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A Landau–Kolmogorov Inequality for Orlicz Spaces

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In this paper we prove that the Landau–Kolmogorov inequality for functions on the half line holds for any Orlicz space with the constants, which are best possible for L_∞-space.

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1 INTRODUCTION

The Landau–Kolmogorov inequality

$$\|f^{(k)}\|_\infty^n \leq K(k,n)\|f\|_\infty^{n-k}\|f^{(n)}\|_\infty^k, \tag{1}$$

where $0 < k < n$, is well known and has many interesting applications and generalizations (see [1–6, 15, 18–21]). Its study was initiated by Landau [11] and Hadamard [7] (the case $n = 2$). For functions on the whole real line $\mathbb{R}$, Kolmogorov [9] succeeded in finding in explicit form the best possible constants $K(k,n) = C_{k,n}$ in (1), and Stein proved in [20] that inequality (1) still holds for L_p-norm, $1 \leq p < \infty$, with these constants (the same situation also happens for an arbitrary Orlicz

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norm [1]). The best constants $C^+_{k,n}$ for the half line $\mathbb{R}_+ = [0, \infty)$ are not known in explicit form except for $n = 2, 3, 4$ (see [11, 13]), but an algorithm exists for their computation (Schoenberg and Cavaretta [17]). In this paper, essentially developing the Stein method [20], we prove that, for the half line, inequality (1) still holds for an arbitrary Orlicz norm with the constants $C^+_{k,n}$.

2 RESULTS

Let $G = \mathbb{R}, \mathbb{R}_+$ or $[a, b], \Phi : [0, +\infty) \to [0, +\infty]$ be an arbitrary Young function [10, 12–14], i.e., $\Phi(0) = 0, \Phi(t) \geq 0, \Phi(t) \not\equiv 0$ and Φ is convex. Denote by

$$\overline{\Phi}(t) = \sup_{s \geq 0} \{ts - \Phi(s)\}$$

the Young function conjugate to Φ and $L_\Phi(G)$-the space of measurable functions u such that

$$|\langle u, v\rangle| = \left| \int_G u(x)v(x)dx \right| < \infty$$

for all v with $\rho(v, \overline{\Phi}) < \infty$, where

$$\rho(v, \overline{\Phi}) = \int_G \overline{\Phi}(|v(x)|)dx.$$

Then $L_\Phi(G)$ is a Banach space with respect to the Orlicz norm

$$\|u\|_{\Phi,G} = \sup_{\rho(v,\overline{\Phi}) \leq 1} \left| \int_G u(x)v(x)dx \right|,$$

which is equivalent to the Luxemburg norm

$$\|f\|_{(\Phi,G)} = \inf \left\{ \lambda > 0 : \int_G \Phi(|f(x)|/\lambda)dx \leq 1 \right\} < \infty.$$

Recall that $\|\cdot\|_{(\Phi,G)} = \|\cdot\|_{L_p(G)}$ where $\Phi(t) = t^p$ with $1 \leq p < \infty$, and $\|\cdot\|_{(\Phi,G)} = \|\cdot\|_{L_\infty(G)}$ when $\Phi(t) = 0$ for $0 \leq t \leq 1$ and $\Phi(t) = \infty$ for $t > 1$.

We have the following results [13–14]:

LEMMA 1 *Let $u \in L_\Phi(G)$ and $v \in L_{\overline{\Phi}}(G)$. Then*

$$\int_G |u(x)v(x)|dx \leq \|u\|_{\Phi,G}\|v\|_{(\overline{\Phi},G)}.$$

LEMMA 2 *Let $u \in L_\Phi(\mathbb{R})$ and $v \in L_1(\mathbb{R})$. Then*

$$\|u * v\|_{\Phi,\mathbb{R}} \leq \|u\|_{\Phi,\mathbb{R}}\|v\|_1.$$

LEMMA 3 [5, p. 37] *Let $n \geq 1$. If $f \in L_{1,loc}(\mathbb{R}_+)$ has a generalized n-th derivative $g \in L_{1,loc}(\mathbb{R}_+)$, then f can be redefined on a set of measure zero so that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} = g$ a.e. on $\mathbb{R}_+$.*

THEOREM 1 *Let Φ be an arbitrary Young function, f and its generalized derivative $f^{(n)}$ be in $L_\Phi(\mathbb{R}_+)$. Then $f^{(k)} \in L_\Phi(\mathbb{R}_+)$ for all $k \in \{1, \ldots, n-1\}$ and*

$$\|f^{(k)}\|^n_{\Phi,\mathbb{R}_+} \leq C^+_{k,n}\|f\|^{n-k}_{\Phi,\mathbb{R}_+}\|f^{(n)}\|^k_{\Phi,\mathbb{R}_+}. \tag{2}$$

Proof We divide our proof into two steps.

