Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2010, Article ID 707146, 34 pages
doi:10.1155/2010/707146

Research Article

Estimating L-Functionals for Heavy-Tailed Distributions and Application

Abdelhakim Necir and Djamel Meraghni

Laboratory of Applied Mathematics, Mohamed Khider University of Biskra, 07000 Biskra, Algeria

Correspondence should be addressed to Abdelhakim Necir, necirabdelhakim@yahoo.fr

Received 5 October 2009; Accepted 21 January 2010

Academic Editor: Ričardas Zitikis

Copyright © 2010 A. Necir and D. Meraghni. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

L-functionals summarize numerous statistical parameters and actuarial risk measures. Their sample estimators are linear combinations of order statistics (L-statistics). There exists a class of heavy-tailed distributions for which the asymptotic normality of these estimators cannot be obtained by classical results. In this paper we propose, by means of extreme value theory, alternative estimators for L-functionals and establish their asymptotic normality. Our results may be applied to estimate the trimmed L-moments and financial risk measures for heavy-tailed distributions.

1. Introduction

1.1. L-Functionals

Let X be a real random variable (rv) with continuous distribution function (df) F. The corresponding L-functionals are defined by

$$L(J) := \int_0^1 J(s)Q(s)ds, \tag{1.1}$$

where $Q(s) := \inf\{x \in \mathbb{R} : F(x) \geq s\}, 0 < s \leq 1$, is the quantile function pertaining to df F and J is a measurable function defined on $[0,1]$ (see, e.g. Serfling, [1]). Several authors have used the quantity $L(J)$ to solve some statistical problems. For example, in a work by Chernoff et al. [2] the L-functionals have a connection with optimal estimators of location and scale parameters in parametric families of distributions. Hosking [3] introduced the L-moments as a new approach of statistical inference of location, dispersion, skewness, kurtosis, and other aspects of shape of probability distributions or data samples having finite means. Elamir and Seheult [4] have defined the trimmed L-moments to answer some questions related to *heavy-tailed* distributions for which means do not exist, and therefore the L-moment method cannot be applied. In the case where the trimming parameter equals one, the first four theoretical trimmed L-moments are

$$m_i := \int_0^1 J_i(s)Q(s)ds, \quad i = 1,2,3,4, \tag{1.2}$$

where

$$J_i(s) := s(1-s)\phi_i(s), \quad 0 < s < 1, \tag{1.3}$$

with ϕ_i polynomials of order $i-1$ (see Section 4). A partial study of statistical estimation of trimmed L-moments was given recently by Hosking [5].

Deriving asymptotics of complex statistics is a challenging problem, and this was indeed the case for a decade since the introduction of the distortion risk measure by Denneberg [6] and Wang [7]; see also Wang [8]. The breakthrough in the area was offered by Jones and Zitikis [9], who revealed a fundamental relationship between the distortion risk measure and the classical L-statistic, thus opening a broad gateway for developing statistical inferential results in the area (see, e.g., Jones and Zitikis [10, 11]; Brazauskas et al. [12, 13] and Greselin et al. [14]). These works mainly discuss CLT-type results. We have been utilizing the aforementioned relationship between distortion risk measures and L-statistics to develop a statistical inferential theory for distortion risk measures in the case of heavy-tailed distributions.

Indeed L-functionals have many applications in actuarial risk measures (see, e.g., Wang [8, 15, 16]). For example, if $X \geq 0$ represents an insurance loss, the distortion risk premium is defined by

$$\Pi(X) := \int_0^\infty g(1-F(x))dx, \tag{1.4}$$

where g is a non decreasing concave function with $g(0) = 0$ and $g(1) = 1$. By a change of variables and integration by parts, $\Pi(X)$ may be rewritten into

$$\Pi(X) = \int_0^1 g'(1-s)Q(s)ds, \tag{1.5}$$

where g' denotes the Lebesgue derivative of g. For heavy-tailed claim amounts, the empirical estimation with confidence bounds for $\Pi(X)$ has been discussed by Necir et al. [17] and Necir and Meraghni [18]. If $X \in \mathbb{R}$ represents financial data such as asset log-returns, the distortion risk measures are defined by

$$H(X) := \int_{-\infty}^0 (g(1-F(x))-1)dx + \int_0^\infty g(1-F(x))dx. \tag{1.6}$$

Likewise, by integration by parts it is shown that

$$H(X) = \int_0^1 g'(1-s)Q(s)ds. \tag{1.7}$$

Wang [8] and Jones and Zitikis [9] have defined the *risk two-sided deviation* by

$$\Delta_r(\mathrm{X}) := \int_0^1 J_r(s)Q(s)ds, \quad 0 < r < 1, \tag{1.8}$$

with

$$J_r(s) := \frac{r}{2}\frac{s^{1-r} - (1-s)^{1-\mathrm{r}}}{s^{1-r}(1-s)^{1-r}}, \quad 0 < s < 1. \tag{1.9}$$

As we see, $\Pi(X)$, $H(X)$, and $\Delta_r(\mathrm{X})$ are L-functionals for specific weight functions. For more details about the distortion risk measures one refers to Wang [8, 16]. A discussion on their empirical estimation is given by Jones and Zitikis [9].

1.2. Estimation of L-Functionals and Motivations

In the sequel let $\xrightarrow{p}$ and $\xrightarrow{\mathcal{D}}$, respectively, stand for convergence in probability and convergence in distribution and let $\mathcal{N}(0, \eta^2)$ denote the normal distribution with mean 0 and variance η^2.

The natural estimators of quantity $L(J)$ are linear combinations of order statistics called L-statistics. For more details on this kind of statistics one refers to Shorack and Wellner [19, page 260]. Indeed, let $(X_1, \dots, X_n)$ be a sample of size $n \geq 1$ from an rv X with df F, then the sample estimator of $L(J)$ is

$$\widehat{L}_n(J) := \int_0^1 J(s)Q_n(s)ds, \tag{1.10}$$

where $Q_n(s) := \inf\{x \in \mathbb{R} : F_n(x) \geq s\}$, $0 < s \leq 1$, is the empirical quantile function that corresponds to the empirical df $F_n(x) := n^{-1}\sum_{i=1}^n \mathbb{I}\{X_i \leq x\}$ for $x \in \mathbb{R}$, pertaining to the sample $(X_1, \dots, X_n)$ with $\mathbb{I}(\cdot)$ denoting the indicator function. It is clear that $\widehat{L}_n(J)$ may be rewritten into

$$\widehat{L}_n(J) = \sum_{i=1}^n a_{i,n}X_{i,n}, \tag{1.11}$$

where $a_{i,n} := \int_{(i-1)/n}^{i/n} J(s)ds$, $i = 1, \dots, n$, and $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based upon the sample $(X_1, \dots, X_n)$. The first general theorem on the asymptotic normality of $\widehat{L}_n(J)$ is established by Chernoff et al. [2]. Since then, a large number of authors have studied the asymptotic behavior of L-statistics. A partial list consists of Bickel [20], Shorack [21, 22], Stigler [23, 24], Ruymgaart and Van Zuijlen [25], Sen [26], Boos [27], Mason [28], and Singh [29]. Indeed, we have

$$\sqrt{n}\left(\widehat{L}_n(J) - L(J)\right) \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, \sigma^2(J)\right), \quad \text{as } n \longrightarrow \infty, \tag{1.12}$$

provided that

$$\sigma^2(J) := \int_0^1 \int_0^1 (\min(s,t) - st) J(s) J(t) dQ(s) Q(t) < \infty. \tag{1.13}$$

In other words, for a given function J, condition (1.13) excludes the class of distributions F for which $\sigma^2(J)$ is infinite. For example, if we take $J = 1$, $L(J)$ is equal to the expected value EX and hence the natural estimator of $\widehat{L}_n(J)$ is the sample mean $\overline{X}_n$. In this case, result (1.12) corresponds to the classical central limit theorem which is valid only when the variance of F is finite. How then can be construct confidence bounds for the mean of a df when its variance is infinite? This situation arises when df F belongs to the domain of attraction of α-stable laws (heavy-tailed) with characteristic exponent $\alpha \in (1,2)$; see Section 2. This question was answered by Peng [30, 31] who proposed an alternative asymptotically normal estimator for the mean. Remark 3.3 below shows that this situation also arises for the sample trimmed L-moments m_i when $1/2 < \alpha < 2/3$ and for the sample risk two-sided deviation $\Delta_r(X)$ when $1/(r+1/2) < \alpha < 1/r$ for any $0 < r < 1$. To solve this problem in a more general setting, we propose, by means of the extreme value theory, asymptotically normal estimators of L-functionals for heavy-tailed distributions for which $\sigma^2(J) = \infty$.

The remainder of this paper is organized as follows. Section 2 is devoted to a brief introduction on the domain of attraction of α-stable laws. In Section 3 we define, via the extreme value approach, a new asymptotically normal estimator of L-functionals and state our main results. Applications to trimmed L-moments, risk measures, and related quantities are given in Section 4. All proofs are deferred to Section 5.

2. Domain of Attraction of α-Stable Laws

A df is said to belong to the domain of attraction of a stable law with stability index $0 < \alpha \leq 2$, notation: $F \in D(\alpha)$, if there exist two real sequences $A_n > 0$ and C_n such that

$$A_n^{-1}\left(\sum_{i=1}^{n} X_i - C_n\right) \xrightarrow{\mathfrak{D}} S_\alpha(\sigma, \delta, \mu), \quad \text{as } n \longrightarrow \infty, \tag{2.1}$$

where $S_\alpha(\sigma, \delta, \mu)$ is a stable distribution with parameters $0 < \alpha \leq 2$, $-1 \leq \delta \leq +1$, $\sigma > 0$ and $-\infty < \mu < +\infty$ (see, e.g., Samorodnitsky and Taqqu [32]). This class of distributions was introduced by Lévy during his investigations of the behavior of sums of independent random variables in the early 1920s [33]. $S_\alpha(\sigma, \delta, \mu)$ is a rich class of probability distributions that allow skewness and thickness of tails and have many important mathematical properties. As shown in early work by Mandelbrot (1963) and Fama [34], it is a good candidate to accommodate heavy-tailed financial series and produces measures of risk based on the tails of distributions, such as the Value-at-Risk. They also have been proposed as models for many types of physical and economic systems, for more details see Weron [35]. This class of distributions have nice heavy-tail properties. More precisely, if we denote by $G(x) := \mathbb{P}(|X| \leq x) = F(x) - F(-x), x > 0$, the df of $Z := |X|$, then the tail behavior of $F \in D(\alpha)$, for $0 < \alpha < 2$, may be described by the following

(i) The tail $1 - G$ is regularly varying at infinity with index $-\alpha$. That is

$$\lim_{t \to \infty} \frac{(1 - G(xt))}{(1 - G(t))} = x^{-\alpha}, \quad \text{for any } x > 0. \tag{2.2}$$

(ii) There exists $0 \leq p \leq 1$ such that

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = p, \qquad \lim_{x \to \infty} \frac{F(-x)}{1 - G(x)} = 1 - p =: q. \tag{2.3}$$

Let, for $0 < s < 1, K(s) := \inf\{x > 0 : G(x) \geq s\}$ be the quantile function pertaining to G and $Q_1(s) := \max(-Q(1 - s), 0)$ and $Q_2(s) := \max(Q(s), 0)$. Then Proposition A.3 in a work by Csörgő et al. [36] says that the set of conditions above is equivalent to the following.

(i′) $K(1 - \cdot)$ is regularly varying at 0 with index $-1/\alpha$. That

$$\lim_{s \downarrow 0} \frac{K(1 - xs)}{K(1 - s)} = x^{-1/\alpha}, \quad \text{for any } x > 0. \tag{2.4}$$

(ii′) There exists $0 \leq p \leq 1$ such that

$$\lim_{s \downarrow 0} \frac{Q_1(1 - s)}{K(1 - s)} = p^{1/\alpha}, \qquad \lim_{s \downarrow 0} \frac{Q_2(1 - s)}{K(1 - s)} = (1 - p)^{1/\alpha} =: q^{1/\alpha}. \tag{2.5}$$

Our framework is a second-order condition that specifies the rate of convergence in statement (i′). There exists a function A, not changing sign near zero, such that

$$\lim_{s \downarrow 0} (A(s))^{-1} \left(\frac{K(1 - xs)}{K(1 - s)} - x^{-1/\alpha} \right) = x^{-1/\alpha} \frac{x^{\varrho} - 1}{\varrho}, \quad \text{for any } x > 0, \tag{2.6}$$

where $\varrho \leq 0$ is the second-order parameter. If $\varrho = 0$, interpret $(x^{\varrho} - 1)/\varrho$ as $\log x$. The second-order condition for heavy-tailed distributions has been introduced by de Haan and Stadtmüller [37].

3. Estimating L-Functionals When $F \in D(\alpha)$

3.1. Extreme Quantile Estimation

The right and left extreme quantiles of small enough level t of df F, respectively, are two reals x_R and x_L defined by $1 - F(x_R) = t$ and $F(x_L) = t$, that is, $x_R = Q(1 - t)$ and $x_L = Q(t)$.

