$\mathcal{H}_\infty$ CONSTANT GAIN STATE FEEDBACK STABILIZATION OF STOCHASTIC HYBRID SYSTEMS WITH WIENER PROCESS

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This paper considers the stabilization problem of the class of continuous-time linear stochastic hybrid systems with Wiener process. The $\mathcal{H}_\infty$ state feedback stabilization problem is treated. A state feedback controller with constant gain that does not require access to the system mode is designed. LMI-based conditions are developed to design the state feedback controller with constant gain that stochastically stabilizes the studied class of systems and, at the same time, achieve the disturbance rejection of a desired level. The minimum disturbance rejection is also determined. Numerical examples are given to show the usefulness of the proposed results.

1. Introduction

Systems with abrupt changes in their dynamics that result from causes like connections or disconnections of some components, failures in the components, are more often met in practice. The occurrence of the abrupt changes is random in more cases. Analysis and design of these systems cannot be done using the linear invariant system theory since it is unable to model adequately such systems. These practical systems have been modeled by the class of linear systems with Markovian jumps that we will term in this paper as stochastic hybrid systems. This class of systems has two components in the state vector. The first component of this state vector takes values in $\mathbb{R}^n$, evolves continuously in time, and represents the classical state vector that is usually used in the modern control theory. The second one takes values in a finite set and switches in a random manner between a finite number of states (see Mariton [10], Boukas and Liu [4], Boukas [2], and the references therein). This component is represented by a continuous-time Markov process. Usually the state vector of the class of stochastic hybrid systems is denoted by $(x(t), r_t)$. Examples of such systems can be found in manufacturing systems, power systems, telecommunications systems, and so forth.

This class of systems has attracted a lot researchers and many problems have been tackled and solved. Among these problems, we quote those of stability, stabilizability, $\mathcal{H}_\infty$ control, and filtering. For more details on what has been done on this class of systems,

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we refer the reader to the recent books by Boukas and Liu [4] and Boukas [2] and the references therein. These two books present a good review of the literature of the subject up to 2004.

The stabilization problem of the class of linear systems with Markovian jumping parameters has attracted a lot of researchers, and many contributions have been reported in the literature. For more details on this topics and the contributions to the subject, we refer the reader to [2, 3, 4, 5, 6, 7, 14, 15] and the references therein. Most of the results consider state feedback controllers that have mode-dependent gains. This access to the mode may not be possible in some circumstances, which limits the use of such controllers. One alternative consists of estimating the mode and using this estimate to compute the controllers gain. A second alternative that we will develop in this paper consists of developing state feedback controllers with constant gain that do not depend on the system mode. To the best of our knowledge, the case of stabilization with state feedback controller with constant gain for continuous-time systems with Markovian jumps and multiplicative noise has never been studied and our objective in this paper is to study the $\mathscr{H}_\infty$ stabilization of such class of systems.

Our goal in this paper consists of designing a state feedback controller with constant gain that stochastically stabilizes the class of systems we are studying and, at the same time, rejects the disturbance with a desired level $\gamma > 0$. We are also interested in determining the minimum level of the disturbance rejection. In this paper, we will solve these two problems and develop LMI conditions that we can use to determine the state feedback controller that stochastically stabilizes the class of systems of stochastic hybrid systems with multiplicative noise and guarantees the minimum disturbance rejection.

For the deterministic hybrid systems, there exist many contributions to different subjects. Among the problems that are linked with our works, the ones of stability and stabilizability are quoted. The idea of handling the problem is different from the one used in this paper. For more details on this direction of research, we refer to the works of Pettersson and Lennartson [12, 13] and the references therein.

The rest of the paper is organized as follows. In Section 2, the problem we are considering is stated and some useful definitions are given. Section 3 gives the main results of the paper that synthesize the state feedback controller with constant gain. In Section 4, some numerical examples are provided to show the usefulness of the proposed results.