Step 1 We begin to prove (2) with the assumption that $f^{(k)} \in L_\Phi(\mathbb{R}_+)$, $k = 0, 1, \ldots, n$.

Fix $0 < k < n$. Let $\varepsilon > 0$ be given. We choose a function $v_\varepsilon \in L_{\overline{\Phi}}(\mathbb{R}_+)$, $\rho(v_\varepsilon, \overline{\Phi}) \leq 1$ such that

$$\left|\int_0^\infty f^{(k)}(x)v_\varepsilon(x)dx\right| \geq \|f^{(k)}\|_{\Phi,\mathbb{R}_+} - \varepsilon. \tag{3}$$

Put

$$F_\varepsilon(x) = \int_0^\infty f(x+y)v_\varepsilon(y)dy.$$

Then $F_\varepsilon(x) \in L_\infty(\mathbb{R}_+)$ by virtue of Lemma 1, and it is easy to check that

$$F_\varepsilon^{(r)}(x) = \int_0^\infty f^{(r)}(x+y)v_\varepsilon(y)dy, r = 0, 1, \dots, n \tag{4}$$

in the $\mathcal{D}'(0, \infty)$ sense.

Since $\rho(v_\varepsilon, \overline{\Phi}) \leq 1$, $\|v_\varepsilon\|_{(\overline{\Phi},\mathbb{R}_+)} \leq 1$. So, for all $x \in \mathbb{R}_+$, clearly,

$$|F_\varepsilon^{(r)}(x)| \leq \|f^{(r)}(x+\cdot)\|_{\Phi,\mathbb{R}_+}\|v_\varepsilon\|_{(\overline{\Phi},\mathbb{R}_+)} \leq \|f^{(r)}\|_{\Phi,\mathbb{R}_+}.$$

Now we prove the continuity of $F_\varepsilon^{(r)}$ on $\mathbb{R}_+$. We show this for $r = 0$ by contradiction: Assume that for some $\delta > 0$, a point x^0 and a sequence $\{t_m\}$ in $\mathbb{R}$ with $x^0 + t_m \geq 0$ and $t_m \to 0$ we have

$$\left|\int_0^\infty (f(x^0 + t_m + y) - f(x^0 + y))v_\varepsilon(y)dy\right| \geq \delta, m \in \mathbb{N}. \tag{5}$$

Since $f \in L_\Phi(\mathbb{R}_+)$ we easily get $f \in L_{1,loc}(\mathbb{R}_+)$. Then $f(x^0 + t_m + \cdot) \to f(x^0 + \cdot)$ in $L_1[0,j]$ for any $j = 1, 2, \dots$. Therefore, there exists a subsequence, denoted again by $\{t_m\}$, such that $f(x^0 + t_m + y) \to f(x^0 + y)$ a.e. in $[0,j]$. So, there exists a subsequence (for simplicity of notation we assume that it coincides with $\{t_m\}$) such that $f(x^0 + t_m + y) \to f(x^0 + y)$ a.e. in $[0, \infty)$.

For simplicity of notations we consider only the case when $x^0 = 0$. Because inequality (2) holds for f if and only if it holds for f/C, where C is an arbitrary positive number, without loss of generality we may assume that $\rho(2f, \Phi) < \infty$. By the Young inequality we get

$$\begin{aligned} &|f(t_m + y) - f(y)||v_\varepsilon(y)| \\ &\quad \leq \Phi(|f(t_m + y) - f(y)|) + \overline{\Phi}(|v_\varepsilon(y)|) \\ &\quad \leq \tfrac{1}{2}\Phi(2|f(y)|) + \tfrac{1}{2}\Phi(2|f(t_m + y)|) + \overline{\Phi}(|v_\varepsilon(y)|). \end{aligned} \tag{6}$$

