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Research Article

A Note on Strong Convergence of Sums of Dependent Random Variables

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For a sequence of dependent square-integrable random variables and a sequence of positive constants $\{b_n,\ n \geq 1\}$, conditions are provided under which the series $\sum_{i=1}^{n}(X_i - EX_i)/b_i$ converges almost surely as $n \to \infty$. These conditions are weaker than those provided by Hu et al. (2008).

1. Introduction and Results

Let $\{X_n,\ n \geq 1\}$ be a sequence of square-integrable random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{b_n,\ n \geq 1\}$ be a sequence of positive constants. The random variables $\{X_n,\ n \geq 1\}$ are not assumed to be independent. Past research has focussed on conditions that ensure the strong convergence of two distinct but related series:

$$\sum_{i=1}^{n} \frac{X_i - EX_i}{b_i}, \qquad b_n^{-1} \sum_{i=1}^{n} (X_i - EX_i). \tag{1.1}$$

If the second sequence converges to 0 almost surely, then $\{X_n,\ n \geq 1\}$ is said to obey the strong law of large numbers (SLLN).

Assume that there exists a sequence of constants $\{\rho_k,\ k \geq 1\}$ such that

$$\sup_{n \geq 1} |\mathrm{Cov}(X_n, X_{n+k})| \leq \rho_k, \quad k \geq 1. \tag{1.2}$$

Our interest is in conditions on the growth rates of $\{\operatorname{Var} X_n,\ n \geq 1\}$, $\{b_n,\ n \geq 1\}$, and $\{\rho_k,\ k \geq 1\}$ which imply strong convergence of the above series.

There is an extensive literature on strong laws for independent random variables. Strong laws have been derived for various dependence structures such as negative association (e.g., Kuczmaszewska [1]), quasi-stationarity (e.g., Móricz [2], Chobanyan et al. [3]), and orthogonality (e.g., Stout [4]).

Hu et al. [5] focus on the strong convergence of the series without imposing strong conditions on the nature of the variances and covariances. Our aim is to weaken their condition on the covariances and establish the following theorem.

Theorem 1.1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $\{\rho_k,\ k \geq 1\}$ such that (1.2) holds. Let $\{b_n,\ n \geq 1\}$ be a sequence of positive constants. Assume that there exists a constant K such that, for all $n \geq 1$,*

$$\frac{n}{b_n} \leq K. \tag{1.3}$$

Suppose that

$$\sum_{n=1}^{\infty} \frac{(\operatorname{Var} X_n)(\log n)^2}{b_n^2} < \infty, \tag{1.4}$$

$$\sum_{k=1}^{\infty} \frac{\rho_k}{k}(\log k)^2 < \infty. \tag{1.5}$$

Then

$$\sum_{i=1}^{n} \frac{X_i - EX_i}{b_i} \quad \text{converges a.s. as } n \longrightarrow \infty. \tag{1.6}$$

To motivate the general nature of our result consider the following example. Let $\{X_n\}$ be a sequence of zero mean random variables where

$$X_n = \xi_n + \nu_n, \tag{1.7}$$

where $\{\xi_n\}$ is a stationary time series with autocovariance function $\{\gamma_k\}$ and $\{\nu_n\}$ is a sequence of independent, zero mean random variables distributed independently of $\{\xi_n\}$. Let $\operatorname{Var}(\nu_n) = \sigma_n^2$. Thus what we observe is an underlying stationary series disturbed by a noise process with variance that can depend on n.

We have $\operatorname{Var}(X_n) = \gamma_0 + \sigma_n^2$ and $\operatorname{Cov}(X_n, X_{n+k}) = \gamma_k (= \rho_k)$, $k \geq 1$. Condition (3.1) in Theorem 1 of Hu et al. [5], which is the same as (1.4), is a constraint on the σ_n^2 values whereas their condition (3.2)

$$\sum_{k=1}^{\infty} \frac{\rho_k}{k^q} < \infty, \quad \text{for some } q \in [0,1) \tag{1.8}$$

is a constraint on γ_k. In Chapter 2 of Stout [4] the condition on the variances is shown to be close to optimal for sequences of orthogonal random variables. Lyons [6] provides an SLLN for random variables with bounded variances under the condition $\sum_{k=1}^{\infty} \rho_k/k < \infty$. One might conjecture that the condition (1.8) could be relaxed to $\sum \rho_k/k < \infty$. The above theorem, whilst allowing for far more general models than (1.7), moves us closer to this constraint on the ρ_k values.

