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A Characterization of Operator Order *Via* Grand Furuta Inequality

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As an application of the grand Furuta inequality, we shall show a characterization of usual order associated with operator equation and a Kantorovich type order preserving operator inequality by using essentially the same idea of [9].

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1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol: $T > 0$) if T is positive and invertible. The Löwner–Heinz theorem asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p (0 \leq p \leq 1)$. Related to this, Furuta established the following ingenious order preserving operator inequality.

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THEOREM F (Furuta inequality) ([5]) *If* $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}$$

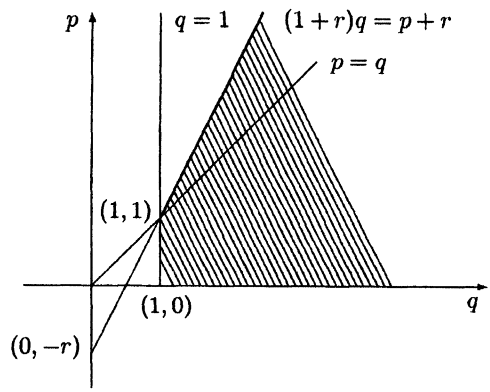
and

(ii) $$(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with*

$$(1+r)q \geq p+r.$$

Alternative proofs of Theorem F have been given in [2, 13], and one-page proof in [7]. The domain drawn for p, q and r in Figure is the best possible one [14] for Theorem F.



FIGURE

As a corollary of [8, Theorem 1.1], Furuta established the following grand Furuta inequality which interpolates Theorem F itself and an inequality equivalent to main theorem of log majorization by Ando–Hiai [1].

THEOREM G (The grand Furuta inequality) ([8]) *If* $A \geq B \geq 0$ *and* A *is invertible, then for each* $t \in [0, 1]$

$$\{A^{r/2}(A^{-t/2}A^pA^{-t/2})^sA^{r/2}\}^{1/q} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{1/q}$$

holds for any $s \geq 0$, $p \geq 0$, $q \geq 1$ *and* $r \geq t$ *with* $(s-1)(p-1) \geq 0$ *and* $(1-t+r)q \geq (p-t)s+r$.

An alternative proof of Theorem G in [4] and one-page proof in [11] and the best possibility of Theorem G is shown in [15], and two very simple proofs of the best possibility of Theorem G are in [16] and [5].

We recall the celebrated Kantorovich inequality: If a positive operator A on a Hilbert space H satisfies $M \geq A \geq m > 0$, then

$$(A^{-1}x, x) \leq \frac{(M+m)^2}{4Mm}(Ax, x)^{-1}$$

for every unit vector $x \in H$. The number $((M+m)^2/4Mm)$ is called the Kantorovich constant. Related to an extension of the Kantorovich inequality, Furuta [10] showed the following order preserving operator inequality:

Theorem A *If* $A \geq B \geq 0$ *and* $M \geq A \geq m > 0$, *then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m, M, p)A^p \geq B^p \quad \textit{holds for all } p \geq 1,$$

where

$$K_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}.$$

The order between positive invertible operators A and B defined by $\log A \geq \log B$ is said to be chaotic order $A \gg B$ in [3] which is a weaker order than usual order $A \geq B$. In [17], Yamazaki and Yanagida showed the following chaotic order version of Theorem A:

Theorem B *If* $\log A \geq \log B$ *and* $M \geq A \geq m > 0$, *then*

$$\left(\frac{M}{m}\right)^{p} A^p \geq K_+(m, M, p+1)A^p \geq B^p \quad \textit{holds for all } p > 0,$$

Moreover, Yamazaki and Yanagida gave a new characterization of chaotic order by means of the Kantorovich constant.

THEOREM C *Let A and B be invertible positive operators and $M \geq A \geq m > 0$. Then the following properties are mutually equivalent*:

$$\text{(I)} \qquad A \gg B \ (\text{i.e., } \log A \geq \log B).$$

$$\text{(II)} \qquad \frac{(M^p + m^p)^2}{4M^p m^p} A^p \geq B^p \quad \text{holds for all } p \geq 0.$$

In this paper, as an application of the grand Furuta inequality, we shall show a characterization of usual order associated with operator equation and a Kantorovich type order preserving operator inequality which interpolates Theorem A and Theorem B by using essentially the same idea of [9]. Also, we present a Kantorovich type inequality which is parallel result with Theorem C.