Since $\Phi(2|f|), \overline{\Phi}(|v_\varepsilon|) \in L_1(\mathbb{R}_+)$ and $t_m \to 0$, there are positive numbers M and h such that for all $m \in \mathbb{N}$

$$\int_{y>M} \left(\Phi(2|f(y)|) + \Phi(2|f(t_m+y)|) + \overline{\Phi}(|v_\varepsilon(y)|)\right)dy < \frac{\delta}{2} \tag{7}$$

and

$$\int_B \Phi(2|f(y)|)dy < \frac{\delta}{6}, \int_B \Phi(2|f(t_m+y)|)dy < \frac{\delta}{6}, \int_B \overline{\Phi}(|v_\varepsilon(y)|)]dy < \frac{\delta}{6} \tag{8}$$

if $B \subset \mathbb{R}_+$, $\mathrm{mes}(B) < h$. On the other hand, by the Egorov theorem, there is a set $A \subset [0, M]$, $\mathrm{mes}(A) < h$ such that $f(t_m + y)v_\varepsilon(y)$ uniformly converges to $f(y)v_\varepsilon(y)$ on $[0, M]\backslash A$. Therefore, applying (6) and (8), we have

$$\begin{aligned}
&\overline{\lim_{m\to\infty}} \int_0^M |f(t_m+y) - f(y)||v_\varepsilon(y)|dy \\
&\quad \le \overline{\lim_{m\to\infty}} \int_{[0,M]\backslash A} |f(t_m+y) - f(y)||v_\varepsilon(y)|dy \\
&\quad\quad + \overline{\lim_{m\to\infty}} \int_A |f(t_m+y) - f(y)||v_\varepsilon(y)|dy \\
&\quad = \overline{\lim_{m\to\infty}} \int_A |f(t_m+y) - f(y)||v_\varepsilon(y)|dy \le \frac{\delta}{12} + \frac{\delta}{12} + \frac{\delta}{6} = \frac{\delta}{3}.
\end{aligned} \tag{9}$$

Combining (7), (9) and using (6), we get for sufficiently large m

$$\int_0^\infty |(f(t_m+y) - f(y))v_\varepsilon(y)|dy < \delta,$$

which contradicts (5). The cases $1 \le r \le n$ are proved similarly. The continuity of $F_\varepsilon^{(r)}$ has been proved.

The functions $F_\varepsilon^{(r)}$ are continuous and bounded on $\mathbb{R}_+$. Therefore, it follows from the Landau–Kolmogorov inequality and (3)-(4) that

$$(\|f^{(k)}\|_{\Phi,\mathbb{R}_+} - \varepsilon)^n \leq |F_\varepsilon^{(k)}(0)|^n \leq \|F_\varepsilon^{(k)}\|_\infty^n \\ \leq C_{k,n}^+ \|F_\varepsilon\|_\infty^{n-k} \|F_\varepsilon^{(n)}\|_\infty^k. \qquad (10)$$

On the other hand,

$$\|F_\varepsilon\|_\infty \leq \|f(x+\cdot)\|_{\Phi,\mathbb{R}_+} \|\nu_\varepsilon(\cdot)\|_{(\overline{\Phi},\mathbb{R}_+)} \leq \|f\|_{\Phi,\mathbb{R}_+}, \qquad (11)$$

$$\|F_\varepsilon^{(n)}\|_\infty \leq \|f^{(n)}(x+\cdot)\|_{\Phi,\mathbb{R}_+} \|\nu_\varepsilon(\cdot)\|_{(\overline{\Phi},\mathbb{R}_+)} \leq \|f^{(n)}\|_{\Phi,\mathbb{R}_+}. \qquad (12)$$

Combining (10)–(12), we get

$$(\|f^{(k)}\|_{\Phi,\mathbb{R}_+} - \varepsilon)^n \leq C_{k,n}^+ \|f\|_{\Phi,\mathbb{R}_+}^{n-k} \|f^{(n)}\|_{\Phi,\mathbb{R}_+}^k.$$

By letting $\varepsilon \to 0$ we have (2).

