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Research Article

Viscosity Solutions and American Option Pricing in a Stochastic Volatility Model of the Ornstein-Uhlenbeck Type

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We study the valuation of American-type derivatives in the stochastic volatility model of Barndorff-Nielsen and Shephard (2001). We characterize the value of such derivatives as the unique viscosity solution of an integral-partial differential equation when the payoff function satisfies a Lipschitz condition.

1. Introduction

In their seminal paper, Barndorff-Nielsen and Shephard [1] introduced a model that has been shown to describe particularly well financial assets for which log-returns have heavy tail distributions and display long-range dependence. In this model, the volatility of the asset is described by an Ornstein-Uhlenbeck-type process with a pure jump Lévy process acting as the background driving process. An empirical study was made in [1] and showed from exchange rate data that suitable distributions for the Lévy process are the so-called generalized inverse gaussian distributions from which well-understood examples are the normal inverse gaussian (studied in [2]) and the gamma distribution.

The BNS model has been studied from different points of view. Benth et al. [3] considered the problem of optimal portfolio selection. Nicolato and Vernados [4] have studied European option pricing and described the set of equivalent martingale measures under this model. To evaluate these types of options, the authors propose the transform-based method and a simple Monte Carlo method.

In this paper, we consider the pricing of American options with the use of integral-partial differential equations (IPDEs). Although our technique can be simplified and used for European options and certain path-dependent options such as barrier options (see [5] for a definition and examples), we will mainly concentrate on American type derivatives which have not been studied for this model. The main difficulty in this case is the lack of Lipschitz continuity of some of the coefficients of the IPDE.

The question of whether observed option prices can be calibrated and shown to reproduce stylized features such as smiles using this model is of great practical and theoretical relevance. However the majority of exchange-traded options are of the American type and there do not exist any established numerical methods to compute option prices for this model. One could use Monte-Carlo methods; however it is generally more efficient to characterize the option value function as a solution of a variational inequality and to discretize this inequality in order to get an approximation of the value function. Whereas the existence of a solution to the IPDE suggests the use of finite difference schemes, the uniqueness of the solution is a particularly crucial property which insures the convergence of any such numerical scheme to the correct value function. The characterization of the value function as the unique solution of an IPDE is thus the first step to achieve this goal. The design and implementation of a numerical scheme are beyond the scope of this paper; however we refer the reader to the paper of Levendorskiĭ et al. [6] and references therein for some ideas on how this problem could be approached.

The connection between viscosity solutions of IPDEs and Lévy processes has been studied in the literature by various authors. Pham [7] considered a general stopping time problem of a controlled jump diffusion processes. However, his results do not apply here because the Lipschitz condition on the coefficients is not satisfied in our current setting. Cont and Voltchkova [5] studied barrier options and Barles et al. [8] established the connection between viscosity solutions and backward stochastic differential equations. In these papers, the stock price considered is modeled by a stochastic differential equation with jumps driven by a Lévy process. The main difference between the BNS model and these models is the presence of stochastic volatility. However, we will see that the lack of smoothness of the solution to our IPDE will also lead us to consider the notion of viscosity solutions as presented in [9].

The rest of the paper is organized as follows. In Section 2, we present the model and recall the results of Nicolato and Vernados [4] regarding the set of equivalent martingale measures. Section 3 is devoted to the continuity of the value function. In Section 4, we prove that the value function is the viscosity solution of the associated IPDE, and the uniqueness of the solution is presented in Section 5.

2. Lévy Processes and the BNS Model

Let $T > 0$. We consider the stochastic volatility model of Barndorff-Nielsen and Shephard [1] for the price process of an asset, denoted by $S = \{S_t\}_{0\le t\le T}$ and defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$. We thus assume that the log-return $X_t = \log(S_t)$ of the asset satisfies the following stochastic differential equation:

$$dX_t = (\mu + \beta V_t)dt + \sqrt{V_t}dB_t + \rho dZ_{\lambda t} \tag{2.1}$$

with

$$dV_t = -\lambda V_t\, dt + dZ_{\lambda t} \tag{2.2}$$

in which $\mu, \beta \in \mathbb{R}$, $\lambda > 0$ and $\rho \leq 0$. $B = \{B_t\}_{0\leq t\leq T}$ is a Brownian motion, and $Z = \{Z_t\}_{0\leq t\leq T}$ is the background driving Lévy process (BDLP) under the physical measure $\mathbb{P}$. In this model, Z has no gaussian component and the increments are positive. Z and B are assumed to be independent, and $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ is the usual filtration generated by the pair (B, Z). The positivity of the jumps of Z insure that the process V is always positive. We denote by W the Lévy measure of Z.

Suppose that $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ under which S is a martingale. We are interested in American-type derivatives of the form

$$U_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_T, \tau\geq t} \mathbf{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)} h(X_\tau) \mid \mathcal{F}_t\right] \tag{2.3}$$

in which h is the payoff function, and $\mathcal{T}_T$ is the set of all stopping times with values less or equal to T. Since $\{X_t\}_{0\leq t\leq T}$ and $\{V_t\}_{0\leq t\leq T}$ are Markov processes, U_t can be written as a function of (x, v, t), say,

$$U_t = u(x, v, t) = \sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}_{\mathbb{Q}}\left(e^{-r\tau} h\left(X_\tau^{x,v}\right)\right) \tag{2.4}$$

in which $(X_t^{x,v})_{t\geq 0}$ is the process X for which $X_0 = x$ and $V_0 = v$. We also denote by $(V_t^v)_{t\geq 0}$ the process V starting at $V_0 = v$ and at $t = 0$.

2.1. Equivalent Martingale Measures

We start by summarizing the results of Nicolato and Vernados [4] concerning the set of equivalent martingale measures. In order to do so, we define the set

$$\mathcal{Y}' = \left\{y : [0, \infty) \longrightarrow [0, \infty); \int_0^\infty \left(\sqrt{y(x)} - 1\right)^2 w(x)dx < \infty\right\} \tag{2.5}$$

and $\mathcal{M}'$ as the set of all equivalent martingale measures $\mathbb{Q}$ such that Z is still a Lévy process under $\mathbb{Q}$ independent of B, possibly with a different marginal distribution.

As in [4], we impose the following conditions on the process Z:

(C1) the process Z is given by the characteristic triplet $(0, 0, W)$ so that the cumulant transform is given by

$$\kappa(\theta) = \log\{\mathbf{E}[\exp(\theta Z_1)]\} = \int_0^\infty \left(e^{\theta z} - 1\right) W(dz), \tag{2.6}$$

for values of θ, for which this expression is defined;

(C2) $\widehat{\theta} = \sup\{\theta \in \mathbb{R} \mid \kappa(\theta) < \infty\} > 0$;

(C3) $\lim_{\theta\to\widehat{\theta}} \kappa(\theta) = \infty$.