2. KANTOROVICH TYPE OPERATOR INEQUALITIES

Firstly we shall show the following characterizations of usual order associated with operator equation.

THEOREM 1 *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent*:

(I) $A \geq B$.

(II) *For each $t \in [0, 1]$, $p \geq 1$ and $s \geq 1$ such that $(p-t)s \geq t$, there exists a unique invertible positive contraction T such that*

$$TA^{(p-t)s}T = (A^{-t/2}B^p A^{-t/2})^s.$$

(III) *For all $p \geq 2$, there exists a unique invertible positive contraction T such that*

$$TA^{p-1}T = A^{-1/2}B^p A^{-1/2}.$$

As an application of Theorem 1, we obtain the following Kantorovich type order preserving operator inequality:

THEOREM 2 *Let A and B be positive and invertible operators on a Hilbert space H satisfying $M \geq A \geq m > 0$. Then the following* *assertions are mutually equivalent*:

(I) $A \geq B$.

(II) *For each* $t \in [0,1]$,

$$\frac{(M^{(p-t)s} + m^{(p-t)s})^2}{4M^{(p-t)s}m^{(p-t)s}} A^{(p-t)s} \geq (A^{-t/2}B^pA^{-t/2})^s$$

holds for any $p \geq 1$ *and* $s \geq 1$ *such that* $(p-t)s \geq t$.

(III)
$$\left(\frac{(M^{(p-1)s} + m^{(p-1)s})^2}{4M^{(p-1)s}m^{(p-1)s}}\right)^{1/s} A^p \geq B^p$$

holds for any $s \geq 1$ *and* $p \geq 1/s+1$.

(IV)
$$\left(\frac{M}{m}\right)^{p-1} A^p \geq B^p \quad \textit{holds for all } p \geq 1.$$

By Theorem 2, we have the following corollary which is a parallel result with Theorem C.

Corollary 3 *If* $A \geq B \geq 0$ *and* $M \geq A \geq m > 0$, *then*

$$\frac{(M^{p-1} + m^{p-1})^2}{4m^{p-1}M^{p-1}} A^p \geq B^p \quad \textit{holds for all } p \geq 2.$$

Let A and B be positive invertible operators on a Hilbert space H. We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ which interpolates usual order $A \geq B$ and choatic order $A \gg B$ continuously. The following theorem is easily obtained by Theorem 2.

Theorem 4 *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $A^\delta \geq B^\delta$ *for* $\delta \in (0,1]$ *and* $M \geq A \geq m > 0$, *then*

$$\left(\frac{(M^{(p-\delta)s} + m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{1/s} A^p \geq B^p$$

holds for all $s \geq 1$ *and* $p \geq (1/s+1)\delta$.

Remark 5 Theorem 4 interpolates Theorems A and B by means of the Kantorovich constant. Let A and B be positive invertible operators and $M \geq A \geq m > 0$. Then the following assertions holds:

(i) $A \geq B$ implies $(M/m)^{p-1}A^p \geq B^p$ for all $p \geq 1$.
(ii) $A^\delta \geq B^\delta$ implies $((M^{(p-\delta)s} + m^{(p-\delta)s})^2/4m^{(p-\delta)s}M^{(p-\delta)s})^{1/s}A^p \geq B^p$ for all $s \geq 1$ and $p \geq ((1/s)+1)\delta$.
(iii) $\log A \geq \log B$ implies $(M/m)^p A^p \geq B^p$ for all $p > 0$.

It follows that the Kantorovich constant of (ii) interpolates the scalar of (i) and (iii) continuously. In fact, if we put $\delta = 1$ and $s \to +\infty$ in (ii), then we have (i), also if we put $\delta \to 0$ and $s \to +\infty$ in (ii), then we have (iii).