Step 2 To complete the proof, it remains to show that $f^{(k)} \in L_\Phi(\mathbb{R}_+), \forall k \in \{1, \dots, n-1\}$ if $f, f^{(n)} \in L_\Phi(\mathbb{R}_+)$. By Lemma 3 we can assume that $f, f', \dots, f^{(n-1)}$ are continuous on $\mathbb{R}_+$ and $f^{(n-1)}$ is absolutely continuous on $\mathbb{R}_+$.

We define for $k = 0, 1, \dots, n$,

$$f_{(k)}(x) = \begin{cases} f^{(k)}(x), & x \in [0, \infty) \\ 0, & x \in (-\infty, 0). \end{cases}$$

Let $\psi \in C_0^\infty(0,\infty)$, $\psi \geq 0$, $\psi(x) = 0$ for $x \geq 1$ and $\int_{\mathbb{R}} \psi(x)dx = 1$. We put $\psi_\lambda(x) = 1/\lambda\psi(x/\lambda)$, $\lambda > 0$ and $f_\lambda = f_{(0)} * \psi_\lambda$.

Fix $b > 0$. Then $\forall \varphi \in C_0^\infty(b, \infty)$ we have for $0 < \lambda < b$ and $k = 1, \dots, n$

$$\begin{aligned}
\langle f_\lambda^{(k)}, \varphi \rangle &= (-1)^k \langle f_\lambda, \varphi^{(k)} \rangle \\
&= (-1)^k \int_0^\infty \left(\int_0^\infty f_{(0)}(x-y)\psi_\lambda(y)dy \right) \varphi^{(k)}(x)dx \\
&= (-1)^k \int_0^\lambda \left(\int_b^\infty f_{(0)}(x-y)\varphi^{(k)}(x)dx \right) \psi_\lambda(y)dy \\
&= \int_0^\lambda \left(\int_b^\infty f^{(k)}(x-y)\varphi(x)dx \right) \psi_\lambda(y)dy \\
&= \int_b^\infty \left(\int_0^\lambda f^{(k)}(x-y)\psi_\lambda(y)dy \right) \varphi(x)dx \\
&= \int_b^\infty (f_{(k)} * \psi_\lambda)(x)\varphi(x)dx \\
&= \langle f_{(k)} * \psi_\lambda, \varphi \rangle.
\end{aligned}$$

So, we have proved that for $0 < \lambda < b$ and $k = 1, \dots, n$

$$f_\lambda^{(k)} = f_{(k)} * \psi_\lambda \tag{13}$$

in the $\mathcal{D}'(b, \infty)$ sense. Therefore, for $0 < \lambda < b$ we have

$$\begin{aligned}
\|(f_{(0)} * \psi_\lambda)^{(n)}\|_{\Phi,[b,\infty)} &= \| f_{(n)} * \psi_\lambda\|_{\Phi,[b,\infty)} \\
&\le \| f_{(n)} * \psi_\lambda\|_{\Phi,\mathbb{R}} \le \| f_{(n)}\|_{\Phi,\mathbb{R}} \\
&= \| f_{(n)}\|_{\Phi,\mathbb{R}_+} = \| f^{(n)}\|_{\Phi,\mathbb{R}_+}.
\end{aligned} \tag{14}$$

On the other hand, using $(f_{(0)} * \psi_\lambda)^{(k)} = f_{(0)} * \psi_\lambda^{(k)} \in L_\Phi(\mathbb{R})$, $\forall k = 0, 1, \dots, n$ and the proved in Step 1 Landau–Kolmogorov inequality for functions on $[b, \infty)$, we get for $k = 1, \dots, n-1$,

$$\| f_\lambda^{(k)} \|_{\Phi,[b,\infty)}^n \le C_{k,n}^+ \| f_\lambda \|_{\Phi,[b,\infty)}^{n-k} \| f_\lambda^{(n)} \|_{\Phi,[b,\infty)}^k.$$