Remark 2.1. Assumption (C2) implies that there exists $\widehat{\theta}_0 > 0$ such that

$$\int_0^\infty \left(e^{\widehat{\theta}_0 z} - 1\right) W(dz) < \infty. \tag{2.7}$$

For $z > 0$ and $n \geq 1$, we have $0 < z^n \leq (n!/\widehat{\theta}_0^n)(e^{\widehat{\theta}_0 z} - 1)$, so that $\mu_n := \int_0^\infty z^n W(dz) < \infty$. Furthermore, Assumption (C2) is a sufficient condition for the process Z to have finite moments of all orders.

The following theorem was proved in [4].

Theorem 2.2. *For all $\mathbb{Q} \in \mathcal{M}'$, there exists $y \in \mathcal{Y}'$ such that*

$$dX_t = \left(r - \lambda\kappa^y(\rho) - \frac{1}{2}V_t\right)dt + \sqrt{V_t}dB_t^{\mathbb{Q}} + \rho dZ_{\lambda t}, \tag{2.8}$$

in which

$$\kappa^y(\theta) = \int_0^\infty \left(e^{\theta x} - 1\right) y(x) w(x) dx, \tag{2.9}$$

and $B_t^{\mathbb{Q}} = B_t - \int_0^t (\sqrt{V_s})^{-1}(r - \mu - (\beta + 1/2)V_s - \lambda\kappa^y(\rho))ds$ and $Z_{\lambda t}$ are, respectively, a Brownian motion and a Lévy process under $\mathbb{Q}$. $w^y(x) = y(x)w(x)$ is the Lévy density of Z_1 under $\mathbb{Q}$ and $\kappa^y(\theta)$ is the cumulant function.

In the remaining part of this paper, all expectations will be with respect to a chosen EMM $\mathbb{Q}$, unless specified otherwise, and W and B will denote the associated Lévy measure and the Brownian motion associated to $\mathbb{Q}$.

Let $\mathcal{O} = \mathbb{R} \times \mathbb{R}_+ \times [0, T)$ and assume for a moment that u is Lipschitz in (x, v) and

$$u \in C^{2,1,1}(\mathcal{O}), \tag{2.10}$$

that is u is differentiable with respect to v and t, and twice differentiable with respect to x. We can then apply Itô's formula to U to find

$$dU_t = \left(\frac{\partial u}{\partial t} + \mathcal{L}[u]\right)dt + \frac{\partial u}{\partial x}\sqrt{V_t}dB_t + d\mathcal{V}_t, \tag{2.11}$$

in which

$$\begin{aligned}\mathcal{L}[u] = &\left(r - \frac{1}{2}v - \lambda\kappa^y(\rho) + \lambda\rho\mu_1\right)\frac{\partial u}{\partial x} - \lambda(v - \mu_1)\frac{\partial u}{\partial v} + \frac{1}{2}v\frac{\partial^2 u}{\partial x^2} \\ &+ \lambda\int_0^\infty \left(u(x + \rho z, v + z, t) - u(x, v, t) - \left(\rho z\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial v}\right)\right)W(dz),\end{aligned} \tag{2.12}$$

and $\mathcal{V}_t$ is the $\mathbb{Q}$-martingale given by

$$\begin{aligned} d\mathcal{V}_t = \int_0^\infty \Bigg(& u(X_{t-} + \rho z, V_{t-} + z, t) - u(X_{t-}, V_{t-}, t) \\ & - \left(\rho z \frac{\partial u}{\partial x}(X_{t-}, V_{t-}, t) + z \frac{\partial u}{\partial v}(X_{t-}, V_{t-}, t)\right)\Bigg) \widetilde{N}(dz, \lambda dt) \\ & + \int_0^\infty \left(\rho z \frac{\partial u}{\partial x}(X_{t-}, V_{t-}, t) + z \frac{\partial u}{\partial v}(X_{t-}, V_{t-}, t)\right) \widetilde{N}(dz, \lambda dt) \end{aligned} \tag{2.13}$$

in which $\widetilde{N}(dz, dt) = N(dz, dt) - W(dz)dt$, and $N(dz, dt)$ is the random measure of the process Z. Since $\int_0^t (\partial u/\partial x)\sqrt{V_t} dB_t$ is a $\mathbb{Q}$-martingale, if it can be shown that $e^{-rt}U_t$ is also a martingale, we can then expect u to satisfy the following integral-partial differential equation (IPDE):

$$\frac{\partial u}{\partial t}(x, v, t) + \mathcal{L}[u](x, v, t) - ru(x, v, t) = 0 \tag{2.14}$$

if $u(x, v, t) > h(x)$. Otherwise $u(x, v, t) = h(x)$ and this IPDE can be written as

$$\max\left(\frac{\partial u}{\partial t}(x, v, t) + \mathcal{L}[u](x, v, t) - ru(x, v, t), h(x) - u(x, v, t)\right) = 0. \tag{2.15}$$

It is clear also that the function satisfies

$$u(x, v, t) = h(x) \quad \text{for } v = 0 \text{ or } t = T. \tag{2.16}$$

Condition (2.10) is in fact very restrictive and most of the time not satisfied. Despite this problem, we will see that u can still be regarded as a solution of this equation in a weaker sense.

3. Continuity of the Value Function

Recall the definition of the value of an American option with payoff h:

$$u(x, v, t) = \sup_{\tau \in \mathcal{T}_{T-t}} \mathbf{E}\left(e^{-r\tau} h\left(X_\tau^{x,v}\right)\right). \tag{3.1}$$

In the rest of this paper, we will assume that h is positive and satisfies the Lipschitz condition, in other words $\exists K > 0$ such that for all $(x_1, x_2) \in \mathbb{R}^2$

$$|h(x_1) - h(x_2)| \leq K|x_1 - x_2|. \tag{3.2}$$

For instance, the payoff function for an American put with strike $\widetilde{X} > 0$ is $h(x) = \max(\widetilde{X} - \exp(x), 0)$ and satisfies this condition.

Our goal is to show that the function u satisfies the IPDE (2.15) in some weak sense. In order to give meaning to this IPDE for a function u that does not satisfy basic differentiability conditions, we introduce the idea of viscosity solutions following Crandall and Lions [10]. Let $\mathcal{W}$ be the set of functions $f : \overline{\mathcal{O}} \to \mathbb{R}$ that satisfy

$$\sup_{(x,v),(x',v')\in\mathbb{R}\times\mathbb{R}_+} \frac{\left|f(x,v,t) - f(x',v',t)\right|}{1 + |x - x'| + |v - v'|} < \infty \quad \forall t \in [0,T]. \tag{3.3}$$

Definition 3.1. The function $u \in C^0(\overline{\mathcal{O}}) \cap \mathcal{W}$ is a viscosity subsolution (supersolution) of (2.15)-(2.16) if for all $(x,v,t) \in \mathcal{O}$ and for all $\psi \in \mathcal{W} \cap C^{2,1,1}(\mathcal{O})$ such that

(i) $\psi(x,v,t) = u(x,v,t)$ and

(ii) for all $(x',v',t') \in \mathcal{O}$ $\psi(x',v',t') \geq u(x',v',t')$ $(\leq)$,

$$\max\left(\frac{\partial\psi(x,v,t)}{\partial t} + \mathcal{L}[\psi](x,v,t) - r\psi(x,v,t); h(x) - u(x,v,t)\right) \geq 0 \quad (\leq), \tag{3.4}$$

$$u(x,v,t) = h(x) \quad \text{for } v = 0 \text{ or } t = T. \tag{3.5}$$

The function u is a viscosity solution if it is both subsolution and supersolution.