2. Problem statement

We consider a dynamical system defined in a fundamental probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and assume that its dynamics is described by the following differential equations:

$$\begin{aligned} dx(t) &= A(r_t)x(t)dt + B(r_t)u(t)dt + B_\omega(r_t)\omega(t)dt + \mathbb{W}(r_t)x(t)dw(t), \quad x(0) = x_0, \\ y(t) &= C_y(r_t)x(t) + D_y(r_t)u(t) + B_y(r_t)\omega(t), \\ z(t) &= C_z(r_t)x(t) + D_z(r_t)u(t) + B_z(r_t)\omega(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $x_0 \in \mathbb{R}^n$ is the initial state, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t) \in \mathbb{R}^l$ is the system external disturbance, $w(t)$ is a standard Wiener process that is assumed to be independent of the Markov process $\{r_t,\ t \geq 0\}$ which is a continuous-time Markov process taking values in a finite space $\mathscr{S} = \{1,\dots,N\}$ and that describes the evolution of the mode at time t, and when $r_t = i$, the matrices $A(i)$, $B(i)$, $B_\omega(i)$, $\mathbb{W}(i)$, $C_y(i)$, $D_y(i)$, $B_y(i)$, $C_z(i)$, $D_z(i)$, and $B_z(i)$ are given matrices with appropriate dimensions.

Remark 2.1. For the existence and uniqueness of the solution of (2.1), we refer the reader to Kushner [9], Arnold [1], Has'minskiĭ [8], and the references therein.

The system disturbance $\omega(t)$ is assumed to belong to $\mathscr{L}_2[0,\infty)$, which means that the following holds:

$$\mathbb{E}\left[\int_0^\infty \omega^\top(t)\omega(t)dt\right] < \infty. \tag{2.2}$$

This implies that the disturbance has finite energy.

The Markov process $\{r_t,\ t \geq 0\}$ takes values in the finite set $\mathscr{S}$ and, in addition, the switching between the different modes is described by the following probability transitions:

$$\mathbb{P}[r_{t+h} = j \mid r_t = i] = \begin{cases} \lambda_{ij}h + o(h) & \text{when } r_t \text{ jumps from } i \text{ to } j, \\ 1 + \lambda_{ij}h + o(h) & \text{otherwise,} \end{cases} \tag{2.3}$$

where λ_{ij} is the transition rate from mode i to mode j with $\lambda_{ij} \geq 0$ when $i \neq j$ and $\lambda_{ii} = -\sum_{j=1,\,j\neq i}^{N}\lambda_{ij}$, and $o(h)$ is such that $\lim_{h\to 0} o(h)/h = 0$.

For system (2.1), when $u(t) \equiv 0$ for all $t \geq 0$, we have the following definitions.

Definition 2.2. System (2.1) is said to be

(i) stochastically stable (SS) if there exists a finite positive constant $T(x_0,r_0)$ such that the following holds for any initial conditions (x_0,r_0):

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2 dt | x_0, r_0\right] \leq T(x_0,r_0); \tag{2.4}$$

(ii) mean square stable (MSS) if

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0 \tag{2.5}$$

holds for any initial conditions (x_0,r_0);

(iii) mean exponentially stable (MES) if there exist positive constants α and β such that the following holds for any initial conditions (x_0,r_0):

$$\mathbb{E}[\|x(t)\|^2 \mid x_0,r_0] \leq \alpha\|x_0\|e^{-\beta t}. \tag{2.6}$$

Definition 2.3 [2]. System (2.1) with $u(t) \equiv 0$ is said to be internally mean square quadratically stable (MSQS) if there exists a symmetric and positive-definite matrix $P > 0$ satisfying the following for every $i \in \mathscr{S}$:

$$A^\top(i)P + PA(i) + \mathbb{W}(i)P\mathbb{W}(i) < 0. \tag{2.7}$$

By virtue of Definition 2.2, it is obvious that internally MSQS means that system (2.1) is MSQS in case $\omega(t) \equiv 0$, that is, system (2.1) is free of input disturbance. Likewise, we can give the following definitions.