The estimation of extreme quantiles for heavy-tailed distributions has got a great deal of interest, see for instance Weissman [38], Dekkers and de Haan [39], Matthys and Beirlant [40] and Gomes et al. [41]. Next, we introduce one of the most popular quantile estimators. Let $k = k_n$ and $\ell = \ell_n$ be sequences of integers (called trimming sequences) satisfying $1 < k <$ n, $1 < \ell < n$, $k \to \infty$, $\ell \to \infty$, $k/n \to 0$ and $\ell/n \to 0$, as $n \to \infty$. Weissman's estimators of extreme quantiles x_R and x_L are defined, respectively, by

$$
\begin{aligned}
\widehat{x}_L = \widehat{Q}_L(t) &:= \left(\frac{k}{n}\right)^{1/\alpha_L} X_{k,n} t^{-1/\alpha_L}, \quad \text{as } t \downarrow 0, \\
\widehat{x}_R = \widehat{Q}_R(1-t) &:= \left(\frac{\ell}{n}\right)^{1/\alpha_R} X_{n-\ell,n} t^{-1/\alpha_R}, \quad \text{as } t \downarrow 0,
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\widehat{\alpha}_L = \widehat{\alpha}_L(k) &:= \left(\frac{1}{k}\sum_{i=1}^{k}\log^{+}(-X_{i,n}) - \log^{+}(-X_{k,n})\right)^{-1}, \\
\widehat{\alpha}_R = \widehat{\alpha}_R(\ell) &:= \left(\frac{1}{\ell}\sum_{i=1}^{\ell}\log^{+}(X_{n-i+1,n}) - \log^{+}(X_{n-\ell,n})\right)^{-1}
\end{aligned}
\tag{3.2}
$$

are two forms of Hill's estimator [42] for the stability index α which could also be estimated, using the order statistics $Z_{1,n} \leq \cdots \leq Z_{n,n}$ associated to a sample $(Z_1, \ldots, Z_n)$ from Z, as follows:

$$
\widehat{\alpha} = \widehat{\alpha}(m) := \left(\frac{1}{m}\sum_{i=1}^{m}\log^{+}(Z_{n-i+1,n}) - \log^{+}(Z_{n-m,n})\right)^{-1}, \tag{3.3}
$$

with $\log^{+}u := \max(0, \log u)$ and $m = m_n$ being an intermediate sequence fulfilling the same conditions as k and ℓ. Hill's estimator has been thoroughly studied, improved, and even generalized to any real-valued tail index. Its weak consistency was established by Mason [43] assuming only that the underlying distribution is regularly varying at infinity. The almost sure convergence was proved by Deheuvels et al. [44] and more recently by Necir [45]. The asymptotic normality has been investigated, under various conditions on the distribution tail, by numerous workers like, for instance, Csörgő and Mason [46], Beirlant and Teugels [47], and Dekkers et al. [48].

3.2. A Discussion on the Sample Fractions k and ℓ

Extreme value-based estimators rely essentially on the numbers k and ℓ of lower- and upper-order statistics used in estimate computation. Estimators $\widehat{\alpha}_L$ and $\widehat{\alpha}_R$ have, in general, substantial variances for small values of k and ℓ and considerable biases for large values of k and ℓ. Therefore, one has to look for optimal values for k and ℓ, which balance between these two vices.

Numerically, there exist several procedures for the thorny issue of selecting the numbers of order statistics appropriate for obtaining good estimates of the stability index α;

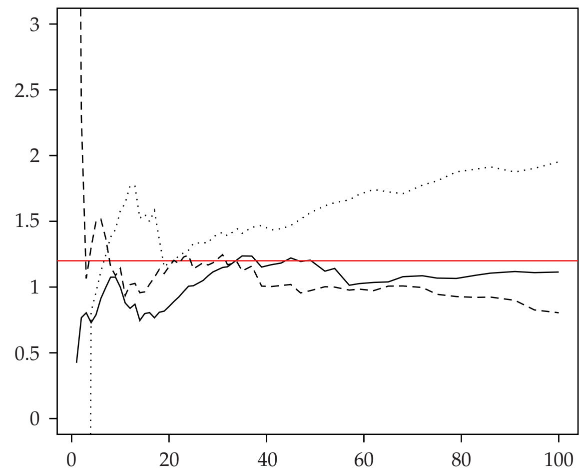

Figure 1: Plots of Hill estimators, as functions of the number of extreme statistics, $\widehat{\alpha}$ (solid line), $\widehat{\alpha}_R$ (dashed line), and $\widehat{\alpha}_L$ (dotted line) of the characteristic exponent α of a stable distribution skewed to the right, based on 1000 observations with 50 replications. The horizontal line represents the true value of $\alpha = 1.2$.

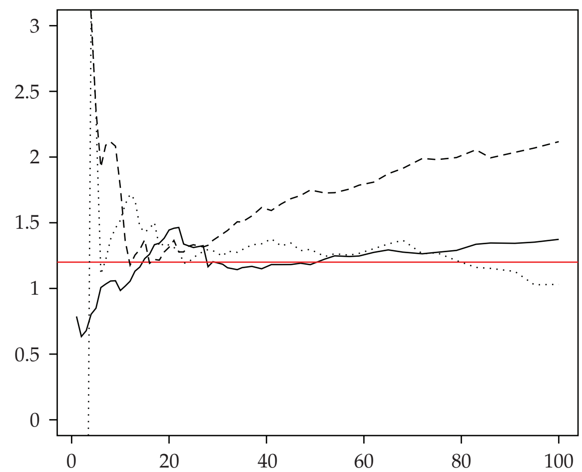

Figure 2: Plots of Hill estimators, as functions of the number of extreme statistics, $\widehat{\alpha}$ (solid line), $\widehat{\alpha}_R$ (dashed line), and $\widehat{\alpha}_L$ (dotted line) of the characteristic exponent α of a stable distribution skewed to the left, based on 1000 observations with 50 replications. The horizontal line represents the true value of $\alpha = 1.2$.

see, for example, Dekkers and de Haan [49], Drees and Kaufmann [50], Danielsson et al. [51], Cheng and Peng [52] and Neves and Alves [53]. Graphically, the behaviors of $\widehat{\alpha}_L$, $\widehat{\alpha}_R$, and $\widehat{\alpha}$ as functions of k, ℓ and m, respectively, are illustrated by Figures 1, 2, and 3 drawn by means of the statistical software R [54]. According to Figure 1, $\widehat{\alpha}_R$ is much more suitable than $\widehat{\alpha}_L$ when estimating the stability index of a distribution which is skewed to the right ($\delta > 0$) whereas Figure 2 shows that $\widehat{\alpha}_L$ is much more reliable than $\widehat{\alpha}_R$ when the distribution is skewed to the left ($\delta < 0$). In the case where the distribution is symmetric ($\delta = 0$), both estimators seem to be equally good as seen in Figure 3. Finally, it is worth noting that, regardless of the distribution skewness, estimator $\widehat{\alpha}$, based on the top statistics pertaining to the absolute value of X, works well and gives good estimates for the characteristic exponent α.

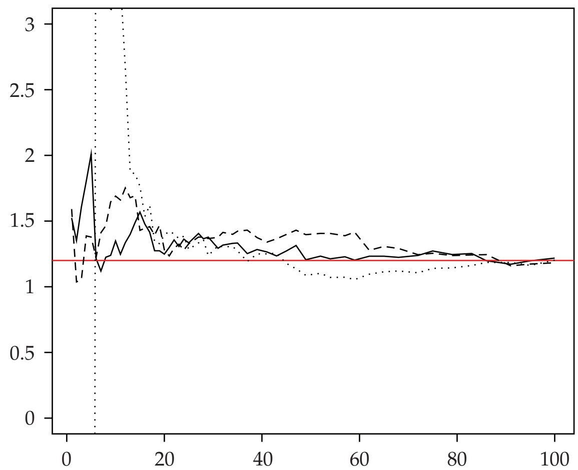

Figure 3: Plots of Hill estimators, as functions of the number of extreme statistics, $\hat{\alpha}$ (solid line), $\hat{\alpha}_R$ (dashed line) and $\hat{\alpha}_L$ (dotted line) of the characteristic exponent α of a symmetric stable distribution, based on 1000 observations with 50 replications. The horizontal line represents the true value of $\alpha = 1.2$.

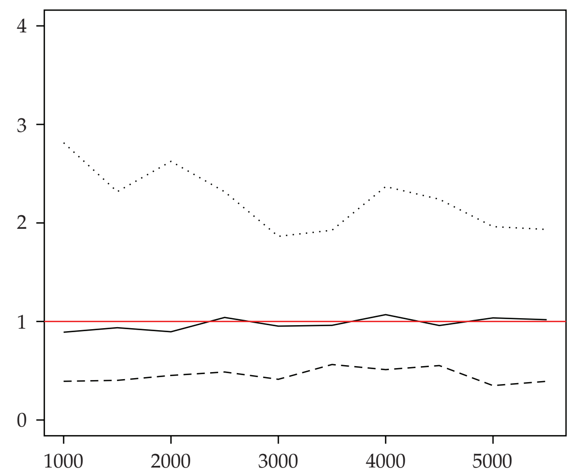

Figure 4: Plots of the ratios of the numbers of extreme statistics, as functions of the sample size, for a stable symmetric distribution $S_{1.2}(1, 0, 0)$ (solid line), a stable distribution skewed to the right $S_{1.2}(1, 0.5, 0)$ (dashed line) and a stable distribution skewed to the left $S_{1.2}(1, -0.5, 0)$ (dotted line).

It is clear that, in general, there is no reason for the trimming sequences k and ℓ to be equal. We assume that there exists a positive real constant θ such that $\ell/k \to \theta$ as $n \to \infty$. If the distribution is symmetric, the value of θ is equal to 1; otherwise, it is less or greater than 1 depending on the sign of the distribution skewness. For an illustration, see Figure 4 where we plot the ratio θ for several increasing sample sizes.

3.3. Some Regularity Assumptions on J

For application needs, the following regularity assumptions on function J are required:

(H1) J is differentiable on $(0,1)$,

(H2) $\lambda := \lim_{s \downarrow 0} \dfrac{J(1-s)}{J(s)} < \infty$,

(H3) both $J(s)$ and $J(1-s)$ are regularly varying at zero with common index $\beta \in \mathbb{R}$.

(H4) there exists a function $a(\cdot)$ not changing sign near zero such that

$$\lim_{t \downarrow 0} \frac{J(xt)/J(t) - x^{\beta}}{a(t)} = x^{\beta} \frac{x^{\omega} - 1}{\omega}, \quad \text{for any } x > 0, \tag{3.4}$$

where $\omega \leq 0$ is the second-order *parameter.*

The three remarks below give more motivations to this paper.

Remark 3.1. Assumption (H3) has already been used by Mason and Shorack [55] to establish the asymptotic normality of trimmed L-statistics. Condition (H4) is just a refinement of (H3) called the second order condition that is required for quantile function K in (2.6).

Remark 3.2. Assumptions (H1)–(H4) are satisfied by all weight functions $(J_i)_{i=2,4}$ with $(\beta, \lambda) = (1, \pm 1)$ (see Section 4.1) and by function J_r in (1.9) with $(\beta, \lambda) = (r-1, -1)$. These two examples show that the constants β and λ may be positive or negative depending on application needs.

Remark 3.3. L-functionals $L(J)$ exist for any $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ such that $1/\alpha - \beta < 1$. However, Lemma 5.4 below shows that for $1/\alpha - \beta > 1/2$ we have $\sigma^2(J) = \infty$. Then, recall (1.3); whenever $1/2 < \alpha < 2/3$, the trimmed L-moments exist however $\sigma^2(J_i) = \infty, i = 1, \ldots, 4$. Likewise, recall (1.9); whenever $1/(r+1/2) < \alpha < 1/r$, the two-sided deviation $\Delta_r(X)$ exists while $\sigma^2(J_r) = \infty$.