Remark 3.2. As noted in [5, page 317] the condition $\psi \in \mathcal{W}$ is sufficient to have a well-defined integral term in $\mathcal{L}[\psi]$. In fact if $\psi \in \mathcal{W} \cap C^{2,1,1}$, then

$$\begin{aligned} &\int_0^\infty \left(\psi(x+\rho z, v+z, t) - \psi(x,v,t) - \left(\rho z\frac{\partial\psi}{\partial x}(x,v,t) + z\frac{\partial\psi}{\partial v}(x,v,t)\right)\right)W(dz) \\ &\quad \leq \int_{z<\eta} Cz^2 W(dz) + \int_\eta^\infty C(1+|z|)W(dz) < \infty \end{aligned} \tag{3.6}$$

for any $\eta > 0$.

An important property of viscosity solutions is the continuity of the function. It is the content of the following proposition.

Proposition 3.3. *When h satisfies the Lipschitz condition* (3.2)*, the function u is continuous and in $\mathcal{W}$.*

Proof. In this proof, we will assume for simplicity that $r = 0$. The generalization to $r > 0$ is straightforward. Throughout, C is a positive constant that can change from line to line.

We start by showing the continuity of u with respect to (x,v), uniformly in t. We have the following representation of the volatility process:

$$V_t^v = ve^{-\lambda t} + \int_0^t e^{-\lambda s} dZ_{\lambda s}. \tag{3.7}$$

We define the integrated variance process started with $V_0 = v$ by

$$V_t^{v,*} = \int_0^t V_s^v \, ds. \tag{3.8}$$

By (2.2), we find that

$$V_t^v dt = \frac{1}{\lambda}\left(-dV_t^v + dZ_{\lambda t}\right), \tag{3.9}$$

so that we have the following representation of the integrated variance process

$$V_t^{v,*} = \frac{1}{\lambda}\left(v - V_t^v\right) + \frac{1}{\lambda}\int_0^t dZ_{\lambda s} \tag{3.10}$$

$$= v\varepsilon(t) + \int_0^t \varepsilon(s) dZ_{\lambda s}, \tag{3.11}$$

in which $\varepsilon(t) = (1 - e^{-\lambda t})/\lambda$.

We also have the following identities:

$$\begin{aligned}
X_t^{x,v} &= x - \lambda\kappa(\rho)t - \frac{1}{2}V_t^{v,*} + \int_0^t \sqrt{V_s^v}\, dB_s + \rho Z_{\lambda t}, \\
\Delta V_t &:= V_t^{v'} - V_t^v = \Delta v e^{-\lambda t}, \\
\Delta V_t^* &:= V_t^{v',*} - V_t^{v,*} = \Delta v \varepsilon(t), \\
\Delta X_t &:= X_t^{x',v'} - X_t^{x,v} = x' - x - \frac{1}{2}\Delta v\varepsilon(t) + \int_0^t \left(\sqrt{V_s^{v'}} - \sqrt{V_s^v}\right) dB_s \\
&:= \Delta x - \frac{1}{2}\Delta v \varepsilon(t) + M_t^{v,v'},
\end{aligned} \tag{3.12}$$

with $\Delta x = x' - x$ and $\Delta v = v' - v$.

Using the Lipschitz condition on h, we obtain

$$\begin{aligned}
\left|u(x',v',t) - u(x,v,t)\right| &= \left|\sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}\, h\left(X_\tau^{x',v'}\right) - \sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}\, h(X_\tau^{x,v})\right| \\
&\le \sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}\left|h\left(X_\tau^{x',v'}\right) - h(X_\tau^{x,v})\right| \\
&\le C \sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}\left|X_\tau^{x',v'} - X_\tau^{x,v}\right|.
\end{aligned} \tag{3.13}$$

Then,

$$\left|u(x',v',t)-u(x,v,t)\right| \le C\left(|\Delta x|+|\Delta v|+\sup_{\tau\in\mathcal{T}_{T-t}}\mathbf{E}\left|M_\tau^{v,v'}\right|\right). \tag{3.14}$$

Letting $\mathcal{G}=\sigma(\{Z_s\}_{0\le s\le T})$, the σ-field generated by the BDLP Z up to the maturity T, we find that $\{M_t^{v,v'}\}_{t\ge 0}$ is a $\mathcal{G}\vee\mathcal{F}_t$-martingale. Thus, $\{|M_t^{v,v'}|\}_{t\ge 0}$ is a $\mathcal{G}\vee\mathcal{F}_t$-submartingale and Doob's theorem applies. In other words,

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_{T-t}}\mathbf{E}\left|M_\tau^{v,v'}\right| &\le \mathbf{E}\left(\sup_{\tau\in\mathcal{T}_{T-t}}\mathbf{E}\left(\left|M_\tau^{v,v'}\right|\mid\mathcal{G}\right)\right)\\ &\le \mathbf{E}\left(\mathbf{E}\left(\left|M_{T-t}^{v,v'}\right|\mid\mathcal{G}\right)\right)\le\sqrt{\mathbf{E}\left(\mathbf{E}\left(\left(M_{T-t}^{v,v'}\right)^2\mid\mathcal{G}\right)\right)}.\end{aligned} \tag{3.15}$$

Also,

$$\begin{aligned}\mathbf{E}\left(\left(M_{T-t}^{v,v'}\right)^2\mid\mathcal{G}\right) &= \int_0^{T-t}\left(V_s^{v'}-2\sqrt{V_s^{v'}V_s^{v}}+V_s^{v}\right)ds\\ &= \int_0^{T-t}\Delta v e^{-\lambda s}ds+2\int_0^{T-t}V_s^v-\sqrt{(V_s^v)^2+V_s^v\Delta v e^{-\lambda s}}ds\\ &= \int_0^{T-t}\Delta v e^{-\lambda s}ds+2\int_0^{T-t}\frac{-V_s^v\Delta v e^{-\lambda s}}{V_s^v+\sqrt{(V_s^v)^2+V_s^v\Delta v e^{-\lambda s}}}ds\\ &\le \int_0^{T-t}3|\Delta v|e^{-\lambda s}ds = 3|\Delta v|\varepsilon(T-t)\le 3|\Delta v|T.\end{aligned} \tag{3.16}$$

And thus we proved the continuity of u in (x,v) uniformly in t since

$$\left|u(x',v',t)-u(x,v,t)\right|\le C\left(|\Delta x|+|\Delta v|+\sqrt{|\Delta v|}\right). \tag{3.17}$$

In particular $u\in\mathcal{W}$ because of the following inequality:

$$\left|u(x',v',t)-u(x,v,t)\right|\le C\left(|\Delta x|+|\Delta v|+\sqrt{|\Delta v|}\right)\le 2C(1+|\Delta x|+|\Delta v|). \tag{3.18}$$

