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Research Article

On Tightness of the Skew Random Walks

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The primary purpose of this paper is to prove a tightness of α-skew random walks. The tightness result implies, in particular, that the α-skew Brownian motion can be constructed as the scaling limit of such random walks. Our proof of tightness is based on a fourth-order moment method.

1. Introduction and Statement of the Main Result

Skew Brownian motion was introduced by Itô and Mckean [1] to furnish a construction of certain stochastic processes related to Feller's classification of second-order differential operators associated with diffusion processes (see also Section 4.2 in [2]). For $\alpha \in (0,1)$, the α-skew Brownian motion is defined as a one-dimensional Markov process with the same transition mechanism as of the usual Brownian motion, with the only exception that the excursions away from zero are assigned a positive sign with probability α and a negative sign with probability $1-\alpha$. The signs form an i.i.d. sequence and are chosen independently of the past history of the process. If $\alpha = 1/2$, the process is the usual Brownian motion.

Formally, the α-skew random walk on $\mathbb{Z}$ starting at 0 is defined as the birth-death Markov chain $S^{(\alpha)} = \{S_k^{(\alpha)};\ k \geq 0\}$ with $S_0^{\alpha} = 0$ and one-step transition probabilities given by

$$P\left(S_{k+1}^{(\alpha)} = m+1 \mid S_k^{(\alpha)} = m\right) = \begin{cases} \alpha & \text{if } m = 0, \\ \dfrac{1}{2}, & \text{otherwise,} \end{cases}$$
$$P\left(S_{k+1}^{(\alpha)} = m-1 \mid S_k^{(\alpha)} = m\right) = \begin{cases} 1-\alpha & \text{if } m = 0, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases} \quad (1.1)$$

In the special case $\alpha = 1/2$, $S^{(1/2)}$ is a simple symmetric random walk on $\mathbb{Z}$. Notice that when $\alpha \neq 1/2$, the jumps (in general, increments) of the random walk are not independent.

Harrison and Shepp [3] asserted (without proof) that the functional central limit theorem (FCLT, for short) for reflecting Brownian motion can be used to construct skew Brownian motion as the limiting process of a suitably modified symmetric random walk on the integer lattice. This result has served as a foundation for numerical algorithms tracking moving particle in a highly heterogeneous porous media; see, for instance, [4–7]. In [5] it was suggested that tightness could be obtained based on second moments; however this is not possible even in the case of simple symmetric random walk. The lack of statistical independence of the increments makes a fourth moment proof all the more challenging. Although proofs of FCLTs in more general frameworks have subsequently been obtained by other methods, for example, by Skorokhod embedding in [8], a self-contained simple proof of tightness for simple skew random walk has not been available in the literature.

The main goal of this paper is to prove the following result. Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+ = [0, \infty)$ into $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. For $n \in \mathbb{N}$, let $X_n^{(\alpha)} \in C(\mathbb{R}_+, \mathbb{R})$ denote the following linear interpolation of $S_{[nt]}^{(\alpha)}$:

$$X_n^{(\alpha)}(t) = \frac{1}{\sqrt{n}}\left(S_{[nt]}^{(\alpha)} + (nt - [nt]) \cdot S_{[nt]+1}^{(\alpha)}\right). \tag{1.2}$$

Here and henceforth $[x]$ denotes the integer part of a real number x.

Theorem 1.1. *For any $\alpha \in (0, 1)$, there exists a constant $C > 0$, such that the inequality*

$$E\left|X_n^{(\alpha)}(t) - X_n^{(\alpha)}(s)\right|^4 \leq C|s - t|^2, \tag{1.3}$$

holds uniformly for all $s, t > 0$, and $n \in \mathbb{N}$.

The results stated above implies the following (see, for instance, [9, page 98]).

Corollary 1.2. *The family of processes $X_n^{(\alpha)}$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}_+, \mathbb{R})$.*

2. Proof of Theorem 1.1

In this section we complete the proof of our main result, Theorem 1.1. In what follows we will use S to denote the simple symmetric random walk $S^{(1/2)}$. The following observations can be found in [3].