3.4. Defining the Estimator and Main Results

We now have all the necessary tools to introduce our estimator of $L(J)$, given in (1.1), when $F \in D(\alpha)$ with $0 < \alpha < 2$. Let $k = k_n$ and $\ell = \ell_n$ be sequences of integers satisfying $1 < k < n$, $1 < \ell < n$, $k \to \infty$, $\ell \to \infty$, $k/n \to 0$, $\ell/n \to 0$, and the additional condition $\ell/k \to \theta < \infty$ as $n \to \infty$. First, we must note that since $1 + \beta - 1/\alpha > 0$ (see Remark 3.3) and since both $\widehat{\alpha}_L$ and $\widehat{\alpha}_R$ are consistent estimators of α (see, Mason [43]), then we have for all large n

$$\mathbb{P}\left(1 + \beta - \frac{1}{\widehat{\alpha}_L} > 0\right) = \mathbb{P}\left(1 + \beta - \frac{1}{\widehat{\alpha}_R} > 0\right) = 1 + o(1). \tag{3.5}$$

Observe now that $L(J)$ defined in (1.1) may be split in three integrals as follows:

$$L(J) = \int_0^{k/n} J(t)Q(t)dt + \int_{k/n}^{1-\ell/n} J(t)Q(t)dt + \int_{1-\ell/n}^{1} J(t)Q(t)dt =: T_{L,n} + T_{M,n} + T_{R,n}. \tag{3.6}$$

Substituting $\widehat{Q}_L(t)$ and $\widehat{Q}_R(1-t)$ for $Q(t)$ and $Q(1-t)$ in $T_{L,n}$ and $T_{R,n}$, respectively and making use of assumption (H3) and (3.5) yield that for all large n

$$\begin{aligned}\int_0^{k/n} J(t)\widehat{Q}_L(t)dt &= \left(\frac{k}{n}\right)^{1/\alpha_L} X_{k,n}\int_0^{k/n} t^{-1/\alpha_L}J(t)dt = (1+o(1))\frac{(k/n)J(k/n)}{1+\beta-1/\widehat{\alpha}_L}X_{k,n},\\ \int_0^{\ell/n} J(1-t)\widehat{Q}_R(1-t)dt &= \left(\frac{\ell}{n}\right)^{1/\alpha_R} X_{n-\ell,n}\int_0^{\ell/n} t^{-1/\alpha_R}J(1-t)dt\\ &= (1+o(1))\frac{(\ell/n)J(1-\ell/n)}{1+\beta-1/\widehat{\alpha}_R}X_{n-\ell,n}.\end{aligned} \tag{3.7}$$

Hence we may estimate $T_{L,n}$ and $T_{R,n}$ by

$$\widehat{T}_{L,n} := \frac{(k/n)J(k/n)}{1+\beta-1/\widehat{\alpha}_L}X_{k,n}, \qquad \widehat{T}_{R,n} := \frac{(\ell/n)J(1-\ell/n)}{1+\beta-1/\widehat{\alpha}_R}X_{n-\ell,n}, \tag{3.8}$$

respectively. As an estimator of $T_{M,n}$ we take the sample one that is

$$\widehat{T}_{M,n} := \int_{k/n}^{1-\ell/n} J(t)Q_n(t)dt = \sum_{i=k+1}^{n-\ell} a_{i,n}X_{i,n}, \tag{3.9}$$

with the same constants $a_{i,n}$ as those in (1.11). Thus, the final form of our estimator is

$$\widehat{L}_{k,\ell}(J) = \frac{(k/n)J(k/n)}{1+\beta-1/\widehat{\alpha}_L}X_{k,n} + \sum_{i=k+1}^{n-\ell} a_{i,n}X_{i,n} + \frac{(\ell/n)J(1-\ell/n)}{1+\beta-1/\widehat{\alpha}_R}X_{n-\ell,n}. \tag{3.10}$$

A universal estimator of $L(J)$ may be summarized by

$$\widehat{L}_n^*(J) = \widehat{L}_{k,\ell}(J)\mathbb{I}\left(\sigma^2(J) = \infty\right) + \widehat{L}_n(J)\mathbb{I}\left(\sigma^2(J) < \infty\right), \tag{3.11}$$

where $\widehat{L}_n(J)$ is as in (1.11). More precisely

$$\widehat{L}_n^*(J) = \widehat{L}_{k,\ell}(J)\mathbb{I}(A(\alpha,\beta)) + \widehat{L}_n(J)\mathbb{I}\left(\overline{A}(\alpha,\beta)\right), \tag{3.12}$$

where $A(\alpha,\beta) := \{(\alpha,\beta) \in (0,2)\times\mathbb{R} : 1/2 < 1/\alpha - \beta < 1\}$ and $\overline{A}(\alpha,\beta)$ is its complementary in $(0,2)\times\mathbb{R}$.

Note that for the particular case $\ell = k$ and $J = 1$ the asymptotic normality of the trimmed mean $\widehat{T}_{M,n}$ has been established in Theorem 1 of Csörgő et al. [56]. The following theorem gives the asymptotic normality of $\widehat{T}_{M,n}$ for more general trimming sequences k and ℓ and weighting function J. For convenience, we set, for any $0 < x < 1/2$ and $0 < y < 1/2$,

$$\sigma^2(x, y; J) := \int_x^{1-y} \int_x^{1-y} (\min(s, t) - st) J(s) J(t) dQ(s) Q(t) < \infty, \tag{3.13}$$

and let $\sigma_n^2(J) := \sigma^2(k/n, \ell/n; J)$.

Theorem 3.4. *Assume that $F \in D(\alpha)$ with $0 < \alpha < 2$. For any measurable function J satisfying assumption (H3) with index $\beta \in \mathbb{R}$ such that $0 < 1/\alpha - \beta < 1$ and for any sequences of integers k and ℓ such that$1 < k < n$, $1 < \ell < n$, $k \to \infty$, $\ell \to \infty$, $k/n \to 0$, and $\ell/n \to 0$, as $n \to \infty$, there exists a probability space (Ω, A, P) carrying the sequence $X_1, X_2, \ldots$ and a sequence of Brownian bridges $\{B_n(s), 0 \le s \le 1, n = 1, 2, \ldots\}$ such that one has for all large n*

$$\frac{\sqrt{n}\left(\widehat{T}_{M,n} - T_{M,n}\right)}{\sigma_n(J)} = -\frac{\int_{k/n}^{1-\ell/n} J(s) B_n(s) ds}{\sigma_n(J)} + o_p(1), \tag{3.14}$$

and therefore

$$\frac{\sqrt{n}\left(\widehat{T}_{M,n} - T_{M,n}\right)}{\sigma_n(J)} \xrightarrow{\mathfrak{D}} \mathcal{N}(0, 1) \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

The asymptotic normality of our estimator is established in the following theorem.

Theorem 3.5. *Assume that $F \in D(\alpha)$ with $0 < \alpha < 2$. For any measurable function J satisfying assumptions (H1)–(H4) with index $\beta \in \mathbb{R}$ such that $1/2 < 1/\alpha - \beta < 1$, and for any sequences of integers k and ℓ such that $1 < k < n$, $1 < \ell < n$, $k \to \infty$, $\ell \to \infty$, $k/n \to 0$, $\ell/n \to 0$, $\ell/k \to \theta < \infty$, and $\sqrt{k} a(k/n) A(k/n) \to 0$ as $n \to \infty$, one has*

$$\frac{\sqrt{n}\left(\widehat{L}_{k,\ell}(J) - L(J)\right)}{\sigma_n(J)} \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, \sigma_0^2\right), \quad \text{as } n \longrightarrow \infty, \tag{3.16}$$

where

$$\sigma_0^2 = \sigma_0^2(\alpha, \beta) := (\alpha\beta + 1)(2\alpha\beta + 2 - \alpha) \times \left(\frac{2\alpha^2 + (\beta\alpha - 1)^2 + 2\alpha(\beta\alpha - 1)}{2\left((1 + \beta)\alpha - 1\right)^4} + \frac{1}{(1 + \beta)\alpha - 1}\right) + 1. \tag{3.17}$$

The following corollary is more practical than Theorem 3.5 as it directly provides confidence bounds for $L(J)$.

Corollary 3.6. *Under the assumptions of Theorem 3.5 one has*

$$\frac{\sqrt{n}\left(\widehat{L}_{k,\ell}(J) - L(J)\right)}{(\ell/n)^{1/2} J(1-\ell/n) X_{n-\ell,n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, V^2\right), \quad \text{as } n \longrightarrow \infty, \tag{3.18}$$

where

$$\begin{aligned} V^2 = V^2(\alpha,\beta,\lambda,\theta,p) := {}& \left(1 + \lambda^{-2}\left(\frac{q}{p}\right)^{-2/\alpha} \theta^{-2\beta+2/\alpha-1}\right) \\ & \times \left(\frac{2\alpha^2 + (\beta\alpha-1)^2 + 2\alpha(\beta\alpha-1)}{2\left((1+\beta)\alpha-1\right)^4} + \frac{1}{(1+\beta)\alpha-1}\right) + 1, \end{aligned} \tag{3.19}$$

with (p,q) as in statement (ii) of Section 2 and (λ,β) as in assumptions (H2)-(H3) of Section 3.

3.5. Computing Confidence Bounds for $L(J)$

The form of the asymptotic variance V^2 in (3.20) suggests that, in order to construct confidence intervals for $L(J)$, an estimate of p is needed as well. Using the intermediate order statistic $Z_{n-m,n}$, de Haan and Pereira [57] proposed the following consistent estimator for p :

$$\widehat{p}_n = \widehat{p}_n(m) := \frac{1}{m}\sum_{i=1}^{n} \mathbb{I}\{X_i > Z_{n-m,n}\}, \tag{3.20}$$

where $m = m_n$ is a sequence of integers satisfying $1 < m < n$, $m \to \infty$, and $m/n \to 0$, as $n \to \infty$ (the same as that used in (3.3)).

Let J be a given weight function satisfying (H1)–(H4) with fixed constants β and λ. Suppose that, for n large enough, we have a realization $(x_1,\ldots,x_n)$ of a sample $(X_1,\ldots,X_n)$ from rv X with df F fulfilling all assumptions of Theorem 3.5. The $(1-\varsigma)$-confidence intervals for $L(J)$ will be obtained via the following steps.

Step 1. Select the optimal numbers k^*, ℓ^*, and m^* of lower- and upper-order statistics used in (3.2) and (3.3).

Step 2. Determine $X_{k^*,n}, X_{n-\ell^*,n}, J(k^*/n), J(1-\ell^*/n)$, and $\theta^* := \ell^*/k^*$.

Step 3. Compute, using (3.2), $\widehat{\alpha}_L^* := \widehat{\alpha}_L(k^*)$ and $\widehat{\alpha}_R^* := \widehat{\alpha}_R(\ell^*)$. Then deduce, by (3.10), the estimate $\widehat{L}_{k^*,\ell^*}(J)$.

Step 4. Use (3.3) and (3.20) to compute $\widehat{\alpha}^* := \widehat{\alpha}(m^*)$ and $\widehat{p}_n^* := \widehat{p}_n(m^*)$. Then deduce, by (3.19), the asymptotic standard deviation

$$V^* := \sqrt{V^2(\widehat{\alpha}^*,\beta,\lambda,\theta^*,\widehat{p}_n^*)}. \tag{3.21}$$

Finally, the lower and upper $(1-\varsigma)$-confidence bounds for $L(J)$, respectively, will be

$$\begin{gathered}\widehat{L}_{k^*,\ell^*}(J) - z_{\varsigma/2}\frac{\sqrt{\ell^*}V^* X_{n-\ell^*,n} J(1-\ell^*/n)}{n}, \\ \widehat{L}_{k^*,\ell^*}(J) + z_{\varsigma/2}\frac{\sqrt{\ell^*}V^* X_{n-\ell^*,n} J(1-\ell^*/n)}{n},\end{gathered} \tag{3.22}$$

where $z_{\varsigma/2}$ is the $(1-\varsigma/2)$ quantile of the standard normal distribution $\mathcal{N}(0,1)$ with $0<\varsigma<1$.

4. Applications

4.1. TL-Skewness and TL-Kurtosis When $F \in D(\alpha)$

When the distribution mean EX exists, the skewness and kurtosis coefficients are, respectively, defined by $L_1 := \mu_3/\mu_2^{3/2}$ and $L_2 := \mu_4/\mu_2^2$ with $\mu_k := E(X-EX)^k$, $k = 2, 3$, and 4 being the centered moments of the distribution. They play an important role in distribution classification, fitting models, and parameter estimation, but they are sensitive to the behavior of the distribution extreme tails and may not exist for some distributions such as the Cauchy distribution. Alternative measures of skewness and kurtosis have been proposed; see, for instance, Groeneveld [58] and Hosking [3]. Recently, Elamir and Seheult [4] have used the trimmed L-moments to introduce new parameters called TL-skewness and TL-kurtosis that are more robust against extreme values. For example, when the trimming parameter equals one, the TL-skewness and TL-kurtosis measures are, respectively, defined by

$$\upsilon_1 := \frac{m_3}{m_2}, \qquad \upsilon_2 := \frac{m_4}{m_2}, \tag{4.1}$$

where $m_i, i = 2, 3, 4$, are the trimmed L-moments defined in Section 1. The corresponding weight functions of (1.3) are defined as follows:

$$\begin{gathered}J_2(s) := 6s(1-s)(2s-1), \\ J_3(s) := \frac{20}{3}s(1-s)\left(5s^2-5s+1\right), \\ J_4(s) := \frac{15}{2}s(1-s)\left(14s^3-21s^2+9s-1\right).\end{gathered} \tag{4.2}$$

If we suppose that $F \in D(\alpha)$ with $1/2 < \alpha < 2/3$, then, in view of the results above, asymptotically normal estimators for υ_1 and υ_2 will be, respectively,

$$\widehat{\upsilon}_1 := \frac{\widehat{m}_3}{\widehat{m}_2}, \qquad \widehat{\upsilon}_2 := \frac{\widehat{m}_4}{\widehat{m}_2}, \tag{4.3}$$

where

$$\widehat{m}_i = \begin{cases} \dfrac{-6(k/n)^2}{2-1/\widehat{\alpha}_L} X_{k,n} + \displaystyle\sum_{j=k+1}^{n-\ell} a_{j,n}^{(i)} X_{j,n} + \dfrac{6(\ell/n)^2}{2-1/\widehat{\alpha}_R} X_{n-\ell,n}, & \text{for } i = 2, \\ \dfrac{20(k/n)^2}{3(2-1/\widehat{\alpha}_L)} X_{k,n} + \displaystyle\sum_{j=k+1}^{n-\ell} a_{j,n}^{(i)} X_{j,n} + \dfrac{20(\ell/n)^2}{3(2-1/\widehat{\alpha}_R)} X_{n-\ell,n}, & \text{for } i = 3, \\ \dfrac{-15(k/n)^2}{2(2-1/\widehat{\alpha}_L)} X_{k,n} + \displaystyle\sum_{j=k+1}^{n-\ell} a_{j,n}^{(i)} X_{j,n} + \dfrac{15(\ell/n)^2}{2(2-1/\widehat{\alpha}_R)} X_{n-\ell,n}, & \text{for } i = 4, \end{cases} \tag{4.4}$$

with $a_{j,n}^{(i)} := \int_{(j-1)/n}^{j/n} J_i(s)ds$, $i = 2,3,4$, and $j = 1,\dots,n$.