Hence, combining (13), (14) we obtain for all $0 < \lambda < b$, $k = 1, \dots, n-1$,

$$\begin{aligned} \| f_{(k)} * \psi_\lambda \|^n_{\Phi,[b,\infty)} &\le C^+_{k,n} \| f_{(0)} * \psi_\lambda \|^{n-k}_{\Phi,[b,\infty)} \| f_{(n)} * \psi_\lambda \|^k_{\Phi,[b,\infty)} \\ &\le C^+_{k,n} \| f_{(0)} * \psi_\lambda \|^{n-k}_{\Phi,\mathbb{R}} \| f_{(n)} * \psi_\lambda \|^k_{\Phi,\mathbb{R}} \\ &\le C^+_{k,n} \| f \|^{n-k}_{\Phi,[0,\infty)} \| f^{(n)} \|^k_{\Phi,[0,\infty)}. \end{aligned} \tag{15}$$

On the other hand, because $f_{(k)}$ is continuous on $\mathbb{R}_+$, we easily get

$$\lim_{\lambda \to 0} f_{(k)} * \psi_\lambda(x) = f_{(k)}(x) = f^{(k)}(x), \forall x > 0. \tag{16}$$

Indeed, for $\lambda \le x$ we have from the continuity of $f_{(k)}$ at x that

$$\begin{aligned} | f_{(k)} * \psi_\lambda(x) - f_{(k)}(x) | &= \left| \int_{\mathbb{R}} f_{(k)}(x-y)\psi_\lambda(y)dy - \int_{\mathbb{R}} f_{(k)}(x)\psi_\lambda(y)dy \right| \\ &\le \int_0^\lambda | f_{(k)}(x-y) - f_{(k)}(x) | \psi_\lambda(y)dy \\ &= \int_0^\lambda | f^{(k)}(x-y) - f^{(k)}(x) | \psi_\lambda(y)dy \\ &\le \sup_{0 \le y \le \lambda} | f^{(k)}(x-y) - f^{(k)}(x) | \to 0 \text{ as } \lambda \to 0. \end{aligned}$$

For each function $v \in L_{\overline{\Phi}}[b,\infty)$, $\rho(v, \overline{\Phi}) \le 1$ and $0 < \lambda < b$, by (15) and the definition of the Orlicz norm we get

$$\left(\int_b^\infty |(f_{(k)} * \psi_\lambda)(x)v(x)|dx \right)^n \le C^+_{k,n} \| f \|^{n-k}_{\Phi,[0,\infty)} \| f^{(n)} \|^k_{\Phi,[0,\infty)}.$$

Therefore, using Fatou's lemma, (15) and (16) we obtain

$$\begin{aligned} \left| \int_b^\infty (f^{(k)}(x)v(x)dx \right|^n &\le \left(\int_b^\infty \varliminf_{\lambda \to 0} |(f_{(k)} * \psi_\lambda)(x)v(x)|dx \right)^n \\ &\le \left(\varliminf_{\lambda \to 0} \int_b^\infty |(f_{(k)} * \psi_\lambda)(x)v(x)|dx \right)^n \\ &\le C^+_{k,n} \| f \|^{n-k}_{\Phi,[0,\infty)} \| f^{(n)} \|^k_{\Phi,[0,\infty)} \end{aligned}$$

So, by the definition of the Orlicz norm we have

$$\|f^{(k)}\|^n_{\Phi,[b,\infty)} \le C^+_{k,n}\|f\|^{n-k}_{\Phi,[0,\infty)}\|f^{(n)}\|^k_{\Phi,[0,\infty)} < \infty.$$

On the other hand, it follows from the continuity of $f^{(k)}$ on $[0, \infty)$ that $f^{(k)} \in L_\Phi[0, b]$ for any $b > 0$. Therefore,

$$\|f^{(k)}\|_{\Phi,[0,\infty)} \le \|f^{(k)}\|_{\Phi,[0,b]} + \|f^{(k)}\|_{\Phi,[b,\infty)} < \infty.$$

The proof is complete.

Remark 1 To obtain Theorem 1 we have developed the Stein method because, for example, the property $[g(x+h) - g(x)]/h \to g'(x)$ in the L_p mean $(1 \le p < \infty)$, which is used in [16], holds for L_Φ only if Φ satisfies the Δ_2-condition (see [12, 14]).

REMARK 2 By the representation [14]

$$\|u\|_{(\Phi,G)} = \sup_{\|v\|_{\bar{\Phi},G}\le 1} \left|\int_G u(x)v(x)dx\right|,$$

it is easy to see that Theorem 1 still holds for any Luxemburg norm.

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