Proposition 2.1. (a) *$|S^{(\alpha)}|$ has the same distribution as $|S|$ on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. That is, $|S^{(\alpha)}|$ is a simple symmetric random walk on $\mathbb{Z}_+$, reflected at 0.*

(b) *The processes $-S^{(\alpha)}$ and $S^{(1-\alpha)}$ have the same distribution.*

The next statement describes n-step transition probabilities of the skew random walks by relating them to those of S (see, for instance, [5, page 436]).

Proposition 2.2. *For $m \in Z$, $k > 0$*

$$P\left(S_k^{(\alpha)} = m\right) = \begin{cases} \alpha \cdot P(|S_k| = m) & \text{if } m > 0 \\ (1-\alpha) \cdot P(|S_k| = -m) & \text{if } m < 0 \\ P\left(\left|S_k^{(\alpha)}\right| = 0\right) = P(|S_k| = 0) & \text{if } m = 0. \end{cases} \tag{2.1}$$

The following observation is evident from the explicit form of the distribution function of $S_k^{(\alpha)}$, given in Proposition 2.2.

Proposition 2.3. *With probability one,*

$$\begin{aligned} E\left(S_{j+1}^{(\alpha)} - S_j^{(\alpha)} \mid S_j^{(\alpha)}\right) &= (2\alpha - 1)\mathbf{1}_{\{S_j^{(\alpha)}=0\}}, \\ E\left[\left(S_{i+1}^{(\alpha)} - S_i^{(\alpha)}\right)^2 \mid S_i^{(\alpha)}\right] &= 1. \end{aligned} \tag{2.2}$$

To show the result of Theorem 1.1, we will need a corollary to Karamata's Tauberian theorem, which we are going now to state. For a measure μ on $[0,\infty)$, denote by $\widehat{\mu}(\lambda) := \int_0^\infty e^{-\lambda x}\, \mu(dx)$ the Laplace transform of μ. The transform is well defined for $\lambda \in (c,\infty)$, where $c > 0$ is a nonnegative constant, possibly $+\infty$. If μ and ν are measures on $[0,\infty)$ such that $\widehat{\mu}(\lambda)$ and $\widehat{\nu}(\lambda)$ both exist for all $\lambda > 0$, then the convolution $\gamma = \mu * \nu$ has the Laplace transform $\widehat{\gamma}(\lambda) = \widehat{\mu}(\lambda)\widehat{\nu}(\lambda)$ for $\lambda > 0$. If μ is a discrete measure concentrated on $\mathbb{Z}_+$, one can identify μ with a sequence μ_n of its values on $n \in \mathbb{Z}_+$. For such discrete measures, we have the following. (see, e.g., Corollary 8.10 in [10, page 118]).

Proposition 2.4. *Let $\widetilde{\mu}(t) = \sum_{n=0}^{\infty} \mu_n t^n$, $0 \le t < 1$, where $\{\mu_n\}_{n=0}^{\infty}$ is a sequence of nonnegative numbers. For L slowly varying at infinity and $0 \le \theta < \infty$ one has*

$$\widetilde{\mu}(t) \sim (1-t)^{-\theta} L\left(\frac{1}{1-t}\right) \quad \text{as } t \uparrow 1 \tag{2.3}$$

if and only if

$$\sum_{j=0}^{n} \mu_j \sim \frac{1}{\Gamma(\theta)} n^{\theta} L(n) \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

Here and henceforth, $a_n \sim b_n$ for two sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ means $\lim_{n\to\infty} a_n/b_n = 1$.