Theorem 4.1. *Assume that $F \in D(\alpha)$ with $1/2 < \alpha < 2/3$. For any sequences of integers k and ℓ such that $1 < k < n$, $1 < \ell < n$, $k \to \infty$, $\ell \to \infty$, $k/n \to 0$, $\ell/n \to 0$, $\ell/k \to \theta < \infty$, and $\sqrt{k}a(k/n)A(k/n) \to 0$ as $n \to \infty$, one has, respectively, as $n \to \infty$,*

$$\begin{gathered} \frac{\sqrt{n}(\widehat{v}_1 - v_1)}{(\ell/n)^{3/2} X_{n-\ell,n}} \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, V_1^2\right), \\ \frac{\sqrt{n}(\widehat{v}_2 - v_2)}{(\ell/n)^{3/2} X_{n-\ell,n}} \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, V_2^2\right), \end{gathered} \tag{4.5}$$

where

$$V_1^2 := \frac{36}{m_2^2}\left(1 - \frac{9m_3}{10m_2}\right)^2 \sigma^{*2}, \qquad V_2^2 := \frac{225}{4m_2^2}\left(1 - \frac{4m_4}{5m_2}\right)^2 \sigma^{*2}, \tag{4.6}$$

with

$$\sigma^{*2} := \left(1 + (q/p)^{-2/\alpha}\theta^{2/\alpha-3}\right) \times \left(\frac{2\alpha^2 + (\alpha-1)^2 + 2\alpha(\alpha-1)}{2(2\alpha-1)^4} + \frac{1}{2\alpha-1}\right) + 1. \tag{4.7}$$

4.2. Risk Two-Sided Deviation When $F \in D(\alpha)$

Recall that the risk two-sided deviation is defined by

$$\Delta_r(X) := \int_0^1 J_r(s)Q(s)ds, \quad 0 < r < 1, \tag{4.8}$$

where

$$J_r(s) := \frac{r}{2}\frac{s^{1-r} - (1-s)^{1-r}}{s^{1-r}(1-s)^{1-r}}, \quad 0 < s < 1. \tag{4.9}$$

An asymptotically normal estimator for $\Delta_r(X)$, when $1/(r+1/2) < \alpha < 1/r$, is

$$\widehat{\Delta}_r(X) = -\frac{r(k/n)^r}{2r - 4/\widehat{\alpha}_L} X_{k,n} + \sum_{j=k+1}^{n-\ell} a_{j,n}^{(r)} X_{j,n} + \frac{r(\ell/n)^r}{2r - 4/\widehat{\alpha}_R} X_{n-\ell,n}, \tag{4.10}$$

where

$$a_{j,n}^{(r)} = \frac{1}{2}\left[\left(1 - \frac{i}{n}\right)^{-r} - \left(1 - \frac{i-1}{n}\right)^{-r} - \left(\frac{i}{n}\right)^{-r} + \left(\frac{i-1}{n}\right)^{-r}\right], \tag{4.11}$$

$j = 1, \ldots, n$.

Theorem 4.2. *Assume that $F \in D(\alpha)$ with $0 < \alpha < 2$ such that $1/(r+1/2) < \alpha < 1/r$, for any $0 < r < 1$. Then, for any sequences of integers k and ℓ such that$1 < k < n$, $1 < \ell < n$, $k \to \infty, \ell \to \infty$, $k/n \to 0$, $\ell/n \to 0, \ell/k \to \theta < \infty$, and $\sqrt{k}a(k/n)A(k/n) \to 0$ as $n \to \infty$, one has, as $n \to \infty$,*

$$\frac{\sqrt{n}\left(\widehat{\Delta}_r(X) - \Delta_r(X)\right)}{(\ell/n)^{r-1/2} X_{n-\ell,n}} \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, V_r^2\right), \tag{4.12}$$

where

$$V_r^2 := \frac{r^2}{4}\left(1 + \left(\frac{q}{p}\right)^{-2/\alpha} \theta^{2/\alpha - 2r + 1}\right) \times \left(\frac{2\alpha^2 + (r\alpha - \alpha - 1)^2 + 2\alpha(r\alpha - \alpha - 1)}{2(r\alpha - 1)^4} + \frac{1}{r\alpha - 1}\right) + 1. \tag{4.13}$$

5. Proofs

First we begin by the following three technical lemmas.

Lemma 5.1. *Let f_1 and f_2 be two continuous functions defined on $(0,1)$ and regularly varying at zero with respective indices $\kappa > 0$ and $-\tau < 0$ such that $\kappa < \tau$. Suppose that f_1 is differentiable at zero, then*

$$\lim_{x \downarrow 0} \frac{\int_x^{1/2} f_1(s) df_2(s)}{f_1(x) f_2(x)} = \frac{\tau}{\kappa - \tau}. \tag{5.1}$$

Lemma 5.2. *Under the assumptions of Theorem 3.5, one has*

$$\lim_{n \to \infty} \frac{\int_{k/n}^{1-\ell/n} (s(1-s))^{1/2-\nu} J(s) dQ(s)}{(k/n)^{1/2-\nu} J(k/n) Q(k/n)} = \frac{1 + \lambda \theta^{1/2-\nu+\beta-1/\alpha} (q/p)^{1/\alpha}}{\alpha(1/2 - \nu + \beta) - 1}, \tag{5.2}$$

for any $0 < \nu < 1/4$.

Lemma 5.3. *For any $0 < x < 1/2$ and $0 < y < 1/2$ one has*

$$\sigma^2(x, y; J) = xc^2(x) + yc^2(1-y) + \int_x^{1-y} c^2(t)dt - \left(xc(x) + yc(1-y) + \int_x^{1-y} c(t)dt \right)^2, \tag{5.3}$$

where $c(s) := \int_{1/2}^{s} J(t)dQ(t)$, $0 < s < 1/2$.

Lemma 5.4. *Under the assumptions of Theorem 3.5, one has*

$$\lim_{n\to\infty} \frac{(k/n)J^2(k/n)Q^2(k/n)}{\sigma_n^2(J)} = w^2,$$
$$\lim_{n\to\infty} \frac{(\ell/n)J^2(1-\ell/n)Q^2(1-\ell/n)}{\sigma_n^2(J)} = \lambda^2 \left(\frac{q}{p}\right)^{2/\alpha} \theta^{2\beta-2/\alpha+1} w^2, \tag{5.4}$$

where

$$w^2 := \frac{(\alpha\beta+1)(2\alpha\beta+2-\alpha)}{2\left(1+\lambda^2(q/p)^{2/\alpha}\theta^{2\beta-2/\alpha+1}\right)}. \tag{5.5}$$

Proof of Lemma 5.1. Let f_1' denote the derivative of f_1. Applying integration by parts, we get, for any $0 < x < 1/2$,

$$\int_x^{1/2} f_1(s)df_2(s) = f_1\left(\frac{1}{2}\right) f_2\left(\frac{1}{2}\right) - f_1(x)f_2(x) - \int_x^{1/2} f_1'(s)f_2(s)ds. \tag{5.6}$$

Since the product $f_1 f_2$ is regularly varying at zero with index $-\tau+\kappa < 0$, then $f_1(x)f_2(x) \to 0$ as $x \downarrow 0$. Therefore

$$\lim_{x\downarrow 0} \frac{\int_x^{1/2} f_1(s)df_2(s)}{f_1(x)f_2(x)} = -1 - \lim_{x\downarrow 0} \frac{\int_x^{1/2} f_1'(s)f_2(s)ds}{f_1(x)f_2(x)}. \tag{5.7}$$

By using Karamata's representation (see, e.g., Seneta [59]), it is easy to show that

$$xf_1'(x) = \kappa(1+o(1))f_1(x), \quad \text{as } x \downarrow 0. \tag{5.8}$$

Hence

$$\lim_{x\downarrow 0} \frac{\int_x^{1/2} f_1(s)df_2(s)}{f_1(x)f_2(x)} = -1 - \kappa \lim_{x\downarrow 0} \frac{\int_x^{1/2} f_1'(s)f_2(s)ds}{xf_1'(x)f_2(x)}. \tag{5.9}$$

It is clear that (5.8) implies that f_1' is regularly varying at zero with index $\kappa - 1$; therefore $f_1' f_2$ is regularly varying with index $-\tau + \kappa - 1 < 0$. Then, Theorem 1.2.1 by de Haan [60, page 15] yields

$$\lim_{x \downarrow 0} - \frac{\int_x^{1/2} f_1'(s) f_2(s) ds}{x f_1'(x) f_2(x)} = \frac{1}{\kappa - \tau}. \tag{5.10}$$

This completes the proof of Lemma 5.1. □

Proof of Lemma 5.2. We have

$$\begin{aligned} I_n &:= \int_{k/n}^{1-\ell/n} (s(1-s))^{1/2-\nu} J(s) dQ(s) = \int_{k/n}^{1/2} (s(1-s))^{1/2-\nu} J(s) dQ(s) \\ &\quad - \int_{1-\ell/n}^{1/2} (s(1-s))^{1/2-\nu} J(1-s) dQ(1-s) =: I_{1n} - I_{2n}. \end{aligned} \tag{5.11}$$

By taking, in Lemma 5.1, $f_1(s) = (s(1-s))^{1/2-\nu} J(s)$ and $f_2(s) = Q(s)$ with $\kappa = 1/2 - \nu + \beta, \tau = 1/\alpha$, and $x = k/n$, we get

$$\lim_{n \to \infty} \frac{I_{1n}}{(k/n)^{1/2-\nu} J(k/n) Q(k/n)} = \frac{1/\alpha}{1/2 - \nu + \beta - 1/\alpha}. \tag{5.12}$$

Likewise if we take $f_1(s) = (s(1-s))^{1/2-\nu} J(1-s)$ and $f_2(s) = Q(1-s)$ with $\kappa = 1/2 - \nu + \beta, \tau = 1/\alpha$, and $x = \ell/n$, we have

$$\lim_{n \to \infty} \frac{I_{2n}}{(\ell/n)^{1/2-\nu} J(1-\ell/n) Q(1-\ell/n)} = \frac{1/\alpha}{1/2 - \nu + \beta - 1/\alpha}. \tag{5.13}$$

Note that statement (ii′) of Section 2 implies that

$$\lim_{s \downarrow 0} Q(1-s)/Q(s) = -\left(\frac{q}{p}\right)^{1/\alpha}. \tag{5.14}$$

The last two relations, together with assumption (H2) and the regular variation of $Q(1-s)$, imply that

$$\lim_{n \to \infty} \frac{I_{2n}}{(k/n)^{1/2-\nu} J(k/n) Q(k/n)} = -\frac{(q/p)^{1/\alpha} \lambda \theta^{1/2-\nu+\beta-1/\alpha}/\alpha}{1/2 - \nu + \beta - 1/\alpha}. \tag{5.15}$$

This achieves the proof of Lemma 5.2. □

Proof of Lemma 5.3. We will use similar techniques to those used by Csörgő et al. [36, Proposition A.2]. For any $0 < s < 1/2$, we set

$$W_{x,y}(t) := \begin{cases} c(1-y) & \text{for } 1-y \le t < 1, \\ c(t) & \text{for } x < t < 1-y, \\ c(x) & \text{for } 0 < t \le x. \end{cases} \tag{5.16}$$