For long range dependent stationary processes we have $\rho_k = O(k^{-d}L(k))$, where $0 < d < 1$ and $L(\cdot)$ is a slowly varying function. Theorem 1.1 enables the strong convergence result to be extended to processes where the correlation decays at a slower rate than $O(k^{-d})$ for $d > 0$.

Applying Kronecker's lemma the strong law of large numbers result is an immediate consequence of the above theorem.

Corollary 1.2. *Under the conditions of Theorem 1.1, if b_n is monotone increasing, the strong law of large numbers holds, that is,*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i - EX_i)}{b_n} = 0 \quad a.s. \tag{1.9}$$

There are strong law results under weaker conditions than (1.5) but with stronger conditions on the variance (see, e.g., Lyons [6], Chobanyan et al. [3]). Both papers show that if the summands have bounded variance, then (1.5) can be weakened to $\sum_{k=1}^{\infty} \rho_k/k < \infty$. Our approach focusses on the convergence of the series in (1.6) and relies on Kronecker's Lemma to obtain the strong law. If the aim is purely to obtain the SLLN, then alternative conditions might be possible as it is possible to construct sequences $\{x_n\}$ and $\{b_n\}$ such that $b_n^{-1}(x_1 + \cdots x_n) \to 0$ but $\sum_{i=1}^{n} b_k^{-1} x_k$ diverges. For example, take $b_n = n$ and $x_n = (\log n)^{-1}$. Thus we can have the strong law holding but the series in (1.6) diverging.

2. Proofs

Throughout this paper, the symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessarily the same at each appearance. We first prove a number of lemmas that enable us to obtain tighter bounds for key expressions in the proof of Theorem 1 of Hu et al. [5].

Lemma 2.1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $\{\rho_k,\ k \geq 1\}$ such that* (1.2) *holds and a sequence $\{b_n\}$ satisfying* (1.3)*. Then for all $n \geq 0$, $m \geq n+2$,*

$$E\left(\sum_{i=n+1}^{m} \frac{X_i - EX_i}{b_i}\right)^2 \leq \sum_{i=n+1}^{m} \frac{\operatorname{Var} X_i}{b_i^2} + \sum_{k=1}^{m-n-1} \frac{\rho_k}{k} \log\left(1 + \frac{k}{n}\right). \tag{2.1}$$

Proof. For all $n \geq 0$, $m \geq n+2$,

$$
\begin{aligned}
E\left(\sum_{i=n+1}^{m}\frac{X_i - EX_i}{b_i}\right)^2 &= \sum_{i=n+1}^{m}\frac{\operatorname{Var}(X_i)}{b_i^2} + 2\sum_{i=n+1}^{m-1}\sum_{j=i+1}^{m}\frac{\operatorname{Cov}(X_i, X_j)}{b_i b_j} \\
&\leq \sum_{i=n+1}^{m}\frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{i=n+1}^{m-1}\sum_{j=i+1}^{m}\frac{\rho_{j-i}}{ij} \\
&= \sum_{i=n+1}^{m}\frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{i=n+1}^{m-1}\sum_{k=1}^{m-n-1}\frac{\rho_k}{k}\left(\frac{1}{i} - \frac{1}{i+k}\right) \\
&= \sum_{i=n+1}^{m}\frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{k=1}^{m-n-1}\frac{\rho_k}{k}\left(\sum_{i=n+1}^{m-1}\frac{1}{i} - \sum_{i=n+k+1}^{m-1+k}\frac{1}{i}\right) \\
&\leq \sum_{i=n+1}^{m}\frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{k=1}^{m-n-1}\frac{\rho_k}{k}(\log(n+k) - \log(n)) \\
&\leq \sum_{i=n+1}^{m}\frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{k=1}^{m-n-1}\frac{\rho_k}{k}\log\left(1 + \frac{k}{n}\right).
\end{aligned} \tag{2.2}
$$