Definition 2.4. System (2.1) with $u(t) \equiv 0$ is said to be internally SS (MES) if it is SS (MES) in case $\omega(t) \equiv 0$.

Definition 2.5. System (2.1) is said to be stabilizable in the SS (MES, MSQS) sense if there exists a controller of the form (2.10) such that the closed-loop system is SS (MES, MSQS) for all initial conditions (x_0, r_0).

In the rest of this paper, we will deal with the design of controllers that stochastically stabilize the closed-loop systems and guarantee the disturbance rejection with a certain level $\gamma > 0$. Mathematically, we are concerned with the design of a controller that guarantees the following for all $\omega \in \mathscr{L}_2[0,\infty)$:

$$||z(t)||_2 < \gamma\left[||\omega(t)||_2^2 + M(x_0, r_0)\right]^{1/2}, \tag{2.8}$$

where $\gamma > 0$ is a prescribed level of disturbance rejection to be achieved, x_0 and r_0 are the initial conditions of the state vector and the mode, respectively, at time $t = 0$, and $M(x_0, r_0)$ is a constant that depends on the initial conditions (x_0, r_0).

Definition 2.6. Let $\gamma > 0$ be a given positive constant. System (2.1) with $u(t) \equiv 0$ is said to be stochastically stable with γ-disturbance attenuation if there exists a constant $M(x_0, r_0)$ with $M(0, r_0) = 0$, for all $r_0 \in \mathscr{S}$, such that the following holds:

$$\|z\|_2 \triangleq \left[\mathbb{E}\int_0^\infty z^\top(t) z(t)dt \,|\, (x_0, r_0)\right]^{1/2} \leq \gamma\left[\|\omega\|_2^2 + M(x_0, r_0)\right]^{1/2}. \tag{2.9}$$

Definition 2.7. System (2.1) is said to be stabilizable with γ-disturbance in the SS (MES, MSQS) sense if there exists a control law of the form (2.10) such that the closed-loop system under this control law is SS (MES, MSQS) and satisfies (2.9).

The goal of this paper is to design a state feedback controller with constant gain that stochastically stabilizes the class of stochastic hybrid systems with Wiener process we are considering in this paper and, at the same time, rejects the effect of the external disturbance $\omega(t)$ with a desired level $\gamma > 0$. The structure of the controller we will be using here is given by the following expression:

$$u(t) = \mathscr{K}x(t), \tag{2.10}$$

where $\mathscr{K}$ is a constant gain that we have to determine.

Remark 2.8. The class of systems we are treating in this paper can be stabilized by a different class of controllers that may depend on the mode and the state vector system. The one we are using in this paper is a special one that does not use the mode of the system. Therefore, it is a restricted class of controllers. The existence of such controllers is not treated here and it is an open question.

We are mainly concerned with the design of such controller. LMI-based conditions are searched since the design becomes easier and the gain can be obtained by solving the appropriate LMIs using the existing developed algorithms. In the rest of this paper, we will assume complete access to the state vector at time t.

Before closing this section, we give some lemmas that we will use in the rest of the paper.

Lemma 2.9 [11]. *Let H, F, and G be real matrices of appropriate dimensions, then, for any scalar $\varepsilon > 0$, for all matrices F satisfying $F^T F \leq I$,*

$$HFG + G^\top F^\top H^\top \leq \varepsilon HH^\top + \varepsilon^{-1} G^\top G. \tag{2.11}$$

Lemma 2.10 [4]. *The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0 \tag{2.12}$$

is equivalent to

$$R > 0, \quad H - S^\top R^{-1} S > 0, \tag{2.13}$$

where $H = H^\top$, $R = R^\top$, and S is a matrix with appropriate dimension.

3. Main results

Our goal in this paper consists of designing a state feedback controller with constant gain that stochastically stabilizes the class of stochastic hybrid systems with Wiener process we are considering and, at the same time, rejects the external disturbance with a desired level $\gamma > 0$.

