Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2009, Article ID 241539, 4 pages
doi:10.1155/2009/241539

Research Article

On the Moments of Hitting Times for Random Walks on Trees

José Luis Palacios

Departamento de Cómputo Científico y Estadística, Universidad Simón Bolívar, Apartado 89000, Caracas, Venezuela

Correspondence should be addressed to José Luis Palacios, jopala@cesma.usb.ve

Received 24 June 2009; Accepted 31 August 2009

Recommended by Nikolaos E. Limnios

Using classical arguments we derive a formula for the moments of hitting times for an ergodic Markov chain. We apply this formula to the case of simple random walk on trees and show, with an elementary electric argument, that all the moments are natural numbers.

1. Introduction

On a connected undirected graph $G = (V, E)$ such that the edge between vertices i and j is given a resistance r_{ij} (or equivalently, a conductance $C_{ij} = 1/r_{ij}$), we can define the general random walk (GRW) on G as the Markov chain $X_n, n \geq 0$, that from its current vertex v jumps to the neighboring vertex w with probability $p_{vw} = C_{vw}/C(v)$, where $C(v) = \sum_{w:w\sim v} C_{vw}$, and $w \sim v$ means that w is a neighbor of v. There may be a conductance C_{zz} from a vertex z to itself, giving rise to a transition probability from z to itself, though the most studied case of these random walks on graphs, the simple random walk (SRW), excludes the loops and considers all r_{ij}'s to be equal to 1.

The hitting time T_b of the vertex b is the number of jumps that the walk takes until it lands on b, and its kth moment when the walk starts at a is denoted by $E_a T_b^k$. Chen and Zhang [1] found a closed-form formula for the expected hitting times of SRW on trees that yielded as a corollary the fact that these expected times are all natural numbers. Furthermore, Chen [2] studied the moment generating function of hitting times for SRW on trees and showed that the second moments are also natural numbers.

The purpose of this note is to give a recursive formula for the moments of the hitting times of any ergodic finite Markov chain, using as tools classical material found in Kemeny and Snell [3] and Chung [4]. When this formula is applied to the case of SRW on trees, an elementary argument from the electrical approach found in Doyle and Snell [5] shows that all these moments are natural numbers.

To ease the notation, let us assume that $V = \{1, 2, \dots, N\}$ and that we are interested in finding $E_i T_N^k$ for $k \geq 1$ and $1 \leq i \leq N-1$. We consider the vectors $\mathbf{E}T_N^k = [E_1 T_N^k, E_2 T_N^k, \dots, E_{N-1} T_N^k]^t$, $k \geq 1$, and $\mathbf{c} = [1, 1, \dots, 1]^t$, where the superscript t means "transpose." The classical approach to finding $\mathbf{E}T_N$ in an ergodic Markov chain with state space V, due to Kemeny and Snell [3], consists of considering the matrix Q which results from deleting the N-th row and column of the transition probability matrix and finding the "fundamental matrix" $(I-Q)^{-1}$ which yields

$$\mathbf{E}T_N = (I-Q)^{-1}\mathbf{c}, \tag{1.1}$$

on account of the fact that the (i, j) entry of the fundamental matrix is the expected number of visits to state j, $n(j)$, by the walk started at i before hitting the state N. We can write this fact as

$$(I-Q)^{-1}_{ij} = E_i n(j). \tag{1.2}$$

An additional fact that we will need regarding the matrix Q can be found in Kemeny and Snell [3, page 49]:

$$(I-Q)^{-1}Q = (I-Q)^{-1} - I. \tag{1.3}$$

Of the electrical approach we will quote the fact that

$$E_i n(j) = C(j) v_j, \tag{1.4}$$

where $C(j) = \sum_{i \sim j} C_{ij}$ is the sum of all conductances emanating from j and v_j is the voltage at j when a battery is placed between i and N such that the current entering at j is 1 and the voltage at N is 0. The details can be found in Doyle and Snell [5, Section 3.3, page 49].

2. The Formula and Its Corollaries

In the spirit of Corollary 2 in Chung [4, page 64] we find a recurrence for the moments of the hitting times of an ergodic Markov chain. Our formula differs from that of Chung in that his is presented neither as a recurrence nor in vector form, and more importantly, in that it involves neither taboo probabilities nor mean recurrence times, and this latter fact is crucial for our purposes, because for SRW on graphs the mean recurrence time $E_i T_i$ of a vertex i is given by $2|E|/d(i)$, where $d(i)$ is the number of neighbors of i, and for the case of a tree this expression becomes $2(N-1)/d(i)$ which may or may not be a natural number, and we need that all coefficients in the vectors and matrices of our formula to be natural numbers. This recurrence should perhaps be better known and we present it here in the following

Theorem 2.1. *For any ergodic Markov chain with state space $V = \{1, 2, \ldots, N\}$ one has*

$$\mathbf{E}T_N^k = \mathbf{E}T_N + \sum_{s=1}^{k-1} \binom{k}{s} \left[(I - Q)^{-1} - I \right] \mathbf{E}T_N^s, \tag{2.1}$$

for $k \geq 2$.