Then $\sigma^2(x, y; J)$ may be rewritten into

$$\sigma^2(x, y; J) = \int_0^1 W_{x,y}^2(s)ds - \left(\int_0^1 W_{x,y}(s)ds\right)^2, \tag{5.17}$$

and the result of Lemma 5.3 follows immediately. □

Proof of Lemma 5.4. From Lemma 5.3 we may write

$$\frac{\sigma_n^2(J)}{(k/n)J^2(k/n)Q^2(k/n)} = T_{n1} + T_{n2} + T_{n3} + T_{n4}, \tag{5.18}$$

where

$$\begin{gathered} T_{n1} := \frac{(k/n)c^2(k/n)}{(k/n)J^2(k/n)Q^2(k/n)}, \qquad T_{n2} := \frac{(\ell/n)c^2(1-\ell/n)}{(k/n)J^2(k/n)Q^2(k/n)}, \\ T_{n3} := \frac{\int_{k/n}^{1-\ell/n} c^2(t)dt}{(k/n)J^2(k/n)Q^2(k/n)}, \\ T_{n4} := \frac{\left((k/n)c(k/n) + (\ell/n)c(1-\ell/n) + \int_{k/n}^{1-\ell/n} c(t)dt\right)^2}{(k/n)J^2(k/n)Q^2(\mathrm{k}/n)}. \end{gathered} \tag{5.19}$$

By the same arguments as in the proof of Lemma 5.2, we infer that

$$\begin{gathered} \lim_{n\to\infty} \frac{c(k/n)}{J(k/n)Q(k/n)} = \frac{1}{\alpha\beta - 1}, \\ \lim_{n\to\infty} \frac{c(1-\ell/n)}{J(k/n)Q(k/n)} = \frac{\lambda(q/p)^{1/\alpha}\theta^{\beta-1/\alpha}}{\alpha\beta - 1}. \end{gathered} \tag{5.20}$$

Therefore

$$\lim_{n\to\infty} T_{n1} = \frac{1}{(\alpha\beta-1)^2}, \qquad \lim_{n\to\infty} T_{n2} = \frac{\lambda^2(q/p)^{2/\alpha}\theta^{2\beta-2/\alpha+1}}{(\alpha\beta-1)^2}. \tag{5.21}$$

Next, we consider the third term T_{n3} which may be rewritten into

$$T_{n3} = \frac{\int_{k/n}^{1/2} c^2(t)dt}{(k/n)J^2(k/n)Q^2(k/n)} + \frac{\int_{1/2}^{1-\ell/n} c^2(t)dt}{(k/n)J^2(k/n)Q^2(k/n)}. \tag{5.22}$$

Observe that

$$\frac{\int_{k/n}^{1/2} c^2(t)dt}{(k/n)J^2(k/n)Q^2(k/n)} = \left(\frac{c(k/n)}{J(k/n)Q(k/n)}\right)^2 \frac{\int_{k/n}^{1/2} c^2(t)dt}{(k/n)c^2(k/n)}. \tag{5.23}$$

It is easy to verify that function $c^2(\cdot)$ is regularly varying at zero with index $2(\beta - 1/\alpha)$. Thus, by Theorem 1.2.1 by de Haan [60] we have

$$\lim_{n\to\infty} \frac{\int_{k/n}^{1/2} c^2(t)dt}{(k/n)c^2(k/n)} = \frac{\alpha}{2 - 2\alpha\beta - \alpha}. \tag{5.24}$$

Hence

$$\lim_{n\to\infty} \frac{\int_{k/n}^{1/2} c^2(t)dt}{(k/n)J^2(k/n)Q^2(k/n)} = \frac{\alpha}{(\alpha\beta - 1)^2(2 - 2\alpha\beta - \alpha)}. \tag{5.25}$$

By similar arguments we show that

$$\lim_{n\to\infty} \frac{\int_{1/2}^{1-\ell/n} c^2(t)dt}{(k/n)J^2(k/n)Q^2(k/n)} = \frac{\alpha(q/p)^{2/\alpha}\lambda^2\theta^{2\beta-2/\alpha+1}}{(\alpha\beta - 1)^2(2 - 2\alpha\beta - \alpha)}. \tag{5.26}$$

Therefore

$$\lim_{n\to\infty} T_{n3} = \frac{1 + \alpha(q/p)^{2/\alpha}\lambda^2\theta^{2\beta-2/\alpha+1}}{(\alpha\beta - 1)^2(2 - 2\alpha\beta - \alpha)}. \tag{5.27}$$

By analogous techniques we show that $T_{n4} \to 0$ as $n \to \infty$; we omit details. Summing up the three limits of $T_{ni}, i = 1, 2, 3$, achieves the proof of the first part of Lemma 5.4. As for the second assertion of the lemma, we apply a similar procedure. □

5.1. Proof of Theorem 3.4

Csörgő et al. [36] have constructed a probability space (Ω, A, P) carrying an infinite sequence $\xi_1, \xi_2, \ldots$ of independent rv's uniformly distributed on $(0,1)$ and a sequence of Brownian bridges $\{B_n(s), 0 \le s \le 1, n = 1, 2, \ldots\}$ such that, for the empirical process,

$$\varphi_n(s) := n^{1/2}\{\Gamma_n(s) - s\}, \quad 0 \le s \le 1, \tag{5.28}$$

where $\Gamma_n(\cdot)$ is the uniform empirical df pertaining to the sample $(\xi_1,\ldots,\xi_n)$; we have for any $0 \le \nu < 1/4$ and for all large n

$$\sup_{1/n\le s\le 1-1/n} \frac{\left|\varphi_n(s) - B_n(s)\right|}{\left(s(1-s)\right)^{1/2-\nu}} = O_p\left(n^{-\nu}\right). \tag{5.29}$$

For each $n \ge 1$, let $\xi_{1,n} \le \cdots \le \xi_{n,n}$ denote the order statistics corresponding to $(\xi_1,\ldots,\xi_n)$. Note that for each n, the random vector $(Q(\xi_{1,n}),\ldots,Q(\xi_{n,n}))$ has the same distribution as $(X_{1,n},\ldots,X_{n,n})$. Therefore, for $1 \le i \le n$, we shall use the rv's $Q(\xi_{i,n})$ to represent the rv's $X_{i,n}$, and without loss of generality, we shall be working, in all the following proofs, on the probability space above. According to this convention, the term $\widehat{T}_{M,n}$ defined in (3.9) may be rewritten into

$$\widehat{T}_{M,n} = \int_{\xi_{k,n}}^{\xi_{n-\ell,n}} Q(s)\,d\Psi(\Gamma_n(s)), \tag{5.30}$$

where $\Psi(s) := \int_0^s J(t)dt$. Integrating by parts yields

$$\frac{n^{1/2}\left(\widehat{T}_{M,n} - T_{M,n}\right)}{\sigma_n(J)} = \Delta_{1,n} + \Delta_{2,n} + \Delta_{3,n}, \tag{5.31}$$

where

$$\begin{aligned}
\Delta_{1,n} &:= -\frac{n^{1/2}\int_{k/n}^{1-\ell/n}\{\Psi(\Gamma_n(s)) - \Psi(s)\}dQ(s)}{\sigma_n(J)}, \\
\Delta_{2,n} &:= \frac{n^{1/2}\int_{k/n}^{\xi_{k,n}}\{\Psi(\Gamma_n(s)) - \Psi(k/n)\}dQ(s)}{\sigma_n(J)}, \\
\Delta_{3,n} &:= \frac{n^{1/2}\int_{\xi_{n-\ell,n}}^{1-\ell/n}\{\Psi(\Gamma_n(s)) - \Psi(1-\ell/n)\}dQ(s)}{\sigma_n(J)}.
\end{aligned} \tag{5.32}$$

Next, we show that

$$\Delta_{1,n} \xrightarrow{\mathfrak{D}} \mathcal{N}(0,1) \quad \text{as } n \longrightarrow \infty, \tag{5.33}$$

$$\Delta_{i,n} \xrightarrow{p} 0 \quad \text{as } n \longrightarrow \infty \text{ for } i = 2, 3. \tag{5.34}$$

Making use of the mean-value theorem, we have for each n

$$\Psi(\Gamma_n(s)) - \Psi(s) = (\Gamma_n(s) - s)J(\vartheta_n(s)), \tag{5.35}$$

where $\{\vartheta_n(s)\}_{n\geq 1}$ is a sequence of rv's with values in the open interval of endpoints $s \in (0,1)$ and $\Gamma_n(s)$. Therefore

$$\Delta_{1,n} = \frac{-\int_{k/n}^{1-\ell/n} \varphi_n(s) J(\vartheta_n(s)) dQ(s)}{\sigma_n(J)}. \tag{5.36}$$

This may be rewritten into

$$\begin{aligned}\Delta_{1,n} &= -\frac{\int_{k/n}^{1-\ell/n} \varphi_n(s) J(s) dQ(s)}{\sigma_n(J)} - \frac{\int_{k/n}^{1-\ell/n} \varphi_n(s) J(s) \{J(\vartheta_n(s)) - J(s)\} dQ(s)}{\sigma_n(J)} \\ &=: \Delta_{1,n}^{*} + \Delta_{1,n}^{**}.\end{aligned} \tag{5.37}$$

Note that

$$\begin{aligned}&\frac{\int_{k/n}^{1-\ell/n} \left|\varphi_n(s) - B_n(s)\right| J(s) dQ(s)}{\sigma_n(J)} \\ &\quad \leq \sup_{k/n \leq s \leq 1-\ell/n} \frac{\left|\varphi_n(s) - B_n(s)\right|}{(s(1-s))^{1/2-\nu}} \int_{k/n}^{1-\ell/n} (s(1-s))^{1/2-\nu} |J(s)| dQ(s)/\sigma_n(J),\end{aligned} \tag{5.38}$$

for $0 < \nu < 1/4$, which by (5.29) is equal to

$$\frac{O_p(n^{-\nu}) \int_{k/n}^{1-\ell/n} (s(1-s))^{1/2-\nu} |J(s)| dQ(s)}{\sigma_n(J)}. \tag{5.39}$$

Since we have, from Lemmas 5.2 and 5.3,

$$\left(\frac{n}{k}\right)^{\nu} \int_{k/n}^{1-\ell/n} (s(1-s))^{1/2-\nu} |J(s)| dQ(s)/\sigma_n(J) = O(1), \quad \text{as } n \longrightarrow \infty, \tag{5.40}$$

then the right-hand side of the last inequality is equal to $O_p(k^{-\nu})$ which in turn tends to zero as $n \to \infty$. This implies that as $n \to \infty$

$$\Delta_{1,n}^{*} = -\frac{\int_{k/n}^{1-\ell/n} B_n(s) J(s) dQ(s)}{\sigma_n(J)} + o_p(1). \tag{5.41}$$

Next, we show that $\Delta_{1,n}^{**} = o_p(1)$. Indeed, function J is differentiable on $(0,1)$; then by the mean-value theorem, there exists a sequence $\{\vartheta_n^{*}(s)\}_{n\geq 1}$ of rv's with values in the open interval of endpoints $s \in (0,1)$ and $\vartheta_n(s)$ such that for each n we have

$$\Delta_{1,n}^{**} = \frac{\int_{k/n}^{1-\ell/n} \varphi_n(s) J(s) \{\vartheta_n(s) - s\} J'(\vartheta_n^{*}(s)) dQ(s)}{\sigma_n(J)}. \tag{5.42}$$

From inequalities (3.9) and (3.10) by Mason and Shorack [55], we infer that, for any $0 < \rho < 1$, there exists $0 < M_\rho < \infty$ such that for all large n we have

$$\left|J'(\vartheta_n^*(s))\right| \leq \frac{M_\rho |J(s)|}{(s(1-s))}, \tag{5.43}$$

for any $0 < s \leq 1/2$. On the other hand, we have for any $0 < s < 1$

$$|\vartheta_n(s) - s| \leq |\Gamma_n(s) - s|. \tag{5.44}$$

Therefore

$$\left|\Delta_{1,n}^{**}\right| \leq \frac{M_\rho n^{-1/2} \int_{k/n}^{1/2} \left(\left|\varphi_n(s)\right|^2 |J(s)| / s(1-s) \right) dQ(s)}{\sigma_n(J)}. \tag{5.45}$$

This implies, since, for each $n \geq 1$, $E|\varphi_n(s)|^2 < s(1-s)$, that

$$E\left|\Delta_{1,n}^{**}\right| \leq \frac{M_\rho n^{-1/2} \int_{k/n}^{1/2} |J(s)| dQ(s)}{\sigma_n(J)}, \tag{5.46}$$

which tends to zero as $n \to \infty$.