Theorem 3.1. *If system (2.1) with $u(t) \equiv 0$ is internally MSQS, then it is stochastically stable.*

Proof. Let $r_t = i \in \mathscr{S}$. To prove this theorem, we consider a candidate Lyapunov function to be defined as follows:

$$V(x(t), i) = x^\top(t) P x(t), \tag{3.1}$$

where $P > 0$ is a symmetric and positive-definite matrix.

Using the fact that $\sum_{j=1}^{N} \lambda_{ij} P = 0$, the infinitesimal operator $\mathscr{L}$ is given as follows:

$$\begin{aligned} \mathscr{L}V(x(t), i) &= \dot{x}^\top(t) P x(t) + x^\top(t) P \dot{x}(t) + x^\top(t) \mathbb{W}^\top(i) P \mathbb{W}(i) x(t) \\ &= x^\top(t) \left[A^\top(i) P + PA(i) + \mathbb{W}^\top(i) P \mathbb{W}(i) \right] x(t) + 2x^\top(t) P B_\omega(i) \omega(t). \end{aligned} \tag{3.2}$$

Using now Lemma 2.9, we get the following for any $\varepsilon_w(i) > 0$:

$$2x^\top(t) P B_\omega(i) \omega(t) \leq \varepsilon_w^{-1}(i) x^\top(t) P(i) B_\omega(i) B_\omega^\top(i) P(i) x(t) + \varepsilon_w(i) \omega^\top(t) \omega(t). \tag{3.3}$$

Combining this with the expression of $\mathscr{L}V(x(t),i)$ yields

$$\begin{aligned}\mathscr{L}V(x(t),i) &\leq x^\top(t)[A^\top(i)P+PA(i)+\mathbb{W}^\top(i)P\mathbb{W}(i)]x(t)\\ &\quad+\varepsilon_w^{-1}(i)x^\top(t)PB_\omega(i)B_\omega^\top(i)Px(t)+\varepsilon_w(i)\omega^\top(t)\omega(t)\\ &= x^\top(t)[A^\top(i)P+PA(i)+\mathbb{W}^\top(i)P\mathbb{W}(i)]x(t)\\ &\quad+x^\top(t)[\varepsilon_w^{-1}(i)PB_\omega(i)B_\omega^\top(i)P]x(t)+\varepsilon_w(i)\omega^\top(t)\omega(t)\\ &= x^\top(t)\Xi(i)x(t)+\varepsilon(i)\omega^\top(t)\omega(t),\end{aligned} \tag{3.4}$$

with

$$\Xi(i)=A^\top(i)P+PA(i)+\mathbb{W}^\top(i)P\mathbb{W}(i)+\varepsilon_w^{-1}(i)PB_\omega(i)B_\omega^\top(i)P. \tag{3.5}$$

Based on Dynkin's formula, we get the following:

$$\mathbb{E}[V(x(t),i)]-V(x_0,r_0)=\mathbb{E}\left[\int_0^t \mathscr{L}V(x(s),r_s)ds|x_0,r_0\right], \tag{3.6}$$

which, combined with (3.4), yields

$$\mathbb{E}[V(x(t),i)]-V(x_0,r_0)\leq\mathbb{E}\left[\int_0^t x^\top(s)\Xi(r_s)x(s)ds|x_0,r_0\right]+\varepsilon_w(i)\int_0^t\omega^\top(s)\omega(s)ds. \tag{3.7}$$