Proof. For any $k \geq 1$ and any i, $1 \leq i \leq N - 1$, conditioning gives us

$$E_i T_N^k = p_{iN} + \sum_{j \neq N} p_{ij} E_j (1 + T_N)^k = 1 + \sum_{s=1}^{k} \binom{k}{s} \sum_{j \neq N} p_{ij} E_j T_N^s. \tag{2.2}$$

In vector form the above set of equations yields

$$\mathbf{E}T_N^k = \mathbf{c} + \sum_{s=1}^{k} \binom{k}{s} Q \mathbf{E}T_N^s. \tag{2.3}$$

For $k = 1$, (2.3) becomes (1.1). For $k \geq 2$, from (2.3) we obtain

$$(I - Q)\mathbf{E}T_N^k = \mathbf{c} + \sum_{s=1}^{k-1} \binom{k}{s} Q \mathbf{E}T_N^s. \tag{2.4}$$

Solving for $\mathbf{E}T_N^k$ and using (1.1) in (2.4) we get

$$\mathbf{E}T_N^k = \mathbf{E}T_N + \sum_{s=1}^{k-1} \binom{k}{s} (I - Q)^{-1} Q \mathbf{E}T_N^s. \tag{2.5}$$

Finally using (1.3) in (2.5) finishes the proof. □

Now we can give sufficient conditions for all moments to be natural numbers.

Corollary 2.2. *Under the hypotheses of the theorem, if $\mathbf{E}T_N$ is a vector of natural numbers and all the entries of the fundamental matrix are natural numbers, then for all $k \geq 1$, $\mathbf{E}T_N^k$ is a vector of natural numbers.*

Proof. Use induction and the recursion (2.1), and verify that all the summands in the right-hand side of (2.1), under the hypotheses, turn out to be natural numbers. □

Corollary 2.3. *In the case of SRW on trees, $\mathbf{E}T_N^k$ is a vector of natural numbers for all $k \geq 1$.*

Proof. It is known [1] that $\mathbf{E}T_N$ is a vector of natural numbers. By the previous corollary if suffices to prove that all entries of the fundamental matrix are natural numbers. But (1.2) and (1.4) tell us that $(I-Q)^{-1}_{ij} = C(j)v_j$, and for SRW $C(j)$ is just the number of neighbors of j, so all that is left to check is that the voltage v_j is a natural number, when a battery is placed between i and N such that the current entering i is 1 and the voltage $v_N = 0$. Since the graph is a tree, it is immediate from Ohm's law that for all vertices x in the unique path P from i to N, $v_x = d(x, N)$, the distance from x to N. Also, if we denote by $E(P)$ the set of edges in the path P, it is plain to see that all vertices in the connected component of $G - E(P)$ that contains x share the same voltage $v_x = d(x, N)$. So the voltages in all vertices are natural numbers and we are done. □

The condition in Corollary 2.2 that all the entries of $(I-Q)^{-1}$ are natural numbers is not necessary. For example, in the case of SRW in the complete graph K_N, $N \geq 3$ we have that

$$(I-Q)^{-1} = \frac{N-1}{N}\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}, \tag{2.6}$$

so that no entry of the fundamental matrix is a natural number. However, (1.1) gives us that $\mathbf{E}T_N = (N-1, N-1, \ldots, N-1)^t$. Moreover, by symmetry, $\mathbf{E}T_N^s = (c_s, c_s, \ldots, c_s)^t$ for some constant c_s, and it is plain to see that $(I-Q)^{-1}\mathbf{E}T_N^s = ((N-1)c_s, (N-1)c_s, \ldots, (N-1)c_s)^t$. Therefore, induction and formula (2.1) imply that all moments in this case are natural numbers, which can be given explicitely: $c_2 = (N-1)(2N-3)$, $c_3 = (N-1)(6N^2 - 18N + 13)$, $c_4 = (N-1)(24N^3 - 108N^2 + 158N - 75)$, and so forth.

References

[1] H. Chen and F. Zhang, "The expected hitting times for graphs with cutpoints," *Statistics & Probability Letters*, vol. 66, no. 1, pp. 9–18, 2004.
[2] H. Chen, "The generating functions of hitting times for random walk on trees," *Statistics & Probability Letters*, vol. 77, no. 15, pp. 1574–1579, 2007.
[3] J. G. Kemeny and J. L. Snell, *Finite Markov Chains*, Van Nostrand, Princeton, NJ, USA, 1960.
[4] K. L. Chung, *Markov Chains with Stationary Transition Probabilities*, Springer, New York, NY, USA, 1967.
[5] P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*, vol. 22, Mathematical Association of America, Washington, DC, USA, 1984.