We are now in a position to prove the following key proposition. Define a sequence $\{q(k)\}_{k\in\mathbb{Z}_+}$ as follows

$$g(k) = \begin{cases} 0 & \text{if } k \in \mathbb{N} \text{ is odd} \\ \binom{2i}{i} 2^{-2i} & \text{if } k = 2i \in \mathbb{N} \text{ is even.} \end{cases} \tag{2.5}$$

Note that in view of Proposition 2.2,

$$g(k) = P(S_k = 0) = P(|S_k| = 0) = P\left(\left|S_k^{(\alpha)}\right| = 0\right) = P\left(S_k^{(\alpha)} = 0\right). \tag{2.6}$$

Proposition 2.5.
(a) *If* $\mu(j) = g * g(j)$ *then* $\sum_{j=0}^{m} \mu(j) \sim m$.
(b) *If* $\nu(j) = g * g * g * g(j)$ *then* $\sum_{j=0}^{m} \nu(j) \sim m^2$.

Proof. For $t \in (0,1)$, let $\widetilde{g}(t) = \sum_{k=0}^{\infty} g(k)t^k$. Notice that $\widetilde{g}(t)$ is well defined since $g(k) = P(S_k = 0) < 1$ for $k \geq 0$. Since $g(2j) = \binom{2j}{j} 2^{-2j} = (-1)^j \binom{-1/2}{j}$, we have

$$\begin{aligned} \widetilde{g}(t) &= \sum_{k=0}^{\infty} g(k)t^k = \sum_{j=0}^{\infty} \binom{2j}{j} 2^{-2j} t^{2j} = \sum_{j=0}^{\infty} (-1)^j \binom{-\frac{1}{2}}{j} t^{2j} \\ &= \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} \left(-t^2\right)^j = \left(1 - t^2\right)^{-1/2}. \end{aligned} \tag{2.7}$$

Notice that, using the notation of Proposition 2.4, $\widetilde{g}(t) = \widehat{g}(\lambda)$ if $t = e^{-\lambda}$. Therefore, $\widetilde{\mu}(t) = \widetilde{g}^2(t) = (1-t^2)^{-1}$ while $\widetilde{\nu}(t) = \widetilde{g}^4(t) = (1-t^2)^{-2}$. Thus claims (a) and (b) of the proposition follow from Proposition 2.4 applied, respectively, with $\theta = 1$, $L = 1$ for μ and with $\theta = 2$, $L = 1$ for ν. □

The last technical lemma we need is the following claim.

Lemma 2.6. *For integers* $0 < i_1 < i_2 < i_3 < i_4$ *define*

$$\begin{aligned} A(i_1, i_2, i_3) &:= E\left(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)}\right)^2 \left(S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}\right)\left(S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}\right), \\ B(i_1, i_2, i_3, i_4) &:= E\left(S_{i_4+1}^{(\alpha)} - S_{i_4}^{(\alpha)}\right)\left(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)}\right)\left(S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}\right)\left(S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}\right). \end{aligned} \tag{2.8}$$

Then there is a constant $C > 0$ such that

$$\sum_{1\le i_1<i_2<i_3\le k-j} A(i_1,i_2,i_3) \le C\left|k-j\right|^2,$$
$$\sum_{1\le i_1<i_2<i_3<i_4\le k-j} B(i_1,i_2,i_3,i_4) \le C\left|k-j\right|^2. \tag{2.9}$$

Proof. Using Proposition 2.3, the Markov property, and the fact the excursions of $S^{(\alpha)}$ away from zero are the same as excursions of the simple symmetric random walk S, we obtain

$$\begin{aligned} A(i_1,i_2,i_3) &= E\left(S^{(\alpha)}_{i_3+1}-S^{(\alpha)}_{i_3}\right)^2\left(S^{(\alpha)}_{i_2+1}-S^{(\alpha)}_{i_2}\right)\left(S^{(\alpha)}_{i_1+1}-S^{(\alpha)}_{i_1}\right)\mathbf{1}_{\{S^{(\alpha)}_{i_1}=0\}}\mathbf{1}_{\{S^{(\alpha)}_{i_2}=0\}} \\ &= P(S_{i_1}=0)\cdot(2\alpha-1)\cdot P(S_{i_2}=0\mid S_{i_1}=0)\cdot(2\alpha-1) \\ &= (2\alpha-1)^2 g(i_1)g(i_2-i_1). \end{aligned} \tag{2.10}$$