The next step of the proof is to show that

$$\mathbf{E}\sup_{t\le s\le t'}\left|X_s^{x,v}-X_t^{x,v}\right|\longrightarrow 0,\qquad \mathbf{E}\sup_{t\le s\le t'}\left|V_s^{v}-V_t^{v}\right|\longrightarrow 0 \tag{3.19}$$

as $|t-t'| \to 0$. This is easily obtained by first observing that

$$\begin{aligned}
\mathbf{E}\sup_{t\le s\le t'}\left|X_s^{x,v}-X_t^{x,v}\right| &\le \frac{1}{2}\mathbf{E}\sup_{t\le s\le t'}\left|V_s^{v,*}-V_t^{v,*}\right| + \mathbf{E}\sup_{t\le s\le t'}\left|\int_t^s \sqrt{V_y^v}\,dB_y\right| + \rho\mathbf{E}\sup_{t\le s\le t'}|Z_{\lambda s}-Z_{\lambda t}| \\
&\le \frac{1}{2}v\left|\varepsilon(t')-\varepsilon(t)\right| + \mathbf{E}\left|\int_t^{t'} e^{-\lambda(t'-s)}dZ_{\lambda s}\right| \\
&\quad + C\sqrt{\mathbf{E}\left|V_{t'}^{v,*}-V_t^{v,*}\right|} + \rho\mathbf{E}|Z_{\lambda t'}-Z_{\lambda t}| \\
&\le \frac{1}{2}v\left|\varepsilon(t')-\varepsilon(t)\right| + (1+\rho)\mathbf{E}|Z_{\lambda t'}-Z_{\lambda t}| \\
&\quad + \sqrt{\frac{1}{2}v|\varepsilon(t')-\varepsilon(t)| + \mathbf{E}|Z_{\lambda t'}-Z_{\lambda t}|}.
\end{aligned} \tag{3.20}$$

As for the process V,

$$\begin{aligned}
\mathbf{E}\sup_{t\le s\le t'}\left|V_s^v-V_t^v\right| &\le \left|1-e^{-\lambda(t'-t)}\right|\mathbf{E}\left|V_t^v\right| + \mathbf{E}\left|\int_t^{t'} e^{-\lambda(t'-s)}dZ_{\lambda s}\right| \\
&\le \left|1-e^{-\lambda(t'-t)}\right|\mathbf{E}\left|V_t^v\right| + \mathbf{E}|Z_{\lambda t'}-Z_{\lambda t}|.
\end{aligned} \tag{3.21}$$

Since $V_t^v \le v + Z_{\lambda T}$ for all $t \le T$,

$$\mathbf{E}\sup_{t\le s\le t'}\left|V_s^v-V_t^v\right| \le C(v+\ \mathbf{E}\,Z_{\lambda T})\left|t'-t\right| + \mathbf{E}|Z_{\lambda t'}-Z_{\lambda t}|, \tag{3.22}$$

and we need to show that $\mathbf{E}|Z_{\lambda t'}-Z_{\lambda t}| \to 0$ when $|t'-t| \to 0$.

We mentioned earlier that condition (C2) implies that the moments of Z_t are finite for all orders. Thus Z is uniformly integrable. Since Z is also continuous in probability, it is continuous in $\mathcal{L}_1$, and the conclusion follows.

Let us now show continuity with respect to time. Let $0 \le t \le t' \le T$. Take $\tau \in \mathfrak{T}_{T-t}$ and define $\tau' = \tau \wedge (T-t')$. Then,

$$\begin{aligned}
\mathbf{E}\left(e^{-r\tau}h\left(X_\tau^{x,v}\right)\right) &= \mathbf{E}\left(e^{-r\tau'}h\left(X_{\tau'}^{x,v}\right)\right) + \mathbf{E}\left(e^{-r\tau}h\left(X_\tau^{x,v}\right) - e^{-r\tau'}h\left(X_{\tau'}^{x,v}\right)\right) \\
&\le u(x,v,t') + \mathbf{E}\left(e^{-r\tau}h\left(X_\tau^{x,v}\right) - e^{-r\tau'}h\left(X_{\tau'}^{x,v}\right)\right).
\end{aligned} \tag{3.23}$$

From this inequality, we readily find that

$$\left|u(x,v,t')-u(x,v,t)\right| \leq \mathbf{E} \sup_{T-t'\leq s\leq T-t} \left|X_s^{x,v}-X_{T-t'}^{x,v}\right| \tag{3.24}$$

which converges to zero as $|t-t'| \to 0$.

Global continuity follows from the following inequality:

$$\left|u(x',v',t')-u(x,v,t)\right| \leq \left|u(x',v',t')-u(x,v,t')\right| + \left|u(x,v,t')-u(x,v,t)\right| \tag{3.25}$$

and the fact that the first bound is independent of t'. □

4. Viscosity Solutions

This section is devoted to the viscosity solution property of the value function u. In order to prove that u is a viscosity solution of (2.15), we need the following dynamic programming principle. It is a consequence of the martingale property of the Snell envelope stopped before its optimal stopping time and it is the key property needed in the proof of the subsolution property.

Lemma 4.1. *Let $\epsilon > 0$, $(x,v,t) \in \mathcal{O}$, and define the stopping time*

$$\tau^{\epsilon} = \inf\left\{0 \leq s \leq T-t \mid e^{-rs}u\left(X_s^{x,v}, V_s^{v}, t+s\right) - \epsilon \leq e^{-rs}h\left(X_s^{x,v}\right)\right\}. \tag{4.1}$$

Then,

$$u(x,v,t) = \mathbf{E}\left[e^{-r\tau^{\epsilon}}u\left(X_{\tau^{\epsilon}}^{x,v}, V_{\tau^{\epsilon}}^{v}, t+\tau^{\epsilon}\right)\right]. \tag{4.2}$$

Proof. For some constant C, we have that

$$n\mathbf{E}\left(\left|\mathbf{1}_{\{h(X_\tau)\geq n\}}h(X_\tau)\right|\right) \leq \mathbf{E}\left(h(X_\tau)^2\right) \leq C + C\mathbf{E}\left(X_\tau^2\right) \tag{4.3}$$

for all $\tau \in \mathcal{T}_T$. We know that $X_\tau = X_0 + r\tau + V_\tau^* + \int_0^\tau \sqrt{V_s}dB_s + \rho Z_{\lambda\tau}$ and that $0 \leq V_\tau^* \leq V_T^* \leq (1/\lambda)(V_0+Z_{\lambda T})$ from (3.11). As a result, $X_\tau^2 \leq 4(X_0+rT)^2 + 4(1/\lambda^2)(V_0+Z_{\lambda T})^2 + 4(\int_0^\tau \sqrt{V_s}dB_s)^2 + 4\rho^2 Z_{\lambda T}^2 \leq C + CZ_{\lambda T}^2 + C(\int_0^\tau \sqrt{V_s}dB_s)^2$ for some constant C large enough. Hence $\mathbf{E}X_\tau^2 \leq C+C\mathbf{E}Z_{\lambda T}^2+C\mathbf{E}V_T^* < \infty$ for all $\tau \in \mathcal{T}_T$. As a consequence, $\sup_{\tau\in\mathcal{T}_T}\mathbf{E}(|\mathbf{1}_{\{h(X_\tau)\geq n\}}h(X_\tau)|)$ converges to 0 as n grows to infinity, that is, the collection $\{e^{-r\tau}h(X_\tau) : \tau \in \mathcal{T}_T\}$ is uniformly integrable. Hence we find that the process $(e^{-rs}h(X_s))_{0\leq s\leq T}$ is of Class D, and we can apply the results of [11] to get the result. □

The proof of the solution property of u makes use of the following lemma.

