\le r_0 \Rightarrow g(t,s)a_2 \le \alpha s, \quad \text{a.e. in } t\in[0,1] \tag{2.15}$$

$$g(t,0)=0, \quad \text{a.e. in } t\in[0,1] \tag{2.16}$$

$$\exists_{R_0>0}\forall_{s\ge 0} s\ge R_0 \Rightarrow g(t,s)a_1 \ge \beta s, \quad \text{a.e. in } t\in[0,1] \tag{2.17}$$

or

$$\exists_{r_0>0}\forall_{0\le s} s\le r_0 \Rightarrow g(t,s)a_1 \le \beta s, \quad \text{a.e. in } t\in[0,1] \tag{2.18}$$

$$\exists_{s_0>0}\exists_{\delta>0}\forall_{s\ge 0}|s-s_0|<\delta \Rightarrow g(t,s)<0, \quad \text{a.e. in } t\in[0,1] \tag{2.19}$$

THEOREM 5 *Suppose a continuous function $g_0 : \boldsymbol{R}\to\boldsymbol{R}$ satisfies condition* (2.14).

(i) *If $g:[0,1]\times\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ is a Caratheodory function satisfying conditions* (2.15)–(2.17) *then there exists nonzero, nonnegative solution of* (2.13).

(ii) *If $g:[0,1]\times\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ is a Caratheodory function satisfying conditions* (2.18), (2.19) *then there exists nonzero, nonegative solution of* (2.13) *such that $\|u\|_0<s_0$.*

Proof It is well known (*cf.* [1,4]) that there exists a linear continuous operator $T: L^1(0,1) \to C^1[0,1]$ such that

$$Tf = u \Leftrightarrow \begin{cases} u''(t) + (p'(t)/p(t))u'(t) + f(t) = 0, & \text{a.e. on } (0,1) \\ u \in \mathcal{S}. \end{cases}$$

given by

$$(Tf)(t) = \int_0^1 \hat{G}(t,s)f(s)p(s)ds$$

where $\hat{G}: [0,1] \times [0,1] \to \mathbf{R}$ is the Green function. We can see (*cf.* [4]) that $\hat{G}$ is continuous on $[0,1] \times [0,1]$, $\hat{G}(t,s) \geq 0$ and $\hat{G}(t,s) = \hat{G}(s,t)$ for $(s,t) \in [0,1] \times [0,1]$.

We begin by proving (i). Let $G: [0,+\infty) \times C^1[0,1] \to L^1(0,1)$ be defined as in the proof of Theorem 2 by (2.4) where $\tilde{G}: C^1[0,1] \to L^1(0,1)$ is given by $\tilde{G}(u)(t) = g(t,u(t))g_0(u'(t))$, and $f: [0,+\infty) \times C^1[0,1] \to C^1[0,1]$ be a completely continuous vector field given by $f(\lambda,u) = u - TG(\lambda,u)$. As in the Step 1 of the proof of Theorem 1 we can show that if $f(\lambda,u) = 0$ then $u \geq 0$. The reasoning similar to that in the Step 2 of the proof of Theorem 1 gives that for every compact interval $[a,b] \subset [0,+\infty) \backslash \{\mu_0/\alpha\}$ there exists $r > 0$ such that

$$\forall_{(\lambda,u) \in [a,b] \times \overline{K(0,r)}} f(\lambda,u) = 0 \Rightarrow u = 0.$$

As in the Step 3 of Theorem 1 we can observe that the point $((\mu_0/\alpha),0)$ satisfies condition (1.1) and $d^- = 1$. Now we are going to show that $d^+ = 0$ and $s[f,(\mu_0/\alpha)] = -1$.