Therefore,

$$\sum_{1\le i_1<i_2<i_3<\le k-j} A(i_1,i_2,i_3) \le \sum_{i_3=0}^{[k-j]}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2-1} g(i_2-i_1)g(i_1). \tag{2.11}$$

Using Proposition 2.5, we obtain

$$\begin{aligned} \sum_{i_3=0}^{[k-j]}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2-1} g(i_2-i_1)g(i_1) &= \sum_{i_3=0}^{[k-j]}\sum_{i_2=0}^{i_3-1} g*g(i_2) \le \sum_{i_3=0}^{[k-j]}\sum_{i_2=0}^{[k-j]} g*g(i_2) \\ &\le C_1\left|k-j\right|^2, \end{aligned} \tag{2.12}$$

for some constant $C_1 > 0$ and any $k, j \in \mathbb{N}$.

Similarly,

$$\begin{aligned} B(i_1,i_2,i_3,i_4) &= (2\alpha-1)^4\cdot P(S_{i_1}=0)\cdot\prod_{a=1}^{3} P(S_{i_a+1}=0\mid S_{i_a}=0) \\ &= (2\alpha-1)^4 g(i_1)g(i_2-i_1)g(i_3-i_2)g(i_4-i_3). \end{aligned} \tag{2.13}$$

Hence, using again Proposition 2.5,

$$\sum_{0\le i_1<i_2<i_3<i_4} B(i_1,i_2,i_3,i_4) \le \sum_{i_4=0}^{[k-j]} g*g*g*g(i_4) \le C_2\left|k-j\right|^2, \tag{2.14}$$

for some constant $C_2 > 0$ and any $k, j \in \mathbb{N}$.

To conclude the proof of the lemma, set $C := \max\{C_1, C_2\}$. □

We are now in a position to complete the proof of our main result.

Completion of the Proof of Theorem 1.1

First consider the case where $s = j/n < k/n = t$ are grid points. Then

$$
\begin{aligned}
E\left|\frac{S^{(\alpha)}_{[nt]}}{\sqrt{n}} - \frac{S^{(\alpha)}_{[ns]}}{\sqrt{n}}\right|^4 &= \frac{1}{n^2}E\left|S^{(\alpha)}_k - S^{(\alpha)}_j\right|^4 = \frac{1}{n^2}E\left|\sum_{i=j}^{k-1}\left(S^{(\alpha)}_{i+1} - S^{(\alpha)}_i\right)\right|^4 \\
&= \frac{1}{n^2}\sum_{i=j}^{k-1}E\left(S^{(\alpha)}_{i+1} - S^{(\alpha)}_i\right)^4 + \frac{1}{n^2}\sum_{i_1<i_2\le k-j}E\left(S^{(\alpha)}_{i_1+1} - S^{(\alpha)}_{i_1}\right)^2\left(S^{(\alpha)}_{i_2+1} - S^{(\alpha)}_{i_2}\right)^2 \\
&\quad + \frac{1}{n^2}\sum_{i_1<i_2<i_3\le k-j}E\left(S^{(\alpha)}_{i_3+1} - S^{(\alpha)}_{i_3}\right)^2\left(S^{(\alpha)}_{i_2+1} - S^{(\alpha)}_{i_2}\right)\left(S^{(\alpha)}_{i_1+1} - S^{(\alpha)}_{i_1}\right) \\
&\quad + \frac{1}{n^2}\sum_{i_1<i_2<i_3<i_4\le k-j}E\left(\prod_{a=1}^{4}S^{(\alpha)}_{i_a+1} - S^{(\alpha)}_{i_a}\right) \\
&\le \frac{1}{n^2}\sum_{i=j}^{k-1}1 + \frac{1}{n^2}\binom{k-j}{2}\binom{k-j}{2} + \frac{1}{n^2}C_1|k-j|^2 + \frac{1}{n^2}C_2|k-j|^2 \\
&\le C_3|t-s|^2,
\end{aligned}
\tag{2.15}
$$

for a large enough constant $C_3 > 0$.