Lemma 4.2. *Let $t \leq T$ and $\epsilon > 0$. Suppose that $u(x,v,t) - h(x) > \epsilon$. Then $\mathbb{Q}(\tau^{\epsilon} < s) \to 0$ when $s \to 0$.*

Proof. Let $\eta > \epsilon$ such that $\eta < u(x,v,t) - h(x)$.

First we show that $e^{-r\tau^\epsilon} u(X^{x,v}_{\tau^\epsilon}, V^v_{\tau^\epsilon}, \tau^\epsilon) - e^{-r\tau^\epsilon} h(X^{x,v}_{\tau^\epsilon}) \leq \epsilon$ almost surely. For some sequence $s_n \downarrow \tau^\epsilon$, $e^{-rs_n} u(X^{x,v}_{s_n}, V^v_{s_n}, s_n) \leq e^{-rs_n} h(X^{x,v}_{s_n}) + \epsilon$ for n large enough. In this case, since $(X^{x,v}_{s_n}, V^v_{s_n})$ converges to $(X^{x,v}_{\tau^\epsilon}, V^v_{\tau^\epsilon})$ in $\mathcal{L}^1$, we can take a subsequence if necessary and find that $|u(X^{x,v}_{s_n}, V^v_{s_n}, s_n) - u(X^{x,v}_{\tau^\epsilon}, V^v_{\tau^\epsilon}, \tau_\epsilon)| \to 0$ and $h(X^{x,v}_{s_n}) \to h(X^{x,v}_{\tau^\epsilon})$ a.s. with $n \to \infty$. Taking the limit, we find that

$$\begin{aligned} e^{-r\tau_\epsilon} u\left(X^{x,v}_{\tau^\epsilon}, V^v_{\tau^\epsilon}, \tau^\epsilon\right) &= \lim_{n\to\infty} e^{-rs_n} u\left(X^{x,v}_{s_n}, V^v_{s_n}, s_n\right) \\ &\leq \lim_{n\to\infty} e^{-rs_n} h\left(X^{x,v}_{s_n}\right) + \epsilon \\ &= e^{-r\tau_\epsilon} h\left(X^{x,v}_{\tau^\epsilon}\right) + \epsilon \quad \text{a.s.} \end{aligned} \tag{4.4}$$

Since u is continuous with respect to t, we find that $\eta < e^{-rs} u(x,v,t+s) - e^{-rs} h(x)$ for s small enough. Then, for s small enough,

$$\begin{aligned} \mathbb{Q}(\tau^\epsilon < s) &\leq \mathbb{Q}\left(e^{-r\tau^\epsilon}\left(u(x,v,\tau^\epsilon) - h(x)\right) + e^{-r\tau^\epsilon}\left(h\left(X^{x,v}_{\tau^\epsilon}\right) - u\left(X^{x,v}_{\tau^\epsilon}, V^v_{\tau^\epsilon}, \tau^\epsilon\right)\right) > \eta - \epsilon\right) \\ &\leq \mathbb{Q}\left(e^{-r\tau^\epsilon}\left|u(x,v,\tau^\epsilon) - u\left(X^{x,v}_{\tau^\epsilon}, V^v_{\tau^\epsilon}, \tau^\epsilon\right)\right| + e^{-r\tau^\epsilon}\left|h\left(X^{x,v}_{\tau^\epsilon}\right) - h(x)\right| > \eta - \epsilon\right) \\ &\leq \mathbb{Q}\left(\left|V^v_{\tau^\epsilon} - v\right| > \delta_2\right) + \mathbb{Q}\left(\left|X^{x,v}_{\tau^\epsilon} - x\right| > \delta_3\right) \end{aligned} \tag{4.5}$$

for some constants $\delta_2 > 0$ and $\delta_3 > 0$. By the continuity in probability of the processes X and V, we know that this expression goes to zero when $s \to 0$. □

We can now show that u is a viscosity solution.

Theorem 4.3. *When h satisfies the Lipschitz condition* (3.2)*, u is a viscosity solution of IPDE* (2.15)*.*

Proof. We already know that u is continuous and in $\mathcal{W}$.

Let us start by showing that u is a supersolution of (2.15). Let $(x,v,t) \in \mathcal{O}$ and φ satisfy the conditions given in the above definition of supersolutions. By definition, for any $\Delta t > 0$,

$$\begin{aligned} 0 &\geq e^{-r\Delta t}\mathbf{E}\left(u\left(X^{x,v}_{\Delta t}, V^v_{\Delta t}, t+\Delta t\right)\right) - u(x,v,t) \\ &\geq \mathbf{E}\left(e^{-r\Delta t}\varphi\left(X^{x,v}_{\Delta t}, V^v_{\Delta t}, t+\Delta t\right) - \varphi\left(X^{x,v}_0, V^v_0, t\right)\right) \\ &= \mathbf{E}\left(\int_0^{\Delta t} e^{-rs}\left(-r\varphi + \frac{\partial\varphi}{\partial t} + \mathcal{L}[\varphi]\right)\left(X^{x,v}_s, V^v_s, t+s\right)ds \right. \\ &\qquad \left. + \int_0^{\Delta t} \frac{\partial\varphi}{\partial x}\left(X^{x,v}_s, V^v_s, t+s\right)e^{-rs}\sqrt{V^v_s}\,dB_s + \Psi^{x,v}_{\Delta t} - \Psi^{x,v}_0\right), \end{aligned} \tag{4.6}$$

in which $\Psi^{x,v}$ is the martingale defined by

$$d\Psi_s^{x,v} = e^{-rs}\int_0^\infty \Big(\psi\big(X_{s-}^{x,v}+\rho z, V_{s-}^v+z, t+s\big)-\psi\big(X_{s-}^{x,v}, V_{s-}^v, t+s\big) \\ -z\Big(\rho\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial v}\Big)\big(X_{s-}^{x,v}, V_{s-}^v, t+s\big)\Big)\widetilde{N}(dz,\lambda ds) \\ +e^{-rs}\int_0^\infty z\Big(\rho\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial v}\Big)\big(X_{s-}^{x,v}, V_{s-}^v, t+s\big)\widetilde{N}(dz,\lambda ds). \tag{4.7}$$