Since $V(x(t),i)$ is nonnegative, (3.7) implies

$$\mathbb{E}[V(x(t),i)]+\mathbb{E}\left[\int_0^t x^\top(s)[-\Xi(r_s)]x(s)ds|x_0,r_0\right]\leq V(x_0,r_0)+\varepsilon_w(i)\int_0^t\omega^\top(s)\omega(s)ds, \tag{3.8}$$

which yields

$$\begin{aligned}&\min_{i\in\mathscr{S}}\{\lambda_{\min}(-\Xi(i))\}\mathbb{E}\left[\int_0^t x^\top(s)x(s)ds\right]\\ &\quad\leq\mathbb{E}\left[\int_0^t x^\top(s)[-\Xi(r_s)]x(s)ds\right]\leq V(x_0,r_0)+\varepsilon_w(i)\int_0^\infty\omega^\top(s)\omega(s)ds.\end{aligned} \tag{3.9}$$

This proves that system (2.1) is stochastically stable. □

We now establish what conditions we should satisfy if we want to make system (2.1), with $u(t)=0$ for all $t\geq 0$, stochastically stable and have γ-disturbance rejection. The following theorem gives such conditions.

Theorem 3.2. *Let γ be a given positive constant. If there exists a symmetric and positive-definite matrix $P > 0$ such that the LMI*

$$\begin{bmatrix} J_0(i) & \begin{bmatrix} C_z^\top(i)B_z(i) \\ +PB_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)C_z(i) \\ +B_\omega^\top(i)P \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} < 0 \tag{3.10}$$

holds for every $i \in \mathscr{S}$, where $J_0(i) = A^\top(i)P + PA(i) + \mathbb{W}^\top(i)P\mathbb{W}(i) + C_z^\top(i)C_z(i)$, then system (2.1) with $u(t) \equiv 0$ is stochastically stable and satisfies the following:

$$\|z\|_2 \le \left[\gamma^2\|\omega\|_2^2 + x_0^\top P x_0\right]^{1/2}, \tag{3.11}$$

which means that the system with $u(t) = 0$ for all $t \ge 0$ is stochastically stable with γ-disturbance attenuation.

Proof. Let $r_t = i \in \mathscr{S}$. From (3.10) and using Schur's complement, we get the following inequality:

$$A^\top(i)P + PA(i) + \mathbb{W}^\top(i)P\mathbb{W}(i) + C_z^\top(i)C_z(i) < 0, \tag{3.12}$$

which implies the following since $C_z^\top(i)C_z(i) > 0$:

$$A^\top(i)P + PA(i) + \mathbb{W}^\top(i)P\mathbb{W}(i) < 0. \tag{3.13}$$

Based on Definition 2.3, this proves that the system under study is internally MSQS. Using now Theorem 3.1, we conclude that system (2.1) with $u(t) \equiv 0$ is stochastically stable.

We now prove that (3.11) is satisfied. To this end, we define the following performance function:

$$J_T = \mathbb{E}\left[\int_0^T \left[z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t)\right]dt\right]. \tag{3.14}$$

To prove (3.11), it suffices to establish that J_∞ is bounded, that is,

$$J_\infty \le V(x_0, r_0) = x_0^\top P x_0. \tag{3.15}$$

First of all, notice that for $V(x(t),i) = x^\top(t)Px(t)$, we have

$$\begin{aligned} \mathscr{L}V(x(t),i) &= x^\top(t)\left[A^\top(i)P + PA(i) + \mathbb{W}^\top(i)P\mathbb{W}(i)\right]x(t) \\ &\quad + x^\top(t)PB_\omega(i)\omega(t) + \omega^\top(t)B_\omega^\top(i)Px(t), \\ z^\top(t)z(t) - \gamma^2\omega(t)\omega(t) &= \left[C_z(i)x(t) + B_z(i)\omega(t)\right]^\top\left[C_z(i)x(t) + B_z(i)\omega(t)\right] - \gamma^2\omega(t)\omega(t) \\ &= x^\top(t)C_z^\top(i)C_z(i)x(t) + x^\top(t)C_z^\top(i)B_z(i)\omega(t) \\ &\quad + \omega^\top(t)B_z^\top(i)C_z(i)x(t) + \omega^\top(t)B_z^\top(i)B_z(i)\omega(t) - \gamma^2\omega^\top(t)\omega(t), \end{aligned} \tag{3.16}$$