To conclude the proof of Theorem 1.1, it remains to observe that for nongrid points s and t one can use an approximation by neighbor grid points. In fact, the approximation argument given in [9, pages 100-101] for regular random walks goes through verbatim.

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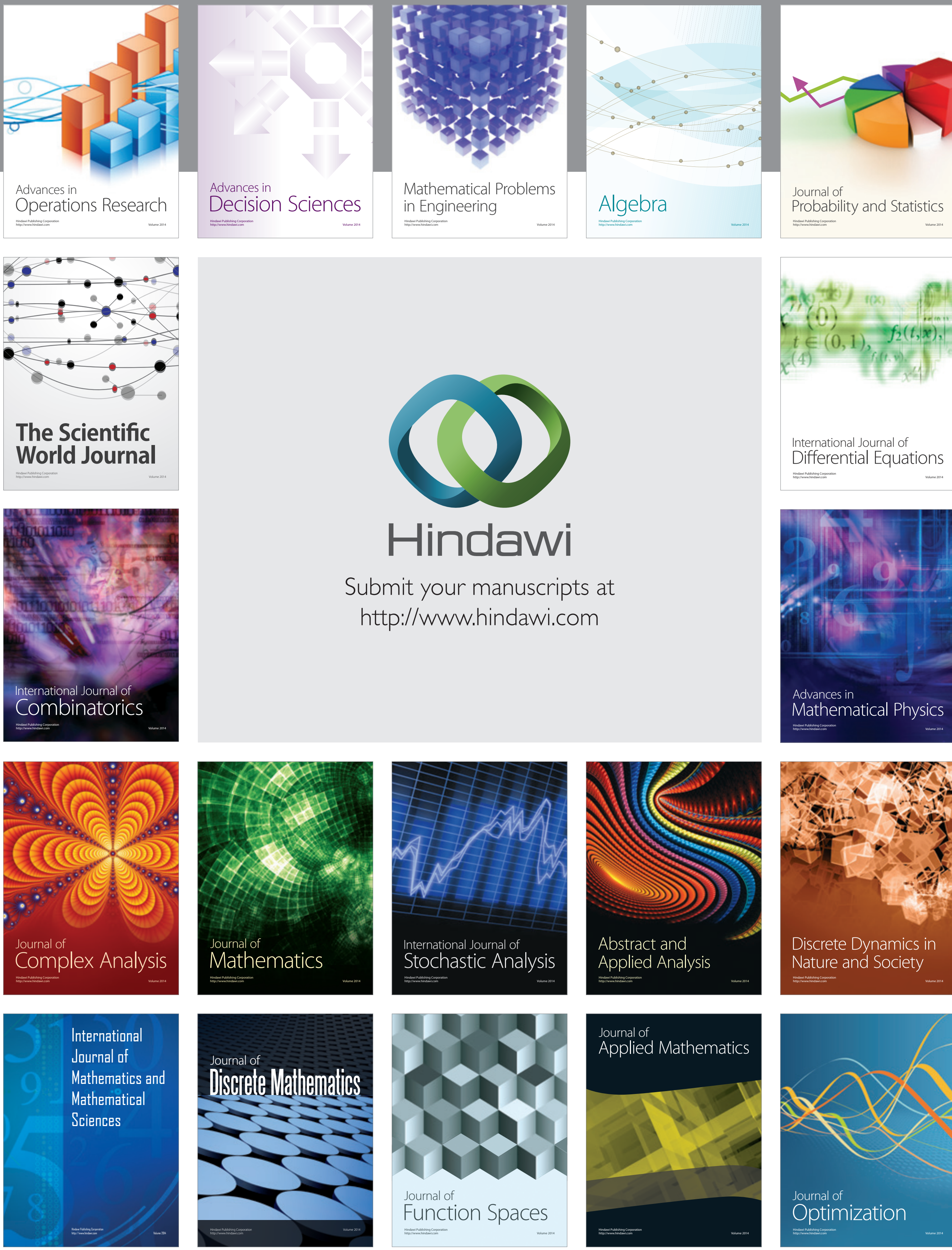
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