Next, we consider the term $\Delta_{2,n}$ which may be rewritten into

$$\Delta_{2,n} = \frac{n^{1/2} \int_{k/n}^{\xi_{k,n}} \{\Psi(\Gamma_n(s)) - \Psi(s)\} dQ(s)}{\sigma_n(J)} + \frac{n^{1/2} \int_{k/n}^{\xi_{k,n}} \{\Psi(s) - \Psi(k/n)\} dQ(s)}{\sigma_n(J)}. \tag{5.47}$$

Making use of the mean-value theorem, we get

$$\Delta_{2,n} = \frac{\int_{k/n}^{\xi_{k,n}} \varphi_n(s) J(\mu_n(s)) dQ(s)}{\sigma_n(J)} + \frac{n^{1/2} \int_{k/n}^{\xi_{k,n}} (s - k/n) J(s_n^*) dQ(s)}{\sigma_n(J)}, \tag{5.48}$$

where $\mu_n(s)$ is a sequence of rv's with values in the open interval of endpoints $s \in (k/n, \xi_{k,n})$ and $\Gamma_n(s)$ and s_n^* a sequence of rv's with values in the open interval of endpoints $s \in (k/n, \xi_{k,n})$ and k/n. Again we may rewrite $\Delta_{2,n}$ into

$$\begin{aligned}
\Delta_{2,n} = {} & \frac{\int_{k/n}^{\xi_{k,n}} \varphi_n(s) \left(J(\mu_n(s)) - J(s)\right) dQ(s)}{\sigma_n(J)} + \frac{\int_{k/n}^{\xi_{k,n}} \varphi_n(s) J(s) dQ(s)}{\sigma_n(J)} \\
& + n^{1/2} J\left(\frac{k}{n}\right) \int_{k/n}^{\xi_{k,n}} \left(s - \frac{k}{n}\right) \left(\frac{J(s_n^*)}{J(k/n)} - 1\right) dQ(s) / \sigma_n(J) \\
& + n^{1/2} J\left(\frac{k}{n}\right) \int_{k/n}^{\xi_{k,n}} \left(s - \frac{k}{n}\right) dQ(s) / \sigma_n(J).
\end{aligned} \tag{5.49}$$

Recall that, as $n \to \infty$, both k and ℓ tend to infinity with $k/n \to 0$ and $\ell/n \to 0$. This implies that

$$\frac{n}{k^{1/2}}\left(\xi_{k,n} - \frac{k}{n}\right) \xrightarrow{\mathfrak{D}} \mathcal{N}(0,1) \quad \text{as } n \longrightarrow \infty, \tag{5.50}$$

$$\frac{n}{l^{1/2}}\left(\xi_{n-l,n} - 1 + \frac{l}{n}\right) \xrightarrow{\mathfrak{D}} \mathcal{N}(0,1) \quad \text{as } n \longrightarrow \infty, \tag{5.51}$$

(see, e.g., Balkema and de Haan [61, page 18]). Next, we use similar arguments to those used in the proof of Theorem 1 by Csörgő et al. [56]. For any $0 < c < \infty$ write

$$\begin{aligned}
\Delta_{2,n}^{(1)}(c) &:= \frac{\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} \left|\varphi_n(s)\right| \left|J(\mu_n(s)) - J(s)\right| dQ(s)}{\sigma_n(J)} \\
&\quad + \frac{\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} |J(s)| \left|\varphi_n(s)\right| dQ(s)}{\sigma_n(J)}, \\
\Delta_{2,n}^{(2)}(c) &:= \frac{\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} n^{1/2} |s - k/n| J(s_n^*) dQ(s)}{\sigma_n(J)} \\
&\quad + \frac{n^{1/2} J(k/n) \int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} (s - k/n)(J(s_n^*)/J(k/n) - 1) dQ(s)}{\sigma_n(J)}.
\end{aligned} \tag{5.52}$$

Notice that by (5.49)

$$\lim_{c\to\infty} \liminf_{n\to\infty} \mathbb{P}\left\{|\Delta_{2,n}| \le \Delta_{2,n}^{(1)}(c) + \Delta_{2,n}^{(2)}(c)\right\} \ge \lim_{c\to\infty} \liminf_{n\to\infty} \mathbb{P}\left\{\left|\xi_{k,n} - \frac{k}{n}\right| \le c\frac{k^{1/2}}{n}\right\}. \tag{5.53}$$

In view of (5.50), this last quantity equals 1. Therefore to establish (5.34) for $i = 2$, it suffices to show that for each $0 < c < \infty$

$$\Delta_{2,n}^{(1)}(c) \xrightarrow{p} 0, \quad \Delta_{2,n}^{(2)}(c) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.54}$$

By the mean-value theorem, there exists $\{\mu_n^*(s)\}_{n\ge1}$ a sequence of rv's with values in the open interval of endpoints s and $\mu_n(s)$ such that for each n we have

$$J(\mu_n(s)) - J(s) = (\mu_n(s) - s) J'(\mu_n^*(s)). \tag{5.55}$$

Since $|\mu_n(s) - s| \le |\Gamma_n(s) - s|$, then by inequality (5.43) we infer that, for any $0 < \rho < 1$, there exists $0 < M'_\rho < \infty$ such that for all large n we have

$$\left|J(\mu_n(s)) - J(s)\right| \le \frac{M'_\rho n^{-1/2}\left|\varphi_n(s)\right||J(s)|}{(s(1-s))}. \tag{5.56}$$

This implies that the first term in $\Delta_{2,n}^{(1)}(c)$ is less than or equal to

$$M'_\rho n^{-1/2}\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)}\left(\frac{\varphi_n^2(s)|J(s)|}{(s(1-s))}\right)dQ(s)/\sigma_n(J) \ . \tag{5.57}$$

Since $E(\varphi_n^2(s)) \le s(1-s)$, then the expected value of the previous quantity is less than or equal to

$$\frac{M'_\rho n^{-1/2}\int_{(1-c)k/n}^{(1+c)k/n}|J(s)|dQ(s)}{\sigma_n(J)}. \tag{5.58}$$

Likewise the expected value of the second term in $\Delta_{2,\mathrm{n}}^{(1)}(c)$ is less than or equal to

$$\frac{\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)}(s(1-s))^{1/2}|J(s)|dQ(s)}{\sigma_n(J)} \le \frac{\left(k/n+c(k^{1/2}/n)\right)^{1/2}\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)}|J(s)|dQ(s)}{\sigma_n(J)}. \tag{5.59}$$

Fix $1 < \varepsilon < \infty$. It is readily verified that, for all large n, the quantity (5.58) is less than

$$\frac{M'_\rho n^{-1/2}\int_{k/\varepsilon n}^{\varepsilon(k/n)}|J(s)|dQ(s)}{\sigma_n(J)}, \tag{5.60}$$

and the right-hand side of (5.59) is less than

$$2\left(\frac{k}{n}\right)^{1/2}\int_{k/\varepsilon n}^{\varepsilon(k/n)}|J(s)|dQ(s)/\sigma_n(J). \tag{5.61}$$

Therefore for all large n

$$E\left(\Delta_{2,n}^{(1)}(c)\right) \le \left(2 + M'_\rho k^{-1/2}\right)\left(\frac{k}{n}\right)^{1/2}\int_{k/\varepsilon n}^{\varepsilon(k/n)}|J(s)|dQ(s)/\sigma_n(J). \tag{5.62}$$

By routine manipulations, as in the proofs of Lemmas 5.1 and 5.2 (we omit details), we easily show that

$$\lim_{n\to\infty}\left(\frac{k}{n}\right)^{1/2}\int_{k/\varepsilon n}^{\varepsilon(k/n)}|J(s)|dQ(s)/\sigma_n(J) = \frac{w}{\alpha\beta-1}\left(\varepsilon^{1/\alpha-\beta} - \varepsilon^{\beta-1/\alpha}\right). \tag{5.63}$$

Since $k \to \infty$, then for any $0 < c < \infty$

$$\limsup_{n\to\infty} E\left(\Delta_{2,n}^{(1)}(c)\right) \le 2\left(\varepsilon^{1/\alpha-\beta} - \varepsilon^{\beta-1/\alpha}\right)w, \tag{5.64}$$

for any fixed $1 < \varepsilon < \infty$. This implies that for all $0 < c < \infty$

$$\lim_{n\to\infty} E\left(\Delta_{2,n}^{(1)}(c)\right) = 0. \tag{5.65}$$

Therefore, by Markov inequality, we have the first result of (5.34).

Now, consider the term $\Delta_{2,n}^{(2)}(c)$. Observe that ns^*/k is a sequence of rv's with values in the open interval of endpoints $n\xi_{k,n}/k$ and 1. On the other hand, (5.49) implies that $n\xi_{k,n}/k \xrightarrow{p} 1$ as $n \to \infty$. Hence $ns^*/k \xrightarrow{p} 1$ as well. Then, it is readily checked that, in view of relation (3.9) by Mason and Shorack [55], we have

$$\frac{\sup_{s\in H_n} J(s_n^*)}{J(k/n)} = O_p(1) \quad \text{as } n \longrightarrow \infty. \tag{5.66}$$

Therefore

$$\Delta_{2,n}^{(2)}(c) = O_p(1) J\left(\frac{k}{n}\right) \int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} n^{1/2}\left|s - \frac{k}{n}\right| dQ(s)/\sigma_n(J). \tag{5.67}$$

Observe that

$$\int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} n^{1/2}\left|s - \frac{k}{n}\right| dQ(s) \le c\left(\frac{k}{n}\right)^{1/2} \int_{k/n-c(k^{1/2}/n)}^{k/n+c(k^{1/2}/n)} dQ(s). \tag{5.68}$$

Hence for all large n

$$\Delta_{2,n}^{(2)}(c) = O_p(1) J\left(\frac{k}{n}\right) \int_{k/\varepsilon n}^{\varepsilon(k/n)} dQ(s)/\sigma_n(J), \tag{5.69}$$

for any fixed $1 < \varepsilon < \infty$, we have

$$\Delta_{2,n}^{(2)}(c) = O_p(1)\left(\varepsilon^{1/\alpha-\beta} - \varepsilon^{\beta-1/\alpha}\right)w. \tag{5.70}$$

this means that $\Delta_{2,n}^{(2)}(c) \to 0$ as $n \to \infty$. By the same arguments and making use of (5.51) we show that $\Delta_{3,n} \xrightarrow{p} 0$ as $n \to \infty$ (we omit details), which achieves the proof of Theorem 3.4.

5.2. Proof of Theorem 3.5

Recall (3.8), (3.9), and (3.10) and write

$$\widehat{L}_{k,\ell}(J) - L(J) = \left(\widehat{T}_{L,n} - T_{L,n}\right) + \left(\widehat{T}_{M,n} - T_{M,n}\right) + \left(\widehat{T}_{R,n} - T_{R,n}\right). \tag{5.71}$$

It is easy to verify that

$$\widehat{T}_{R,n} - T_{R,n} = \frac{(k/n)J(k/n)X_{k,n}\widehat{\alpha}_L}{(1+\beta)\widehat{\alpha}_L - 1} - \int_0^{k/n} J(s)Q(t)dt = S_{n1}^L + S_{n2}^L + S_{n3}^L, \tag{5.72}$$

where

$$\begin{aligned} S_{n1}^L &:= \left(\frac{k}{n}\right) J\left(\frac{k}{n}\right) X_{k,n} \left\{ \frac{\widehat{\alpha}_L}{(1+\beta)\widehat{\alpha}_L - 1} - \frac{\alpha}{(1+\beta)\alpha - 1} \right\}, \\ S_{n2}^L &:= \frac{\alpha(k/n)Q(k/n)J(k/n)}{(1+\beta)\alpha - 1} \left\{ \frac{X_{k,n}}{Q(k/n)} - 1 \right\}, \\ S_{n3}^L &:= \frac{\alpha(k/n)J(k/n)Q(k/n)}{(1+\beta)\alpha - 1} - \int_0^{k/n} J(s)Q(t)dt. \end{aligned} \tag{5.73}$$

Likewise we have

$$\widehat{T}_{R,n} - T_{R,n} = \frac{(\ell/n)J(1-\ell/n)X_{n-\ell,n}\widehat{\alpha}_R}{(1+\beta)\widehat{\alpha}_R - 1} - \int_0^{\ell/n} J(s)Q(1-t)dt = S_{n1}^R + S_{n2}^R + S_{n3}^R, \tag{5.74}$$

where

$$\begin{aligned} S_{n1}^R &:= (k/n)J(k/n)X_{k,n} \left\{ \frac{\widehat{\alpha}_L}{(1+\beta)\widehat{\alpha}_L - 1} - \frac{\alpha}{(1+\beta)\alpha - 1} \right\}, \\ S_{n2}^R &:= \frac{\alpha(\ell/n)Q(1-\ell/n)J(1-\ell/n)}{(1+\beta)\alpha - 1} \left\{ \frac{X_{n-\ell,n}}{Q(1-\ell/n)} - 1 \right\}, \\ S_{n3}^R &:= \frac{\alpha(\ell/n)Q(1-\ell/n)J(1-\ell/n)}{(1+\beta)\alpha - 1} - \int_0^{\ell/n} J(s)Q(1-t)dt. \end{aligned} \tag{5.75}$$

It is readily checked that S_{n1}^R may be rewritten into

$$S_{n1}^L = \frac{\widehat{\alpha}_L \alpha (k/n) J(k/n) X_{k,n}}{((1+\beta)\widehat{\alpha}_L - 1)((1+\beta)\alpha - 1)} \left(\frac{1}{\widehat{\alpha}_L} - \frac{1}{\alpha} \right). \tag{5.76}$$

Since $\widehat{\alpha}_L$ is a consistent estimator of α, then for all large n

$$S_{n1}^L = (1 + o_p(1)) \frac{\alpha^2(k/n)J(k/n)X_{k,n}}{((1+\beta)\alpha - 1)^2}\left(\frac{1}{\widehat{\alpha}_L} - \frac{1}{\alpha}\right). \tag{5.77}$$