which implies the following equality:

$$z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \mathscr{L}V(x(t),i) = \eta^\top(t)\Theta(i)\eta(t), \tag{3.17}$$

with

$$\begin{aligned}\Theta(i) &= \begin{bmatrix} J_0(i) & \begin{bmatrix} C_z^\top(i)B_z(i) \\ +PB_\omega(i)\end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)C_z(i) \\ +B_\omega^\top(i)P\end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I}\end{bmatrix}, \\ \eta^\top(t) &= \begin{bmatrix} x^\top(t) & \omega^\top(t)\end{bmatrix}.\end{aligned} \tag{3.18}$$

Therefore,

$$J_T = \mathbb{E}\left[\int_0^T [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \mathscr{L}V(x(t),i)]dt\right] - \mathbb{E}\left[\int_0^T \mathscr{L}V(x(t),i)dt\right]. \tag{3.19}$$

Using now Dynkin's formula, that is,

$$\mathbb{E}\left[\int_0^T \mathscr{L}V(x(t),i)dt | x_0, r_0\right] = \mathbb{E}[V(x(T),r_T)] - V(x_0,r_0), \tag{3.20}$$

we get

$$J_T = \mathbb{E}\left[\int_0^T \eta^\top(t)\Theta(i)\eta(t)dt\right] - \mathbb{E}[V(x(T),r_T)] + V(x_0,r_0). \tag{3.21}$$

Since $\Theta(i) < 0$ and $\mathbb{E}[V(x(T),r_T)] \geq 0$, (3.23) implies the following:

$$J_T \leq V(x_0,r_0), \tag{3.22}$$

which yields $J_\infty \leq V(x_0,r_0)$, that is, $\|z\|_2^2 - \gamma^2\|\omega\|_2^2 \leq x_0^\top P x_0$.

This gives the desired result: $\|z\|_2 \leq [\gamma^2\|\omega\|_2^2 + x_0^\top P x_0]^{1/2}$. This ends the proof of the theorem. □

First of all, we see how we can design a controller of the form (2.10). Plugging the expression of the controller in the dynamics (2.1), we get

$$\begin{aligned} dx(t) &= \bar{A}(i)x(t)dt + B_\omega(i)w(t)dt + \mathbb{W}(i)x(t)d\omega(t), \\ z(t) &= \bar{C}_z(i)x(t) + B_z(i)w(t),\end{aligned} \tag{3.23}$$

where $\bar{A}(i) = A(i) + B(i)\mathcal{K}$ and $\bar{C}_z(i) = C_z(i) + D_z(i)\mathcal{K}$.

Using now the results of Theorem 3.2, we get the following ones for the stochastic stability and the disturbance rejection of level $\gamma > 0$ for the dynamics of the closed-loop system.

Corollary 3.3. *Let γ be a given positive constant and $\mathcal{K}$ a given gain. If there exists a symmetric and positive-definite matrix $P > 0$ such that the LMI*

$$\begin{bmatrix} \bar{J}_0(i) & \begin{bmatrix} \bar{C}_z^\top(i)B_z(i) \\ +PB_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)\bar{C}_z(i) \\ +B_\omega^\top(i)P \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} < 0 \tag{3.24}$$

holds for every $i \in \mathcal{S}$, with $\bar{J}_0(i) = \bar{A}^\top(i)P + P\bar{A}(i) + \mathbb{W}(i)P\mathbb{W}(i) + \bar{C}_z^\top(i)\bar{C}_z(i)$, then system (2.1) is stochastically stable under the controller (2.10) and satisfies $\|z\|_2 \le [\gamma^2\|\omega\|_2^2 + x_0^\top P x_0]^{1/2}$, which means that the system is stochastically stable with γ-disturbance attenuation.