Since

$$\int_0^{\Delta t}\frac{\partial\psi}{\partial x}\big(X_s^{x,v}, V_s^v, t+s\big)e^{-rs}\sqrt{V_s^v}\,dB_s \tag{4.8}$$

is also a martingale, we have the following inequality:

$$0\ge\int_0^{\Delta t}\mathbf{E}\Big(e^{-rs}\Big(-r\psi+\frac{\partial\psi}{\partial t}+\mathcal{L}[\psi]\Big)\big(X_s^{x,v}, V_s^v, t+s\big)\Big)ds, \tag{4.9}$$

in other words, dividing by Δt and taking the limit as $\Delta t\to 0$,

$$0\ge -r\psi(x,v,t)+\frac{\partial\psi}{\partial t}(x,v,t)+\mathcal{L}[\psi](x,v,t). \tag{4.10}$$

Since, by definition, $u(x,v,t)\ge h(x)$, u satisfies (2.15). To prove that u is a viscosity subsolution of (2.15), let $(x,v,t)\in\mathcal{O}$ and ψ satisfy the conditions of the above definition for subsolutions. If $u(x,v,t)=h(x)$, inequality (3.4) is satisfied. Otherwise, let

$$0<\epsilon<u(x,v,t)-h(x). \tag{4.11}$$

We know from Lemma 4.1 that

$$0=\mathbf{E}\Big(e^{-r(\Delta t\wedge\tau^\epsilon)}u\big(X_{\Delta t\wedge\tau^\epsilon}^{x,v}, V_{\Delta t\wedge\tau^\epsilon}^v, t+(\Delta t\wedge\tau^\epsilon)\big)\Big)-u(x,v,t) \\ \le\mathbf{E}\Big(e^{-r(\Delta t\wedge\tau^\epsilon)}\psi\big(X_{\Delta t\wedge\tau^\epsilon}^{x,v}, V_{\Delta t\wedge\tau^\epsilon}^v, t+(\Delta t\wedge\tau^\epsilon)\big)\Big)-\psi(x,v,t) \\ =\mathbf{E}\Big(\int_0^{\Delta t\wedge\tau^\epsilon}e^{-rs}\Big(-r\psi+\frac{\partial\psi}{\partial t}+\mathcal{L}[\psi]\Big)\big(X_s^{x,v}, V_s^v, t+s\big)ds\Big) \tag{4.12}$$

for any $\Delta t>0$. Knowing that $\mathbb{Q}(\tau^\epsilon<\Delta t)\to 0$ when $\Delta t\to 0$ by Lemma 4.2, dividing the preceding inequality by Δt and taking the limit to 0, we get the desired result by Lebesgue's dominated convergence theorem. □

5. Comparison Principles and Uniqueness of the Solution

In this section, we prove a comparison result from which we obtain the uniqueness of the solution of the IPDE. In proving comparison results for viscosity solutions, the notion of parabolic superjet and subjet as defined by Crandall et al. [10] is particularly useful. Setting $y = (x, v)$, we define the parabolic superjet and its closure by

$$\begin{aligned}
\mathcal{P}^{2,+}u(y,t) = \Big\{ &(p,q,A) \in \mathbb{R}\times\mathbb{R}^2\times\mathcal{S}_2 \text{ such that } u(y',t') - u(y,t) \le p(t'-t) + q\cdot(y'-y) \\
&+\frac{1}{2}(y'-y)^T\cdot A\cdot(y'-y) + o\left(|t'-t| + |y'-y|^2\right) \text{ as } (t',y') \longrightarrow (t,y)\Big\}, \\
\overline{\mathcal{P}}^{2,+}u(y,t) = \Big\{ &(p,q,A) = \lim_{n\to\infty}(p_n,q_n,A_n) \text{ such that} \\
&(p_n,q_n,A_n) \in \mathcal{P}^{2,+}u(y_n,t_n) \text{ and } (y_n,t_n) \longrightarrow (y,t)\left(\lim_{n\to\infty}\right)\Big\},
\end{aligned} \tag{5.1}$$

The subjet and its closure are then defined similarly by

$$\begin{aligned}
\mathcal{P}^{2,-}u(y,t) &= -\mathcal{P}^{2,+}(-u)(y,t), \\
\overline{\mathcal{P}}^{2,-}u(y,t) &= -\overline{\mathcal{P}}^{2,+}(-u)(y,t).
\end{aligned} \tag{5.2}$$

We then have the following lemma which is essentially proved in [8] (Lemma 3.3).

Lemma 5.1. *If the function $u \in C^0(\mathbb{R}\times\mathbb{R}_+\times[0,T])$ is a viscosity subsolution (resp. supersolution) of* (2.15)*, then for all $(x,v,t) \in \mathbb{R}\times\mathbb{R}_+\times[0,T)$ and for all $(p,q,A) \in \overline{\mathcal{P}}^{2,+}u(x,v,t)$ (resp., $\overline{\mathcal{P}}^{2,-}u(x,v,t)$)*

$$\max\left(p + \mathcal{L}_\xi^{q,A}[u,\varphi](x,v,t) - ru(x,v,t);\ h(x) - u(x,v,t)\right) \ge 0 \quad (\le), \tag{5.3}$$

in which

$$\begin{aligned}
\mathcal{L}_\xi^{q,A}[u,\varphi](x,v,t) = &\left(r - \frac{1}{2}v - \lambda\kappa^y(\rho) + \lambda\rho\mu^\eta\right)q^{(1)} - \lambda(v-\mu^\eta)q^{(2)} + \frac{1}{2}vA_{11} \\
&+ \lambda\int_0^\xi\left(\varphi(x+\rho z, v+z,t) - \varphi(x,v,t) - \left(\rho z\frac{\partial\varphi}{\partial x} + z\frac{\partial\varphi}{\partial v}\right)\right)W(dz) \\
&+ \lambda\int_\xi^\infty\left(u(x+\rho z, v+z,t) - u(x,v,t) - \left(\rho z\frac{\partial\varphi}{\partial x} + z\frac{\partial\varphi}{\partial v}\right)\right)W(dz)
\end{aligned} \tag{5.4}$$

for some $\varphi \in C^{2,1,1}$ and $0 < \xi < 1$.

Pham [7] obtains the uniqueness of the solution when the coefficients of $\mathcal{L}$ satisfy Lipschitz conditions on $\mathbb{R}^2 \times [0,T]$. For $\delta > 0$, define $\mathcal{O}^\delta = \mathbb{R}\times(\delta,\infty)\times[0,T)$. Then, the coefficients of $\mathcal{L}$ satisfy the Lipschitz conditions on $\mathcal{O}^{\delta}$, and using the ideas of uniqueness proofs in the literature, we can show a comparison principle on $\mathcal{O}^{\delta}$. This result will then be used to show the uniqueness of the solution on $\mathcal{O}$.

Theorem 5.2. *Let $\epsilon \geq 0$, and let u_1 be a subsolution and u_2 a supersolution of* (2.15) *on $\mathcal{O}^{\delta}$ such that*

$$u_1(x,v,t) \leq u_2(x,v,t) + \epsilon \tag{5.5}$$

for $t = T$ or $v = \delta$. Then $u_1(x,v,t) \leq u_2(x,v,t) + \epsilon e^{r(T-t)}$ for all $(x,v,t) \in \mathcal{O}^{\delta}$.