Moreover, Theorem 4 interpolates Theorem C and Corollary 3 by means of the Kantorovich constant:

(i) $A \geq B$ implies $((M^{p-1} + m^{p-1})^2/4m^{p-1}M^{p-1})A^p \geq B^p$ for all $p \geq 2$.
(ii) $A^\delta \geq B^\delta$ implies $((M^{(p-\delta)s} + m^{(p-\delta)s})^2/4m^{(p-\delta)s}M^{(p-\delta)s})^{1/s}A^p \geq B^p$ for all $s \geq 1$ and $p \geq ((1/s)+1)\delta$.
(iii) $\log A \geq \log B$ implies $((M^p + m^p)^2/4m^pM^p)A^p \geq B^p$ for all $p > 0$.

The Kantorovich constant of (ii) interpolates the scalar of (i) and (iii). In fact, if we put $\delta = 1$ and $s = 1$ in (ii), then we have (i), also if we put $s = 1$ and $\delta \to 0$ in (ii), then we have (iii).

3. PROOF OF THE RESULTS

We need the following lemmas in order to give proofs of the results.

LEMMA 6 ([12]) *If A is positive operator such that $M \geq A \geq m > 0$ and B is a positive contraction, then*

$$\frac{(M+m)^2}{4Mm}A \geq BAB.$$

LEMMA 7 *If $M > m > 0$, then*

$$\lim_{s \to +\infty} \left(\frac{(M^s + m^s)^2}{4m^sM^s} \right)^{1/s} = \frac{M}{m}.$$

Proof Put $x=(M/m)>1$, then it follows from L'Hospital's theorem that

$$\lim_{s\to+\infty}\frac{\log(1+x^s)^2}{s}=\lim_{s\to+\infty}\frac{2x^s\log x}{1+x^s}=\log x^2.$$

Therefore we have

$$\begin{aligned}\lim_{s\to+\infty}\left(\frac{(M^s+m^s)^2}{4m^sM^s}\right)^{1/s}&=\lim_{s\to+\infty}\left(\frac{(1+x^s)^2}{4x^s}\right)^{1/s}\\&=\lim_{s\to+\infty}\left(\frac{(1+x^s)^{2/s}}{4^{1/s}x}\right)=x=\frac{M}{m}.\end{aligned}$$

Proof of Theorem 1 (I) $\Longrightarrow$ (II). Since $A\geq B\geq 0$ and $A>0$, if we put $q=2$ in the grand Furuta inequality, then for $p\geq 1$, $s\geq 1$ and $t\in(0,1]$

$$A^{((p-t)s+r)/2}\geq\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{1/2}\tag{1}$$

holds under the following conditions (2) and (3)

$$r\geq t,\tag{2}$$

$$2(1-t+r)\geq(p-t)s+r.\tag{3}$$

If we moreover put $r=(p-t)s$, then (3) is satisfied and (2) is equivalent to the following

$$(p-t)s\geq t.\tag{4}$$

Therefore, (1) implies that for $t\in(0,1]$, $p\geq 1$ and $s\geq 1$

$$I\geq A^{-(p-t)s/2}\{A^{(p-t)s/2}(A^{-t/2}B^pA^{-t/2})^sA^{(p-t)s/2}\}^{1/2}A^{-(p-t)s/2}\tag{5}$$

holds for the condition (4). Let T be defined by the right hand side of (5). Then it turns out that T is an invertible positive contraction by (5), so that we have

$$A^{(p-t)s/2}TA^{(p-t)s/2}=\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{1/2}.$$

Taking square both sides, we obtain

$$A^{(p-t)s/2}TA^{(p-t)s}TA^{(p-t)s/2} = A^{(p-t)s/2}(A^{-t/2}B^pA^{-t/2})^sA^{(p-t)s/2}.$$

That is, we have the following equation

$$TA^{(p-t)s}T = (A^{-t/2}B^pA^{-t/2})^s.$$

(II) $\Longrightarrow$ (III). Put $t=1$ and $s=1$ in (II).

(III) $\Longrightarrow$ (I). If we put $p=2$ in (III), then we have

$$TAT = A^{-1/2}B^2A^{-1/2},$$

so that it follows that

$$(A^{1/2}TA^{1/2})^2 = A^{1/2}TATA^{1/2} = B^2.$$

By raising each sides to power 1/2, it follows that

$$A \geq A^{1/2}TA^{1/2} = B,$$

and the first inequality holds since $I \geq T \geq 0$.

Whence the proof of Theorem 1 is complete.