In view of Theorems 2.3 and 2.4 of Csörgő and Mason [46], Peng [30], and Necir et al. [17] has been shown that under the second-order condition (2.6) and for all large n

$$\begin{gathered}
\sqrt{k}\alpha\left(\frac{1}{\widehat{\alpha}_L} - \frac{1}{\alpha}\right) = -\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + \sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds + o_p(1), \\
\sqrt{k}\left(\frac{X_{k,n}}{Q(k/n)} - 1\right) = -\alpha^{-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + o_p(1), \\
\frac{X_{k,n}}{Q}\left(\frac{k}{n}\right) = 1 + o_p(1),
\end{gathered} \tag{5.78}$$

where $\{B_n(s), 0 \leq s \leq 1, n = 1, 2, \ldots\}$ is the sequence of Brownian bridges defined in Theorem 3.4. This implies that for all large n

$$\begin{aligned}
S_{n1}^L &= (1 + o_p(1))\frac{\alpha(k^{1/2}/n)J(k/n)Q(k/n)}{((1+\beta)\alpha - 1)^2}\left\{-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + \sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds + o_p(1)\right\}, \\
S_{n2}^L &= \frac{\alpha(k^{1/2}/n)Q(k/n)J(k/n)}{(1+\beta)\alpha - 1}\left\{-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + o_p(1)\right\}.
\end{aligned} \tag{5.79}$$

Then, in view of Lemma 5.3, we get for all large n

$$\begin{aligned}
\frac{\sqrt{n}(S_{n1}^L + S_{n2}^L)}{\sigma_n(J)} = &-\frac{\alpha w}{((1+\beta)\alpha - 1)^2}\left\{-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + \sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right\} \\
&- \frac{w}{(1+\beta)\alpha - 1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + o_p(1).
\end{aligned} \tag{5.80}$$

By the same arguments (we omit details), we show that for all large n

$$\begin{aligned}
\frac{\sqrt{n}(S_{n1}^R + S_{n2}^R)}{\sigma_n(J)} = &\frac{\alpha w_R}{((1+\beta)\alpha - 1)^2}\left\{\sqrt{\frac{n}{\ell}}B_n\left(1 - \frac{\ell}{n}\right) - \sqrt{\frac{n}{\ell}}\int_{1-\ell/n}^{1}\frac{B_n(s)}{1-s}ds\right\} \\
&+ \frac{w_R}{(1+\beta)\alpha - 1}\left\{-\sqrt{\frac{n}{\ell}}B_n\left(1 - \frac{\ell}{n}\right)\right\} + o_p(1),
\end{aligned} \tag{5.81}$$

where $w_R := |\lambda|(q/p)^{1/\alpha}\theta^{\beta-1/\alpha+1/2}w$. Similar arguments as those used in the proof of Theorem 1 by Necir et al. [17] yield that

$$\frac{\sqrt{n}S_{n3}^R}{\sigma_n(J)}=\frac{\sqrt{n}S_{n3}^R}{\sigma_n}(J)=o(1)\quad\text{as } n\longrightarrow\infty. \tag{5.82}$$

Then, by (5.80), (5.81), and (5.82) we get

$$\begin{aligned}\frac{\sqrt{n}\left(\widehat{L}_{k,\ell}(J)-L(J)\right)}{\sigma_n(J)}&=-\frac{\alpha w}{((1+\beta)\alpha-1)^2}\left\{-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right\}\\&\quad-\frac{w}{(1+\beta)\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_p(1)-\frac{\int_{k/n}^{1-\ell/n}J(s)B_n(s)ds}{\sigma_n(J)}\\&\quad+\frac{\alpha w_R}{((1+\beta)\alpha-1)^2}\left\{\sqrt{\frac{n}{\ell}}B_n\left(1-\frac{\ell}{n}\right)-\sqrt{\frac{n}{\ell}}\int_{1-\ell/n}^{1}\frac{B_n(s)}{1-s}ds\right\}\\&\quad+\frac{w_R}{(1+\beta)\alpha-1}\left\{-\sqrt{\frac{n}{\ell}}B_n\left(1-\frac{\ell}{n}\right)\right\}+o_p(1).\end{aligned} \tag{5.83}$$

The asymptotic variance of $\sqrt{n}(\widehat{L}_{k,\ell}(J)-L(J))/\sigma_n(J)$ will be computed by

$$\begin{aligned}\sigma_0^2=\lim_{n\to\infty}\Bigg\{&w^2\frac{\alpha^2}{((1+\beta)\alpha-1)^4}\frac{n}{k}\int_0^{k/n}ds\int_0^{k/n}\frac{\min(s,t)-st}{st}dt\\&+w^2\frac{(1-\beta\alpha)^2}{((1-\beta)\alpha-1)^4}\frac{n}{k}\frac{k}{n}\left(1-\frac{k}{n}\right)+\frac{\int_{k/n}^{1-\ell/n}dc(s)\int_{k/n}^{1-\ell/n}(\min(s,t)-st)dc(s)}{\sigma_n(J)}\\&+w_R^2\frac{(1-\beta\alpha)^2}{((1-\beta)\alpha-1)^4}\frac{n}{\ell}\frac{\ell}{n}\left(1-\frac{\ell}{n}\right)\\&+w_R^2\frac{\alpha^2}{((1+\beta)\alpha-1)^4}\frac{n}{\ell}\int_{1-\ell/n}^{1}ds\int_{1-\ell/n}^{1}\frac{\min(s,t)-st}{st}dt\\&-2w^2\frac{\alpha(1-\beta\alpha)}{((1-\beta)\alpha-1)^4}\frac{n}{k}\int_0^{k/n}\frac{t-(k/n)t}{t}dt\\&+2w\frac{\alpha}{((1+\beta)\alpha-1)^2}\sqrt{\frac{n}{k}}\int_0^{k/n}ds\int_{k/n}^{1-\ell/n}\frac{s-st}{s}dc(t)/\sigma_n(J)\end{aligned}$$

$$
\begin{aligned}
&- 2ww_R \frac{\alpha(1-\beta\alpha)}{((1+\beta)\alpha-1)^4}\sqrt{\frac{n}{k}}\sqrt{\frac{n}{\ell}}\int_0^{k/n}\frac{s-(1-\ell/n)s}{s}ds\\
&+ 2ww_R \frac{\alpha^2}{((1+\beta)\alpha-1)^4}\sqrt{\frac{n}{k}}\sqrt{\frac{n}{\ell}}\int_0^{k/n}ds\int_{1-\ell/n}^{1}\frac{\min(s,t)-st}{st}dt\\
&- 2w\left(\frac{(1-\beta\alpha)}{((1+\beta)\alpha-1)^2}\right)\sqrt{\frac{n}{k}}\int_{k/n}^{1-\ell/n}\left(\frac{k}{n}-s\left(\frac{k}{n}\right)\right)dc(t)/\sigma_n(J)\\
&- 2w\left(\frac{(1-\beta\alpha)}{((1+\beta)\alpha-1)^2}\right)\sqrt{\frac{n}{k}}\int_{k/n}^{1-\ell/n}\left(\frac{k}{n}-s\left(\frac{k}{n}\right)\right)dc(t)/\sigma_n(J)\\
&+ 2ww_R \frac{(1-\beta\alpha)^2}{((1+\beta)\alpha-1)^4}\sqrt{\frac{n}{k}}\sqrt{\frac{n}{\ell}}\left(\frac{k}{n}-\frac{k}{n}\left(1-\frac{\ell}{n}\right)\right)\\
&- 2ww_R \frac{\alpha(1-\beta\alpha)}{((1+\beta)\alpha-1)^4}\int_{1-\ell/n}^{1}\frac{k/n-(k/n)s}{1-s}ds\\
&- 2w_R\left(\frac{(1-\beta\alpha)}{((1+\beta)\alpha-1)^2}\right)\sqrt{\frac{n}{\ell}}\int_{k/n}^{1-\ell/n}\left(s-s\left(1-\frac{\ell}{n}\right)\right)dc(t)/\sigma_n(J)\\
&+ 2w_R\left(\frac{\alpha}{((1+\beta)\alpha-1)^2}\right)\sqrt{\frac{n}{\ell}}\int_{1-\ell/n}^{1}ds\int_{k/n}^{1-\ell/n}\frac{(t-st)}{(1-s)}dc(t)/\sigma_n(J)\\
&-2w_R^2\frac{\alpha(1-\beta\alpha)}{((1+\beta)\alpha-1)^4}\left(\frac{n}{\ell}\right)\int_{1-\ell/n}^{1}\frac{(1-\ell/n)-(1-\ell/n)s}{1-s}ds\Bigg\}.
\end{aligned}
\tag{5.84}
$$

After calculation we get

$$
\begin{aligned}
\sigma_0^2 &= w^2\frac{2\alpha^2}{((1+\beta)\alpha-1)^4}+w^2\frac{(1-\beta\alpha)^2}{((1+\beta)\alpha-1)^4}+1+w_R^2\frac{(1-\beta\alpha)^2}{((1+\beta)\alpha-1)^4}+w_R^2\frac{2\alpha^2}{((1+\beta)\alpha-1)^4}\\
&- 2w^2\frac{\alpha(1-\beta\alpha)}{((1+\beta)\alpha-1)^4}+2w\frac{\alpha}{((1+\beta)\alpha-1)^2}w\\
&- 2w\frac{(1-\beta\alpha)}{((1+\beta)\alpha-1)^2}w+2w_R\frac{\alpha}{((1+\beta)\alpha-1)^2}w_R\\
&- 2w_R\frac{(1-\beta\alpha)}{((1+\beta)\alpha-1)^2}w_R-w_R^2\frac{2\alpha}{((1+\beta)\alpha-1)^4}\\
&= \left(w^2+w_R^2\right)\left[\frac{2\alpha^2+(\beta\alpha-1)^2+2\alpha(\beta\alpha-1)}{((1+\beta)\alpha-1)^4}+\frac{2}{(1+\beta)\alpha-1}\right]+1.
\end{aligned}
\tag{5.85}
$$

Finally, it is easy to verify that

$$w^2 + w_R^2 = \frac{(\alpha\beta + 1)(2\alpha\beta + 2 - \alpha)}{2}. \tag{5.86}$$

This completes the proof of Theorem 3.5.

5.3. Proof of the Corollary

Straightforward by combining Theorem 3.5 and Lemma 5.4, we omit details.

5.4. Proof of Theorem 4.1

We will only present details for the proof concerning the first part of Theorem 4.1. The proof for the second part is very similar. For convenience we set

$$\Delta m_2 := \widehat{m}_2 - m_2, \qquad \Delta m_3 := \widehat{m}_3 - m_3. \tag{5.87}$$

Then we have

$$\widehat{v}_1 - v_1 = \frac{\widehat{m}_3}{\widehat{m}_2} - \frac{m_3}{m_2} = \frac{m_2 \Delta m_3 - m_3 \Delta m_2}{\widehat{m}_2 m_2}. \tag{5.88}$$

Since $\widehat{m}_2$ is consistent estimator of m_2, then for all large n

$$\widehat{v}_1 - v_1 = (1 + o_P(1))\left(\frac{\Delta m_3}{m_2} - \frac{m_3 \Delta m_2}{m_2^2}\right), \tag{5.89}$$

and therefore

$$\frac{\sqrt{n}(\widehat{v}_1 - v_1)}{\sigma_n(J_3)} = \frac{1}{m_2}(C_{1n} + C_{2n}) + o_p(1), \tag{5.90}$$

where

$$C_{1n} := \frac{\sqrt{n}\Delta m_3}{\sigma_n(J_3)}, \qquad C_{2n} := -\frac{m_3 \sqrt{n} \Delta m_2}{m_2 \sigma_n(J_3)}. \tag{5.91}$$

In view of (5.83) we may write that for all large n

$$
\begin{aligned}
C_{1n} = & -\frac{\alpha w_1}{(2\alpha-1)^2}\left\{-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right\} \\
& -\frac{w_1}{2\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_p(1)-\frac{\int_{k/n}^{1-\ell/n}J(s)B_n(s)ds}{\sigma_n(J)} \\
& +\frac{\alpha w_{R,1}}{(2\alpha-1)^2}\left\{\sqrt{\frac{n}{\ell}}B_n\left(1-\frac{\ell}{n}\right)-\sqrt{\frac{n}{\ell}}\int_{1-\ell/n}^{1}\frac{B_n(s)}{1-s}ds\right\} \\
& +\frac{w_{R,1}}{2\alpha-1}\left\{-\sqrt{\frac{n}{\ell}}B_n\left(1-\frac{\ell}{n}\right)\right\}+o_p(1),
\end{aligned}
\tag{5.92}
$$

and, by Lemma 5.4, we infer that $\sigma_n(J_3)/\sigma_n(J_2) \to 10/9$ as $n \to \infty$,

$$
\begin{aligned}
-\frac{10m_2}{9m_3}C_{2n} = & -\frac{\alpha w_1}{(2\alpha-1)^2}\left\{-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right\} \\
& -\frac{w_1}{2\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_p(1)-\frac{\int_{k/n}^{1-\ell/n}J(s)B_n(s)ds}{\sigma_n(J)} \\
& +\frac{\alpha w_{R,1}}{((1+\beta)\alpha-1)^2}\left\{\sqrt{\frac{n}{\ell}}B_n\left(1-\frac{\ell}{n}\right)-\sqrt{\frac{n}{\ell}}\int_{1-\ell/n}^{1}\frac{B_n(s)}{1-s}ds\right\} \\
& +\frac{w_{R,1}}{2\alpha-1}\left\{-\sqrt{\frac{n}{\ell}}B_n\left(1-\frac{\ell}{n}\right)\right\}+o_p(1),
\end{aligned}
\tag{5.93}
$$

with

$$
w_1 := \frac{(\alpha+1)(2\alpha+2-\alpha)}{2\left(1+(q/p)^{2/\alpha}\theta^{3-2/\alpha}\right)}, \qquad w_{R,1} := (q/p)^{1/\alpha}\theta^{3/2-1/\alpha}w_1. \tag{5.94}
$$

By the same arguments as the proof of Theorem 3.5 we infer that

$$
\frac{\sqrt{n}(\hat{v}_1-v_1)}{(\ell/n)^{3/2}X_{n-\ell,n}} \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0,V_1^2\right) \quad \text{as } n \longrightarrow \infty. \tag{5.95}
$$

This achieves the proof of Theorem 4.1.