Proof. An IPDE of the form $(\partial\varphi(x,\vartheta,t)/\partial t) + \mathcal{L}[\varphi](x,\vartheta,t) - r\varphi(x,\vartheta,t) = 0$ for $(x,\vartheta,t) \in \mathcal{O}$ and $\varphi(x,\vartheta,T) = h(x)$ was shown to have a unique solution in [8] when the coefficients of $\mathcal{L}$ satisfy some given Lipschitz conditions. In fact when (x,ϑ,t) and $(x',\vartheta',t') \in \mathcal{O}^{\delta}$, we have $|\sqrt{\vartheta'} - \sqrt{\vartheta}| \leq (1/2\delta)|\vartheta' - \vartheta|$, and so the operator $\mathcal{L}$ satisfies the assumptions made in [8]. The extension of the uniqueness result to our current setting is straightforward, and we only give the main details.

We first show that $u_1 - u_2$ is a subsolution of a related IPDE. Suppose that $\varphi \in \mathcal{W} \cap C^2$ and $u_1 - u_2 - \varphi$ attains a maximum at $(y_0, t_0) \in \mathcal{O}^{\delta}$. Set

$$\begin{aligned} w(y_1,y_2,t,s) &= u_1(y_1,t) - u_2(y_2,s), \\ \phi(y_1,y_2,t,s) &= \frac{1}{2\epsilon}|y_1 - y_2|^2 + \frac{1}{2\alpha}|t-s|^2 + \varphi(y_1,t). \end{aligned} \tag{5.6}$$

Since u_1 and u_2 are in $\mathcal{W}$, the function $w-\phi$ attains its maximum (y_1^*, y_2^*, t^*, s^*) (which depends on ϵ, α) in $\mathcal{O}^{\delta} \times \mathcal{O}^{\delta}$. By a classical argument in the theory of viscosity solutions, we can show that $(1/\epsilon)|y_1^* - y_2^*|^2, (1/\alpha)|t^* - s^*|^2 \longrightarrow 0$ when $\epsilon, \alpha \longrightarrow 0$ and

$$(y_1^*, y_2^*, t^*, s^*) \longrightarrow (y_0, y_0, t_0, t_0) \tag{5.7}$$

when $\epsilon, \alpha \longrightarrow 0$.

Applying Theorem 8.3 of Crandall et al. [9] to the functions w and ϕ, we find matrices Y_1, Y_2 such that

$$\begin{gathered} \left(a + \frac{\partial\varphi}{\partial t}(y_1^*,t^*), b + D\varphi(y_1^*,t^*), Y_1\right) \in \overline{\mathcal{P}}^{2,+} u_1(y_1^*,t^*) \\ (-a,-b,-Y_2) \in \overline{\mathcal{P}}^{2,+}(-u_1)(y_2^*,s^*), \end{gathered} \tag{5.8}$$

with a $= (1/\alpha)(t^* - s^*)$ and $b = (1/\epsilon)(y_1^* - y_2^*)$ and for $0 < \xi < 1$ the inequalities

$$\begin{gathered} \max\Bigg(a + \mathcal{L}_{\xi}^{b,Y_1}[u_1,\varphi](y_1^*,t^*) + \frac{\partial\varphi}{\partial t}(y_1^*,t^*) + v_1^*\frac{\partial^2\varphi}{\partial x^2}(y_1^*,t^*) \\ -ru_1(y_1^*,t^*); h(x_1^*) - u_1(y_1^*,t^*)\Bigg) \geq 0, \\ \max\Big(a + \mathcal{L}_{\xi}^{b,Y_2}[u_2,\varphi](y_2^*,s^*) - ru_2(y_2^*,s^*); h(x_2^*) - u_2(y_2^*,s^*)\Big) \leq 0 \end{gathered} \tag{5.9}$$

are satisfied. Write these two expressions as $\max(A,B)\geq 0$ and $\max(C,D)\leq 0$. Then $\max(A-C,B-D)\geq 0$. Now, $B-D=h(x_1^*)-u_1(y_1^*,t^*)-h(x_2^*)+u_2(y_2^*,s^*)$, and because h is Lipschitz $|h(x_1^*)-h(x_2^*)|\to 0$ when $\epsilon,\alpha\to 0$. Thus $B-D\to u_2(y_0,t_0)-u_1(y_0,t_0)$. On the other hand it was shown in [8] that

$$\begin{aligned}
A-C\leq{}& r\left(u_2\left(y_2^*,s^*\right)-u_1\left(y_1^*,t^*\right)\right)+\frac{1}{\epsilon}\left(\frac{1}{2}-\lambda+\frac{1}{4\delta}\right)\left|y_1^*-y_2^*\right|^2\\
&+\frac{\partial\psi}{\partial t}+\left(r'-\frac{1}{2}v_1^*,v_1^*\right)D\psi\left(y_1^*,t^*\right)+v_1^*\frac{\partial^2\psi}{\partial x^2}\left(y_1^*,t^*\right)\\
&+\lambda\int_0^\infty\left(\psi\left(x_1^*+\rho z,v_1^*+z,t^*\right)-\psi\left(y_1^*,t^*\right)-z(\rho,1)\cdot D\psi\right)W(dz)\\
&+\lambda\int_0^\xi\left(\phi\left(x_1^*+\rho z,v_1^*+z,x_2^*,v_2^*,t^*,s^*\right)-\phi\left(x_1^*,v_1^*,x_2^*,v_2^*,t^*,s^*\right)\right.\\
&\qquad\left.-z(\rho,1)\cdot\left(b+D\psi\left(y_1^*,t^*\right)\right)\right)W(dz)\\
&-\lambda\int_0^\xi\left(\phi\left(x_1^*,v_1^*,x_2^*+\rho z,v_2^*+z,t^*,s^*\right)-\phi\left(x_1^*,v_1^*,x_2^*,v_2^*,t^*,s^*\right)-z(\rho,1)\cdot b_2\right)W(dz),
\end{aligned}\tag{5.10}$$

in which $r'=(r-\lambda\kappa^y(\rho)+\lambda\mu_2)$. Using the fact that $\phi\in\mathcal{W}\cap C^2$ we find by letting $\xi\to 0$ and then $\epsilon,\alpha\to 0$ that

$$A-C\leq -r\left(u_1(y_0,t_0)-u_2(y_0,t_0)\right)+\frac{\partial\psi}{\partial t}+\mathcal{L}\psi.\tag{5.11}$$

Consequently,

$$\max\left(-r(u_1-u_2)(y_0,t_0)+\frac{\partial\psi}{\partial t}(y_0,t_0)+\mathcal{L}\psi(y_0,t_0),-(u_1-u_2)(y_0,t_0)\right)\geq 0.\tag{5.12}$$