To synthesize the controller gain, we transform the LMI (3.24) into a form that can be used easily to compute the gain for every mode $i \in \mathcal{S}$. For this purpose, notice that

$$\begin{bmatrix} \bar{J}_0(i) & \begin{bmatrix} \bar{C}_z^\top(i)B_z(i) \\ +PB_\omega(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)\bar{C}_z(i) \\ +B_\omega^\top(i)P \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} = \begin{bmatrix} \bar{J}_1(i) & PB_\omega(i) \\ B_\omega^\top(i)P & -\gamma^2\mathbb{I} \end{bmatrix} + \begin{bmatrix} \bar{C}_z^\top(i) \\ B_z^\top(i) \end{bmatrix} \begin{bmatrix} \bar{C}_z(i) & B_z(i) \end{bmatrix} \tag{3.25}$$

with

$$\begin{gathered} \bar{J}_0(i) = \bar{A}^\top(i)P + P\bar{A}(i) + \mathbb{W}(i)P\mathbb{W}(i) + \bar{C}_z^\top(i)\bar{C}_z(i), \\ \bar{J}_1(i) = \bar{A}^\top(i)P + P\bar{A}(i) + \mathbb{W}(i)P\mathbb{W}(i). \end{gathered} \tag{3.26}$$

Using now Schur's complement, we show that (3.24) is equivalent to the following inequality:

$$\begin{bmatrix} \bar{J}_1(i) & PB_\omega(i) & \bar{C}_z^\top(i) \\ B_\omega^\top(i)P & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \bar{C}_z(i) & B_z(i) & -\mathbb{I} \end{bmatrix} < 0. \tag{3.27}$$

Since $\bar{A}(i)$ is nonlinear in $\mathcal{K}$ and P, the previous inequality is then nonlinear and therefore it cannot be solved using the existing linear algorithms. To transform it into an LMI, let $X = P^{-1}$. Pre- and postmultiplying this inequality by $\mathrm{diag}[X, \mathbb{I}, \mathbb{I}]$, where $\mathbb{I}$ is an appropriate identity matrix, gives

$$\begin{bmatrix} \bar{J}_X & B_\omega(i) & X\bar{C}_z^\top(i) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \bar{C}_z(i)X & B_z(i) & -\mathbb{I} \end{bmatrix} < 0 \tag{3.28}$$

with $\bar{J}_X = X\bar{A}^\top(i) + \bar{A}(i)X + X\mathbb{W}(i)X^{-1}\mathbb{W}(i)X$.

Notice that

$$\begin{aligned} X\bar{A}^\top(i)+\bar{A}(i)X &= XA^\top(i)+A(i)X+Y^\top B^\top(i)+B(i)Y,\\ X\left[C_z(i)+D_z(i)\mathscr{K}\right]^\top &= XC_z^\top(i)+Y^\top D_z^\top(i), \end{aligned} \tag{3.29}$$

where $Y=\mathscr{K}X$.

Using Schur's complement again, this implies that the previous inequality is equivalent to the following:

$$\begin{bmatrix} J(i) & B_\omega(i) & \begin{bmatrix} XC_z^\top(i) \\ +Y^\top D_z^\top(i) \end{bmatrix} & X\mathbb{W}^\top(i) \\ B_\omega^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 \\ \begin{bmatrix} C_z(i)X \\ +D_z(i)Y \end{bmatrix} & B_z(i) & -\mathbb{I} & 0 \\ \mathbb{W}(i)X & 0 & 0 & -X \end{bmatrix} < 0 \tag{3.30}$$

with $J(i)=XA^\top(i)+A(i)X+Y^\top B^\top(i)+B(i)Y$.

From this discussion we get the following theorem.