Our analysis will be similar to that in Step 3 of the proof of Theorem 1. Let $\lambda_2 > ((\mu_0 - \mu)/m)$ be fixed. Just like before we can observe that for any $\lambda_3 > \lambda_2$ there exists $r \in (0,(r_0/2))$, such that $f(\lambda_2,\cdot)$ may be joined by homotopy with $f(\lambda_3,\cdot)$ on $\overline{K(0,r)}$. Let $\lambda_3 > \lambda_2$ be such that $(\alpha\lambda_3/\mu_0) > 1$. Consider the homotopy $h_1: [0,1] \times \overline{K(0,r)} \to C^1[0,1]$ given by $h_1(t,u) = f(\lambda_3,u) - tu_0$. We will show that $h_1(t,u) \neq 0$ for $t \in (0,1]$ and $u \in \overline{K(0,r)}$. First we should observe that for $\|u\|_1 \leq (r_0/2)$ we have $f(\lambda_3,u) = u - \lambda_3\alpha T|u|$.

On the contrary, assume that $h_1(t,u)=0$ for $t\in(0,1]$ and $\|u\|_1\le r$. We can see that if $u=\lambda_3\alpha T|u|+tu_0$ then $u\ge 0$. Now we have

$$
\begin{aligned}
0 &= \int_0^1 h_1(t,u)(\xi)u_0(\xi)p(\xi)d\xi = \\
&= \int_0^1 u(\xi)u_0(\xi)p(\xi)d\xi - \lambda_3\alpha\int_0^1\left(\int_0^1 \hat{G}(\xi,\eta)|u(\eta)|p(\eta)d\eta\right) \\
&\qquad\qquad u_0(\xi)p(\xi)d\xi - t\int_0^1 u_0^2(\xi)p(\xi)d\xi = \\
&= \int_0^1 u(\xi)u_0(\xi)p(\xi)d\xi \\
&\quad - \lambda_3\alpha\int_0^1\left(\int_0^1 \hat{G}(\eta,\xi)u_0(\xi)p(\xi)d\xi\right)|u(\eta)|p(\eta)d\eta - \\
&\quad - t\int_0^1 u_0^2(\xi)p(\xi)d\xi = \\
&= \int_0^1 u(\xi)u_0(\xi)p(\xi)d\xi - \frac{\lambda_3\alpha}{\mu_0}\int_0^1 u_0(\eta)u(\eta)p(\eta)d\eta - \\
&\quad - t\int_0^1 u_0^2(\xi)p(\xi)d\xi = \\
&= \left(1-\frac{\lambda_3\alpha}{\mu_0}\right)\int_0^1 u(\xi)u_0(\xi)p(\xi)d\xi - t\int_0^1 u_0^2(\xi)p(\xi)d\xi < 0,
\end{aligned}
$$

a contradiction. That is why $d^+=\deg(f(\lambda_2,\cdot), K(0,r),0)=0$.

So by Theorem A there exists connected component C of the set of nontrivial solutions of the equation $f(\lambda,u)=0$ bifurcating from the point $((\mu_0/\alpha),0)$, and $C\subset(0,\mu_2]\times C^1[0,1]$. We can also observe that all solutions belonging to C are nonnegative. Now it suffices to show that there exists $(\lambda,u)\in C$, such that $\lambda\le 1$.

Of course there exists the sequence $\{(\lambda_n,u_n)\}\subset C$ such that $\|u_n\|_1\to+\infty$. If we assume that $\|u_n\|_0\le K$ then

$$|u_n''(t)|\le m_K(t)p(t)a_2\mu_2.$$

and for $u\ge 0$ and $u(0)=u(1)=0$ we have $u_n'(0)\ge 0$ and

$$
\begin{aligned}
u_n'(0)+a_2\mu_2\int_0^1 m_K(t)p(t)dt &\ge |u_n'(t)| \\
&\ge u_n'(0)-a_2\mu_2\int_0^1 m_K(t)p(t)dt.
\end{aligned}
$$

If $\|u_n'\|_0 \to +\infty$ then $u_n'(0) \to +\infty$ and this means that for $t \in (0,1)$ there is $u_n'(t) > 0$ which is impossible since $u(0) = u(1) = 0$.