As shown in [8] (see Lemma 3.8), there exists a function $\chi\geq 1$ such that

$$\frac{\partial\chi}{\partial t}+\mathcal{L}\chi-r\chi<0\tag{5.13}$$

for which the maximum

$$M=\sup_{\mathbb{R}\times\mathbb{R}_+\times[t_1,T]}\left((u_1-u_2)(y,t)-\beta\chi(y,t)\right)e^{r(t-T)}\tag{5.14}$$

is attained at some point (y_0,t_0). Then

$$(u_1-u_2-\beta\chi)(y,t)\leq(u_1-u_2-\beta\chi)(y_0,t_0)e^{r(t_0-t)}.\tag{5.15}$$

Let $\psi(y,t) = \beta\chi(y,t) - (u_1 - u_2 - \beta\chi)(y_0,t_0)e^{r(t_0-t)}$. Then ψ satisfies the properties in the subsolution definition, hence it satisfies (5.12). But

$$\left(\frac{\partial\psi}{\partial t} + \mathcal{L}\psi\right)(y_0,t_0) = \left(\beta\frac{\partial\chi}{\partial t} + r(u_1 - u_2 - \beta\chi) + \beta\mathcal{L}\chi\right)(y_0,t_0) < r(u_1 - u_2)(y_0,t_0). \tag{5.16}$$

Hence, either $(u_1 - u_2)(y_0,t_0) \leq 0$, or $v_0 = \delta$ or $t_0 = T$ and, in this case, $(u_1 - u_2)(y_0,t_0) \leq \epsilon$ by assumption. Hence, we conclude that

$$\begin{aligned}(u_1 - u_2)(y,t) &\leq \beta\chi(y,t) - \beta\chi(y_0,t_0)e^{r(t_0-t)} + (u_1 - u_2)(y_0,t_0)e^{r(t_0-t)}\\ &\leq \beta\chi(y,t) + \epsilon e^{r(t_0-t)}.\end{aligned} \tag{5.17}$$

Sending β to zero we get $u_1 \leq u_2 + \epsilon e^{r(T-t)}$ on $\mathbb{R} \times (\delta,\infty) \times [t_1,T]$. As done in [8], we can repeat this argument as many times as needed to get $u_1 \leq u_2 + \epsilon e^{r(T-t)}$ on $\mathcal{O}^\delta$. □

A solution of (2.15)-(2.16) is said to be minimal if it is less or equal to any other solution of (2.15)-(2.16). To prove uniqueness, we first show that the solution u is minimal.

Theorem 5.3. *u is the minimal viscosity solution of* (2.15)-(2.16).

Proof. Let $\delta > 0$ and define

$$u^\delta(x,v,t) = \sup_{\tau\in\mathcal{T}_{T-t},\tau\leq\tau_\delta} \mathbf{E}\left(e^{-r\tau}h\left(X_\tau^{x,v}\right)\right) \tag{5.18}$$

in which

$$\tau_\delta = \inf\{s \geq 0 : V_s^v \leq \delta\}. \tag{5.19}$$

Then u^δ is a viscosity solution of (2.15) on $\mathcal{O}^\delta$ with boundary conditions

$$u^\delta(x,v,t) = h(x) \quad \text{for } t = T \text{ or } v = \delta. \tag{5.20}$$

The proof of this statement is essentially the same as the proof for the viscosity solution property of u. The main difference is that the maturity T is replaced by τ_δ. Note that $V_s^{\delta'} > \delta$ for $\delta' > \delta e^{\lambda T}$, hence $u^\delta(x,v,t) = u(x,v,t)$ for all $x \in \mathbb{R}, t < T$ and $v > \delta e^{\lambda T}$.

Let $\tilde{u}$ be another viscosity solution of (2.15)-(2.16). Then $\tilde{u}$ is a viscosity solution of (2.15) on $\mathcal{O}^\delta$ with boundary values $\tilde{u}(x,v,t)$ for $t = T$ or $v = \delta$. Also, $\tilde{u}(x,v,t) \geq h(x) = u^\delta(x,v,t)$ for $t = T$ or $v = \delta$. By Theorem 5.2, we find that $\tilde{u} \geq u^\delta$ on $\mathcal{O}^\delta$. In particular, $\tilde{u}(x,v,t) \geq u(x,v,t)$ for $x \in \mathbb{R}, t < T$ and $v > \delta e^{\lambda T}$. Since δ is arbitrary, $\tilde{u} \geq u$ on $\mathcal{O}$. □

Following Pham [7], we denote by $UC_{x,v}(\mathcal{O})$ the set of functions defined on $\mathcal{O}$ uniformly continuous in (x,v), uniformly in t. We have already shown that the function u satisfies

$$\left|u(x',v',t) - u(x,v,t)\right| \leq C\left(\left|x' - x\right| + \left|v' - v\right| + \sqrt{|v' - v|}\right). \tag{5.21}$$

Hence, $u \in UC_{x,v}(\mathcal{O})$. Using the two previous theorems, we can show the uniqueness in $UC_{x,v}(\mathcal{O})$.

Theorem 5.4. *u is the unique viscosity solution of* (2.15)-(2.16) *in* $UC_{x,v}(\mathcal{O})$.

Proof. Let $\tilde{u} \in UC_{x,v}(\mathcal{O})$ be another viscosity solution of (2.15)-(2.16). Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 \le u(x,v,t) - u(x,0,t) = u(x,v,t) - h(x) < \epsilon$ and $0 \le \tilde{u}(x,v,t) - \tilde{u}(x,0,t) = \tilde{u}(x,v,t) - h(x) < \epsilon$ for $v \le \delta$. In particular, $|u(x,v,t) - \tilde{u}(x,v,t)| < \epsilon$ for all x, all t and $v \le \delta$. Furthermore, by Theorem 5.3, we obtain that $u(x,\delta,t) \le \tilde{u}(x,\delta,t) \le u(x,\delta,t) + \epsilon$, and $u(x,v,T) = \tilde{u}(x,v,T)$ by definition. By the comparison principle of Theorem 5.2, we find that $u(x,v,t) \le \tilde{u}(x,v,t) \le u(x,v,t) + \epsilon e^{r(T-t)}$ for all $(x,v,t) \in \mathcal{O}^{\delta}$. Hence, $u(x,v,t) \le \tilde{u}(x,v,t) \le u(x,v,t) + \epsilon e^{rT}$ for all $(x,v,t) \in \mathcal{O}$. Since ϵ is arbitrary, we obtain the desired result. □

To prove the uniqueness of the solution, we first modified the optimal stopping problem by defining it on $\mathcal{O}^{\delta}$ in order to avoid the degeneracy of the infinitesimal generator. This suggests that one may be able to use known numerical schemes for nondegenerate IPDEs for Lévy processes to design a numerical scheme for the value function u^{δ}. When δ is close to 0, we obtain an approximation for the value function u. The reader is referred to the work of Levendorskiĭ et al. [6] and references therein for possible numerical implementation techniques and difficulties associated to them. On the other hand, it is likely that the method of proof of the uniqueness property could be used for a larger class of stochastic volatility models, including the models of Heston [12] and Hull and White [13]. Indeed, all these models have the non-Lipschitz term $\sqrt{V_t}$ in the differential equation for the log-returns. There remain open problems which we leave to future research.

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