□

Lemma 2.2. *For* $0 < e^2 \leq k \leq n$,

$$\log\left(1+\frac{k}{n}\right) \leq \left(\frac{\log k}{\log n}\right)^2. \tag{2.3}$$

Proof. Note that $x/(\log x)^2$ is an increasing function for $x \geq e^2 > 0$. Thus, for $x \geq k > e^2$,

$$\frac{x}{(\log x)^2} \geq \frac{k}{(\log k)^2}. \tag{2.4}$$

Hence for $n \geq k > e^2$,

$$\log\left(1+\frac{k}{n}\right) \leq \frac{k}{n} \leq \left(\frac{\log k}{\log n}\right)^2. \tag{2.5}$$

□

Lemma 2.3. *For* $a > 0$, *define*

$$S_i(a) = \sum_{n=a}^{\infty}\frac{n^i}{2^n}, \quad i = 0, 1, \ldots. \tag{2.6}$$

Then $S_0(a) = 2^{-(a-1)}$, $S_1(a) = (a+1)2^{-(a-1)}$ *and, in general,*

$$S_j(a) = \frac{a^j}{2^{a-1}} + \sum_{i=0}^{j-1} \binom{j}{i} S_i(a). \tag{2.7}$$

Proof. The result for $S_0(a)$ is the sum of a standard geometric progression. The general result follows by noting

$$\begin{aligned} 2S_j(a) &= \sum_{n=a}^{\infty} \frac{n^j}{2^{n-1}} \\ &= \frac{a^j}{2^{a-1}} + \sum_{n=a}^{\infty} \frac{(n+1)^j}{2^n} \\ &= \frac{a^j}{2^{a-1}} + \sum_{n=a}^{\infty} \frac{n^j}{2^n} + \sum_{i=0}^{j-1} \sum_{n=a}^{\infty} \binom{j}{i} \frac{n^i}{2^n}. \end{aligned} \tag{2.8}$$

Thus

$$S_j(a) = \frac{a^j}{2^{a-1}} + \sum_{i=0}^{j-1} \binom{j}{i} S_i(a). \tag{2.9}$$

□

Proof of Theorem 1.1. We will follow the method of proof in Theorem 1 in Hu et al. [5]. To prove (1.6) we first show that $\{\sum_{i=1}^{n}((X_i - EX_i)/b_i),\ n \geq 1\}$ is a Cauchy sequence for convergence in L_2 which will imply convergence in probability. Using Lemmas 2.1 and 2.2,

$$\begin{aligned} &\sup_{m>n} E\left(\sum_{i=1}^{m} \frac{X_i - EX_i}{b_i} - \sum_{i=1}^{n} \frac{X_i - EX_i}{b_i}\right)^2 \\ &= \sup_{m>n} E\left(\sum_{i=n+1}^{m} \frac{X_i - EX_i}{b_i}\right)^2 \\ &\leq \sup_{m>n}\left(\sum_{i=n+1}^{m} \frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{k=1}^{m-n-1} \frac{\rho_k}{k} \log\left(1 + \frac{k}{n}\right)\right), \quad \text{by Lemma 2.1,} \\ &\leq \sum_{i=n+1}^{\infty} \frac{\operatorname{Var} X_i}{b_i^2} + C\sum_{i=1}^{8} \frac{\rho_k}{k} \log\left(1 + \frac{k}{n}\right) + C\sum_{k=9}^{n} \frac{\rho_k}{k}\left(\frac{\log k}{\log n}\right)^2 + \sum_{k=n+1}^{\infty} \frac{\rho_k}{k} \log(k) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.10}$$