Hence we can assume that $\|u_n\|_0 \to +\infty$ and $\|u_n\|_0 > R_0$. Of course we can assume that $\lambda_n \to \bar{\lambda}$. Now we are going to show that $\bar{\lambda} < 1$. We can see that there exists such an integrable function $\gamma \in L^1(0,1)$ that $\gamma > 0$ and $G(\lambda_n, u_n) \geq \lambda_n \beta u_n - \gamma$. For every u_n such that $u_n = TG(\lambda_n, u_n)$ we have

$$\begin{aligned}
&\int_0^1 u_n(t)u_0(t)p(t)dt = \\
&= \int_0^1 \left(\int_0^1 \hat{G}(t,s)G(\lambda_n, u_n)(s)p(s)ds \right) u_0(t)p(t)dt = \\
&= \int_0^1 G(\lambda_n, u_n)(s)p(s) \left(\int_0^1 \hat{G}(s,t)u_0(t)p(t)dt \right) ds = \\
&= \frac{1}{\mu_0} \int_0^1 G(\lambda_n, u_n)(s)p(s)u_0(s)p(s)ds
\end{aligned}$$

Now suppose, contrary to our claim, that $\lambda_n \geq 1$. So we can write

$$\begin{aligned}
\mu_0 \int_0^1 u_n(t)u_0(t)p(t)dt = \int_0^1 G(\lambda_n, u_n)(t)u_0(t)p(t)dt &\geq \\
\geq \lambda_n \beta \int_0^1 u_n(t)u_0(t)p(t)dt &- \\
- \int_0^1 p(t)\gamma(t)u_0(t)dt &\geq \\
\geq \beta \int_0^1 u_n(t)u_0(t)p(t)dt &- \\
- \int_0^1 p(t)\gamma(t)u_0(t)dt &
\end{aligned}$$

and

$$\int_0^1 u_n(t)u_0(t)p(t)dt \leq \frac{1}{\beta - \mu_0} \int_0^1 p(t)\gamma(t)u_0(t)dt$$

which implies

$$\int_0^1 G(\lambda_n, u_n)(t)u_0(t)p(t)dt \leq \frac{\mu_0}{\beta - \mu_0} \int_0^1 p(t)\gamma(t)u_0(t)dt.$$

Let us define $\Gamma : (0,1) \times (0,1) \to \mathbf{R}$ by $\Gamma(s,t) = (\hat{G}(t,s)/u_0(s))$. We can see that Γ is bounded function (see Lemma 1 in [4]).

Hence we have

$$0 \le u_n(t) \le \int_0^1 \hat{G}(t,s)G(\lambda_n, u_n)(s)p(s)ds =$$

$$= \int_0^1 \Gamma(t,s)G(\lambda_n, u_n)(s)u_0(s)p(s)ds \le$$

$$\le \sup_{(s,t)\in(0,1)^2} |\Gamma(s,t)| \frac{\mu_0}{\beta - \mu_0} \int_0^1 p(t)\gamma(t)u_0(t)dt$$

which means that the sequence $\|u_n\|_0$ is bounded, a contradiction. So there must be $\bar{\lambda} < 1$ and there must exist $\lambda < 1$ such that $(\lambda, u) \in C$.

Now we are going to prove (ii). Let $f : [0, +\infty) \times C^1[0,1] \to C^1[0,1]$ be a compact vector field given as in the proof of Theorem 4. We can apply reasoning from (i) to the map f to get the existence of a connected component $C \subset (0, +\infty) \times C^1[0,1]$ of the set on nontrivial zeros of f such that $((\mu_0/\beta), 0) \in \bar{C}$ and $C \cap ((0, +\infty) \times K(0, r_0)) \subset ((0, (\mu_0/\beta)) \times K(0, r_0))$.