5.5. Proof of Theorem 4.2

Theorem 4.2 is just an application of Theorem 3.5, we omit details.

Acknowledgments

The authors thank anonymous referees for their insightful comments and valuable suggestions. The first author is indebted to Ričardas Zitikis for his constructive advice and also for having introduced him (the first author) to the Actuarial Research Group (ARG) in London, Ontario.

References

[1] R. J. Serfling, *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York, NY, USA, 1980.
[2] H. Chernoff, J. L. Gastwirth, and M. V. Johns, "Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation," *Annals of Mathematical Statistics*, vol. 38, pp. 52–72, 1967.
[3] J. R. M. Hosking, "*L*-moments: analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society Serie B*, vol. 52, pp. 105–124, 1990.
[4] E. A. H. Elamir and A. H. Seheult, "Trimmed *L*-moments," *Computational Statistics and Data Analysis*, vol. 43, no. 3, pp. 299–314, 2003.
[5] J. R. M. Hosking, "Some theory and practical uses of trimmed *L*-moments," *Journal of Statistical Planning and Inference*, vol. 137, no. 9, pp. 3024–3039, 2007.
[6] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.
[7] S. S. Wang, "Insurance pricing and increased limits ratemaking by proportional hazards transforms," *Insurance: Mathematics & Economics*, vol. 17, no. 1, pp. 43–54, 1995.
[8] S. Wang, "An actuarial index of the right-tail risk," *North American Actuarial Journal*, vol. 2, no. 2, pp. 88–101, 1998.
[9] B. L. Jones and R. Zitikis, "Empirical estimation of risk measures and related quantities," *North American Actuarial Journal*, vol. 7, no. 4, pp. 44–54, 2003.
[10] B. L. Jones and R. Zitikis, "Testing for the order of risk measures: an application of *L*-statistics in actuarial science," *Metron*, vol. 63, pp. 193–211, 2005.
[11] B. L. Jones and R. Zitikis, "Risk measures, distortion parameters, and their empirical estimation," *Insurance: Mathematics and Economics*, vol. 41, no. 2, pp. 279–297, 2007.
[12] V. Brazauskas, B. L. Jones, M. L. Puri, and R. Zitikis, "Nested *L*-statistics and their use in comparing the riskiness of portfolios," *Scandinavian Actuarial Journal*, vol. 2007, pp. 162–179, 2007.
[13] V. Brazauskas, B. L. Jones, and R. Zitikis, "Robust fitting of claim severity distributions and the method of trimmed moments," *Journal of Statistical Planning and Inference*, vol. 139, no. 6, pp. 2028–2043, 2009.
[14] F. Greselin, M. L. Puri, and R. Zitikis, "*L*-functions, processes, and statistics in measuring economic inequality and actuarial risks," *Statistics and Its Interface*, vol. 2, pp. 227–245, 2009.
[15] S. S. Wang, "Ordering of risks under PH-transforms," *Insurance: Mathematics and Economics*, vol. 18, no. 2, pp. 109–114, 1996.
[16] S. S. Wang, "A class of distortion operators for pricing financial and insurance risks," *Journal of Risk and Insurance*, vol. 67, no. 1, pp. 15–36, 2000.
[17] A. Necir, D. Meraghni, and F. Meddi, "Statistical estimate of the proportional hazard premium of loss," *Scandinavian Actuarial Journal*, vol. 3, pp. 147–161, 2007.
[18] A. Necir and D. Meraghni, "Empirical estimation of the proportional hazard premium for heavy-tailed claim amounts," *Insurance: Mathematics and Economics*, vol. 45, no. 1, pp. 49–58, 2009.
[19] G. R. Shorack and J. A. Wellner, *Empirical Processes with Applications to Statistics*, John Wiley & Sons, New York, NY, USA, 1986.
[20] P. J. Bickel, "Some contributions to the theory of order statistics," in *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, pp. 575–591, University of California Press, Berkeley, Calif, USA, 1967.
[21] G. R. Shorack, "Asymptotic Normality of Linear Combinations of Function of Order Statistics," *Annals of Mathematical Statistics*, vol. 40, pp. 2041–2050, 1969.
[22] G. R. Shorack, "Functions of order statistics," *Annals of Mathematical Statistics*, vol. 43, pp. 412–427, 1972.

[23] S. M. Stigler, "Functions of order statistics," *Annals of Mathematical Statistics*, vol. 43, pp. 412–427, 1969.
[24] S. M. Stigler, "Linear functions of order statistics with smooth weight functions," *Annals of Statistics*, vol. 2, pp. 676–693, 1974.
[25] F. H. Ruymgaart and M. C. A. Van Zuijlen, "Asymptotic normality of linear combinations of functions of order statistics in the non-i.i.d. case," *Indagationes Mathematicae*, vol. 39, no. 5, pp. 432–447, 1977.
[26] P. K. Sen, "An invariance principle for linear combinations of order statistics," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 42, no. 4, pp. 327–340, 1978.
[27] D. D. Boos, "A differential for L-statistics," *Annals of Statistics*, vol. 7, no. 5, pp. 955–959, 1979.
[28] D. M. Mason, "Asymptotic normality of linear combinations of order statistics with a smooth score function," *Annals of Statistics*, vol. 9, pp. 899–908, 1981.
[29] K. Singh, "On asymptotic representation and approximation to normality of L-statistics. I," *Sankhya Series A*, vol. 43, no. 1, pp. 67–83, 1981.
[30] L. Peng, "Estimating the mean of a heavy tailed distribution," *Statistics and Probability Letters*, vol. 52, no. 3, pp. 255–264, 2001.
[31] L. Peng, "Empirical-likelihood-based confidence interval for the mean with a heavy-tailed distribution," *Annals of Statistics*, vol. 32, no. 3, pp. 1192–1214, 2004.
[32] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall, New York, NY, USA, 1994.
[33] P. Lévy, *Calcul des Probabilités*, Gauthier-Villars, Paris, France, 1925.
[34] E. F. Fama, "The Behavior of stock market prices," *Journal of Business*, vol. 38, pp. 34–105, 1965.
[35] R. Weron, "Levy-stable distributions revisited: tail index $>$ 2 does not exclude the Levy-stable regime," *International Journal of Modern Physics C*, vol. 12, no. 2, pp. 209–223, 2001.
[36] M. Csörgő, S. Csörgő, L. Horváth, and D. M. Mason, "Normal and stable convergence of integral functions of the empirical distribution function," *Annals of Probability*, vol. 14, pp. 86–118, 1986.
[37] L. de Haan and U. Stadtmüller, "Generalized regular variation of second order," *Journal of the Australian Mathematical Society*, vol. 61, no. 3, pp. 381–395, 1996.
[38] I. Weissman, "Estimation of parameters and large quantiles based on the k largest observations," *Journal of American Statistical Association*, vol. 73, pp. 812–815, 1978.
[39] A. L. M. Dekkers and L. de Haan, "On the estimation of the extreme-value index and large quantile estimation," *Annals of Statistics*, vol. 17, pp. 1795–1832, 1989.
[40] G. Matthys and J. Beirlant, "Estimating the extreme value index and high quantiles with exponential regression models," *Statistica Sinica*, vol. 13, no. 3, pp. 853–880, 2003.
[41] M. I. Gomes, F. Figueiredo, and S. Mendonça, "Asymptotically best linear unbiased tail estimators under a second-order regular variation condition," *Journal of Statistical Planning and Inference*, vol. 134, pp. 409–433, 2005.
[42] B. M. Hill, "A simple approach to inference about the tail of a distribution," *Annals of Statistics*, vol. 3, pp. 1136–1174, 1975.
[43] D. M. Mason, "Laws of the large numbers for sums of extreme values," *Annals of Probability*, vol. 10, pp. 754–764, 1982.
[44] P. Deheuvels, E. Haeusler, and D. M. Mason, "Almost sure convergence of the Hill estimator," *Mathematical Proceedings of the Cambridge Philosophy Society*, vol. 104, pp. 371–381, 1988.
[45] A. Necir, "A functional law of the iterated logarithm for kernel-type estimators of the tail index," *Journal of Statistical Planning and Inference*, vol. 136, no. 3, pp. 780–802, 2006.
[46] M. Csörgő and D. M. Mason, "Central limit theorems for sums of extreme values," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 98, pp. 547–558, 1985.
[47] J. Beirlant and J. Teugels, "Asymptotic normality of Hill's estimator," in *Extreme Value Theory (Oberwolfach, 1987)*, vol. 51 of *Lecture Notes in Statistics*, pp. 148–155, Springer, New York, NY, USA, 1989.
[48] A. L. M. Dekkers, J. H. J. Einmahl, and L. de Haan, "A moment estimator for the index of an extreme-value distribution," *Annals of Statistics*, vol. 17, pp. 1833–1855, 1989.
[49] A. L. M. Dekkers and L. Dehaan, "Optimal choice of sample fraction in extreme-value estimation," *Journal of Multivariate Analysis*, vol. 47, no. 2, pp. 173–195, 1993.
[50] H. Drees and E. Kaufmann, "Selecting the optimal sample fraction in univariate extreme value estimation," *Stochastic Processes and Their Applications*, vol. 75, no. 2, pp. 149–172, 1998.
[51] J. Danielsson, L. de Haan, L. Peng, and C. G. de Vries, "Using a bootstrap method to choose the sample fraction in tail estimation," *Journal of Multivariate Analysis*, vol. 76, no. 2, pp. 226–248, 2001.
[52] S. Cheng and L. Peng, "Confidence intervals for the tail index," *Bernoulli*, vol. 7, no. 5, pp. 751–760, 2001.

[53] C. Neves and M. I. F. Alves, “Reiss and Thomas’ automatic selection of the number of extremes,” *Computational Statistics and Data Analysis*, vol. 47, no. 4, pp. 689–704, 2004.

[54] R. Ihaka and R. Gentleman, “R: a language for data analysis and graphics,” *Journal of Computational and Graphical Statistics*, vol. 5, no. 3, pp. 299–314, 1996.

[55] D. M. Mason and G. R. Shorack, “Necessary and sufficient conditions for asymptotic normality of trimmed L-statistics,” *Annals of Probability*, vol. 20, pp. 1779–1804, 1992.

[56] S. Csörgő, L. Horváth, and D. M. Mason, “What portion of the sample makes a partial sum asymptotically stable or normal?” *Probability Theory and Related Fields*, vol. 72, no. 1, pp. 1–16, 1986.

[57] L. de Haan and T. T. Pereira, “Estimating the index of a stable distribution,” *Statistics and Probability Letters*, vol. 41, no. 1, pp. 39–55, 1999.

[58] R. A. Groeneveld, “A class of quantile measures for kurtosis,” *American Statistician*, vol. 52, no. 4, pp. 325–329, 1998.

[59] E. Seneta, *Regularly Varying Functions*, vol. 508 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.

[60] L. de Haan, *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*, vol. 32 of *Mathematical Center Tracts*, Center for Mathematics and Computer Science, Amsterdam, The Netherlands, 1970.

[61] A. A. Balkema and L. de Haan, “Limit laws for order statistics,” in *Limit Theorems of Probability Theory (Colloquium, Keszthely, 1974)*, vol. 11 of *Colloquia Mathematica Societatis Janos Bolyai*, pp. 17–22, North-Holland, Amsterdam, The Netherlands, 1975.

[62] M. Csörgő and D. M. Mason, “On the asymptotic distribution of weighted uniform empirical and quantile processes in the middle and on the tails,” *Stochastic Processes and their Applications*, vol. 21, no. 1, pp. 119–132, 1985.