Therefore there exists a random variable $S \in L_2$ such that

$$S_n = \sum_{i=1}^{n} \frac{X_i - EX_i}{b_i} \xrightarrow{p} S. \tag{2.11}$$

Next we will show that $S_{2^n} \longrightarrow S$ a.s. Let $\varepsilon > 0$ be arbitrary. Note

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left\{ \left| \sum_{i=1}^{2^n} \frac{X_i - EX_i}{b_i} - S \right| > \varepsilon \right\} \\
&\quad \le \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \left(\sum_{i=2^n+1}^{\infty} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{k=1}^{\infty} \frac{\rho_k}{k} \log\left(1 + \frac{k}{2^n}\right) \right), \quad \text{by Lemma 2.1}, \\
&\quad = C \sum_{i=3}^{\infty} \sum_{n=1}^{[\log_2 i]} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{2^n} \frac{\rho_k}{k} \log\left(1 + \frac{k}{2^n}\right) + \sum_{k=2^n+1}^{\infty} \frac{\rho_k}{k} \log\left(1 + \frac{k}{2^n}\right) \right) \\
&\quad \le C \sum_{i=2}^{\infty} \sum_{n=1}^{[\log_2 i]} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{k=1}^{\infty} \left(\sum_{n=[\log_2 k]}^{\infty} \frac{\rho_k}{k} \frac{(\log k)^2}{(\log 2^n)^2} + \sum_{n=1}^{[\log_2 k]} \frac{\rho_k}{k} (1 + \log k) \right), \quad \text{by Lemma 2.2}, \\
&\quad \le C \sum_{i=2}^{\infty} \sum_{n=1}^{[\log_2 i]} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{k=1}^{\infty} \frac{\rho_k}{k} \log k + C \sum_{k=1}^{\infty} \frac{\rho_k}{k} (\log k)^2 \\
&\quad < \infty,
\end{aligned} \tag{2.12}$$

where the last line follows by using (1.4) and (1.5). Thus by the Borel Cantelli lemma $S_{2^n} \longrightarrow S$ almost surely. To finish the proof we utilize the generalization of the Rademacher-Menchoff maximal inequality given by Serfling [7] and argue as in Hu et al. [5]. It is sufficient to show that, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left\{ \max_{2^{n-1} < k \le 2^n} \left| S_k - S_{2^{n-1}} \right| > \varepsilon \right\} < \infty. \tag{2.13}$$

Using Serfling's inequality and (3.8) from Hu et al. [5]

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P\left\{\max_{2^{n-1}<k\leq 2^n} |S_k - S_{2^{n-1}}| > \varepsilon\right\} \\
&\leq 1 + C\sum_{n=2}^{\infty}\sum_{i=2^{n-1}+1}^{2^n} \frac{(\operatorname{Var} X_i)(\log i)^2}{b_i^2} + C\sum_{n=2}^{\infty} n^2 \sum_{k=1}^{2^{n-1}-1} \frac{\rho_k}{k} \log\left(1+\frac{k}{2^n}\right) \\
&\leq 1 + C\sum_{i=1}^{\infty} \frac{(\operatorname{Var} X_i)(\log i)^2}{b_i^2} + C\sum_{k=1}^{\infty}\sum_{n=1+[\log k]}^{\infty} n^2 \left(\frac{\rho_k}{k}\frac{k}{2^n}\right) \\
&\leq 1 + C\sum_{i=1}^{\infty} \frac{(\operatorname{Var} X_i)(\log i)^2}{b_i^2} + C\sum_{k=1}^{\infty}\sum_{n=1+[\log k]}^{\infty} \left(\frac{n^2}{2^n}\right)\rho_k \\
&\leq 1 + C\sum_{i=1}^{\infty} \frac{(\operatorname{Var} X_i)(\log i)^2}{b_i^2} + C\sum_{k=1}^{\infty} \frac{\rho_k}{k}(\log k)^2, \quad \text{by Lemma 2.3,} \\
&< \infty.
\end{aligned}
\tag{2.14}
$$

□

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