Let us observe that the projection of the component C on the first factor of the product must be unbounded. Suppose, contrary to our claim, it is not. Then we can observe that for any $r > 0$ there exists $(\lambda, u) \in C$ such that $\|u\|_0 = r$. Of course the above is true also for s_0 given in (2.19). Because there exists $t_0 \in (0,1)$ such that $\|u\|_0 = u(t_0)$ then almost everywhere in the neighbourhood of t_0 we have $u''(t) > 0$ which is impossible in the neighbourhood of the local maximum. This contradiction ends the proof. ■

3. EXAMPLES

Example 3.1 Let us consider the problem

$$\begin{cases} u''(t) + (p'(t)/p(t))u'(t) + \hat{g}(t, u(t)) = 0 \\ u(0) = 0, u(1) = 0 \end{cases} \qquad (\mathcal{M})$$

where $p : [0,1] \to \mathbf{R}$ is a continuous function such that $p|_{(0,1]} \in C^1(0,1]$, $p(0) = 0$ and $p(t) > 0$ for $t \in (0,1]$ and $(1/p) \in L^1(0,1)$.

In [4] Mawhin and Omana proved that there exists a solution of the above problem if

$$\hat{g}(t,0) = 0; \liminf_{s\to+\infty} s^{-1}\hat{g}(t,s) > \mu_0 \text{ and } \limsup_{s\to 0^+} s^{-1}\hat{g}(t,s) < \mu_0 \text{ uniformly with respect to } t \in [0,1].$$

Let $g = \hat{g}$. Let us observe that if $g_0 : \mathbf{R} \to \mathbf{R}$ is given by $g_0(y) = 1$ and we choose $a_1 = a_2 = 1$ then for $\hat{g}$ satisfying the above conditions functions g and g_0 satisfy conditions (2.14)–(2.17). Hence by Theorem 5(i) there exists solution of problem (2.13). Of course it is also the solution of $(\mathcal{M})$. So Theorem 5(i) is a generalization of the above result given in [4].

Example 3.2 Let μ_0 be the minimal eigenvalue of the linear problem

$$\begin{cases} (t^\alpha u'(t))' + \lambda t^\alpha u(t) = 0 & \text{for } t \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

where $0 \le \alpha < 1$. Let us consider the problem

$$\begin{cases} (t^\alpha u'(t))' + a t^\alpha (\mathrm{e}^{u(t)} - 1) P(u(t)) = 0 & \text{for } t \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

where $P : \mathbf{R} \to \mathbf{R}$ is the polynominal such that $P(0) = 1$, there exists $s_0 > 0$ such that $P(s_0) < 0$ and $\lim_{s\to+\infty} P(s) = +\infty$.

If $a < \mu_0$ then by Theorem 2 in [4] there exists a nonnegative solution of the above problem. In the case of $a > \mu_0$ we conclude the existence of a nonnegative solution of the above problem by means of the Theorem 5(ii).

Example 3.3 In [2] authors studied the problem

$$\begin{cases} u'' = \sum_{k=0}^{m} a_k(t, u')u^k \\ u \in \mathcal{S}_0 \end{cases} \tag{3.1}$$

They proved that if m is odd, $a_m(t,0) > 0$ and a_k are continuous functions satisfying

$$|a_k(t,s,y)| \le p_k(t,s) + q_k(t,s)y^2$$

where $p_k, q_k : [0,1] \times \mathbf{R} \to [0, +\infty)$ are bounded on compact subsets of $[0,1] \times \mathbf{R}$ then there exists the solution of (3.1).

Here we consider the case of $m \geq 2$, not necessarily odd, and a_k continuous. Assume that there exist such r_0, R_0, α, $\beta > 0$, that

$$\begin{aligned}
&a_0(t,y) = 0 \\
&\forall_{t \in [0,1]} \forall_{|y| \leq r_0} a_1(t,y) \geq \alpha \\
&\forall_{t \in [0,1]} \forall_{y \in \mathbf{R}} a_m(t,y) \leq -\beta \\
&\forall_{t \in [0,1]} \forall_{|y| \geq R_0} a_k(t,y) \leq 0, \quad k = 2,3,\ldots,m
\end{aligned}$$

Because $-\alpha < 0$ there is of course $-\alpha < \mu_0$, and we can choose R_0, such that $R_0\beta > \mu_0$. Then the conditions (2.1), (2.2), (2.5), (2.6) are satisfied. By Theorem 3 there exists nonnegative solution of (3.1). We can see that none of the Bernstein conditions (*cf.* [2]) need not be satisfied.

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