Proof of Theorem 2

(I) $\Longrightarrow$ (II). The hypothesis $M \geq A \geq m > 0$ ensures $M^{(p-t)s} \geq A^{(p-t)s} \geq m^{(p-t)s} > 0$ for the hypothesis on t, p and s, so the proof is complete by (II) of Theorem 1 and Lemma 6.

(II) $\Longrightarrow$ (III). If we put $t=1$ in (II), then we have (III) by the Löwner–Heinz theorem.

(III) $\Longrightarrow$ (IV). If we put $s \to \infty$, then we have (IV) by Lemma 7.

(IV) $\Longrightarrow$ (I). If we put $p=1$, then we have (I).

Proof of Corollary 3 Put $s=1$ in (III) of Theorem 2.

Proof of Theorem 4 Put $A_1 = A^\delta$ and $B_1 = B^\delta$, then $A_1 \geq B_1 \geq 0$ and $M^\delta \geq A^\delta \geq m^\delta$. By applying (III) of Theorem 2 to A_1 and B_1, it follows that

$$\left(\frac{(M^{\delta(p_1-1)s} + m^{\delta(p_1-1)s})^2}{4m^{\delta(p_1-1)s}M^{\delta(p_1-1)s}}\right)^{1/s} A_1^{p_1} \geq B_1^{p_1} \quad \text{holds for } p_1 \geq \frac{1}{s}+1.$$

Put $p_1 = (p/\delta) \geq (1/s)+1$, then we have the desired inequality

$$\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{1/s} A^p \geq Bp$$

holds for all $s \geq 1$ and $p \geq \left(\frac{1}{s}+1\right)\delta$.

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References

[1] Ando, T. and Hiai, F. (1994). Log-majorization and complementary Golden-Thompson type inequalities, *Linear Alg. and Its Appl.*, **197**, **198**, 113–131.
[2] Fujii, M. (1990). Furuta's inequality and its mean theoretic approach, *J. Operator Theorey*, **23**, 67–72.
[3] Fujii, M., Furuta, T. and Kamei, E. (1993). Furuta's inequality and its application to Ando's theorem, *Linear Alg. and Its Appl.*, **179**, 161–169.
[4] Fujii, M. and Kamei, E. (1996). Mean theoretic approach to the grand Furuta inequality, *Proc. Amer. Math. Soc.*, **124**, 2751–2756.
[5] Fujii, M., Matsumoto, A. and Nakamoto, R. (1999). A short proof of the best possibility for the grand Furuta inequality, *J. Inequal. Appl.*, **4**, 339–344.
[6] Furuta, T. (1987). $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.*, **101**, 85–88.
[7] Furuta, T. (1989). Elementary proof of an order preserving inequality, *Proc. Japan Acad.*, **65**, 126.
[8] Furuta, T. (1995). Extension of the Furuta inequality and Ando–Hiai log-majorization, *Linear Alg. and Its Appl.*, **219**, 139–155.
[9] Furuta, T. (1996). Generalizations of Kosaki trace inequalities and related trace inequalities on chaotic order, *Linear Alg. and Its Appl.*, **235**, 153–161.
[10] Furuta, T. (1998). Operator inequalities associated with Hölder-McCarthy and Kantorovich inequality, *J. Inequality Appl.*, **2**, 137–148.
[11] Furuta, T. (1998). Simplified proof of an order preserving operator inequality, *Proc. Japan Acad.*, **74**, Ser. A.
[12] Furuta, T. and Seo, Y. (1999). An application of generalized Furuta inequality to Kantorovich type inequalities, *Sci. Math.*, **2**, 393–399.
[13] Kamei, E. (1988). A satellite to Furuta's inequality, *Math. Japon.*, **33**, 883–886.
[14] Tanahashi, K. (1996). Best possibility of the Furuta inequality, *Proc. Amer. Math. Soc.*, **124**, 141–146.
[15] Tanahashi, K. (2000). The best possibility of the grand Furuta inequality, *Proc. Amer. Math. Soc.*, **128**, 511–519.
[16] Yamazaki, T. (1999). Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality, *Math. Inequal. Appl.*, **2**, 473–477.
[17] Yamazaki, T. and Yanagida, M. (1999). Characterizations of chaotic order associated with Kantorovich inequality, *Sci. Math.*, **2**, 37–50.