Theorem 3.4. *Let γ be a positive constant. If there exist a symmetric and positive-definite matrix $X>0$ and a matrix Y such that the LMI (3.30) holds for every $i\in\mathscr{S}$, with $J(i)=XA^\top(i)+A(i)X+Y^\top B^\top(i)+B(i)Y$, then system (2.1) under the controller (2.10) with $\mathscr{K}=YX^{-1}$ is stochastically stable and, moreover, the closed-loop system satisfies the disturbance rejection of level γ.*

From a practical point of view, the controller that stochastically stabilizes the class of systems and, at the same time, guarantees the minimum disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

$$\mathrm{P}:\begin{cases} \min\limits_{\substack{\nu>0\\ X>0\\ Y}} \nu \\ \text{such that} \\ \begin{bmatrix} J(i) & B_\omega(i) & \begin{bmatrix} XC_z^\top(i) \\ +Y^\top D_z^\top(i) \end{bmatrix} & X\mathbb{W}^\top(i) \\ B_\omega^\top(i) & -\nu\mathbb{I} & B_z^\top(i) & 0 \\ \begin{bmatrix} C_z(i)X \\ +D_z(i)Y \end{bmatrix} & B_z(i) & -\mathbb{I} & 0 \\ \mathbb{W}(i)X & 0 & 0 & -X \end{bmatrix} < 0, \end{cases} \tag{3.31}$$

where the LMI in the constraints is obtained from (3.30) by replacing γ^2 by ν.

The following corollary gives the results on the design of the controller that stochastically stabilizes system (2.1) and simultaneously guarantees the smallest disturbance rejection level.

Corollary 3.5. *Let $\nu > 0$, $X > 0$, and let Y be the solution of the optimization problem P. Then, the controller (2.10) with $\mathcal{K} = YX^{-1}$ stochastically stabilizes the class of systems under consideration and, moreover, the closed-loop system satisfies the disturbance rejection of level $\sqrt{\nu}$.*

4. Numerical examples

In the previous section, we developed results that determine the state feedback controller that stochastically stabilizes the class of systems we are treating in this paper and, at the same time, rejects the disturbance $w(t)$ with the desired level $\gamma > 0$. The conditions we developed are in the LMI form which makes their resolution easy. In the rest of this section we will give some numerical examples to show the usefulness of our results. Two numerical examples are presented.

Example 4.1. We consider a system of two modes with the following data:

(i) transition probability rates matrix:

$$\Lambda = \begin{bmatrix} -2.0 & 2.0 \\ 3.0 & -3.0 \end{bmatrix}; \tag{4.1}$$

(ii) mode 1:

$$\begin{gathered} A(1) = \begin{bmatrix} 1.0 & -0.4 \\ 0.1 & 1.0 \end{bmatrix}, \qquad B(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad B_\omega(1) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \\ B_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}, \qquad \mathbb{W}(1) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}, \qquad C_z(1) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}, \\ D_z(1) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}; \end{gathered} \tag{4.2}$$

(iii) mode 2:

$$\begin{gathered} A(2) = \begin{bmatrix} -0.1 & -0.3 \\ 0.5 & -0.25 \end{bmatrix}, \qquad B(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad B_\omega(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}, \\ B_z(2) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad \mathbb{W}(2) = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.2 \end{bmatrix}, \qquad C_z(2) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \\ D_z(2) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}. \end{gathered} \tag{4.3}$$

First of all, notice the system is unstable in mode 1 and it is stochastically instable. Letting $\gamma = 5$ and solving the LMI (3.30), we get

$$X = \begin{bmatrix} 0.8325 & -0.0083 \\ -0.0083 & 0.4112 \end{bmatrix}, \qquad Y = \begin{bmatrix} -1.5644 & 0.0876 \\ -0.0785 & -1.0120 \end{bmatrix}, \tag{4.4}$$

which gives the following gains:

$$\mathscr{K} = \begin{bmatrix} -1.8773 & 0.1752 \\ -0.1188 & -2.4635 \end{bmatrix}. \tag{4.5}$$

All the conditions of Theorem 3.4 are satisfied, and therefore the closed-loop system is stochastically stable under the state feedback controller we designed for this system and also it assures the disturbance rejection of level 5.

Example 4.2. To design a stabilizing controller that assures the minimum disturbance rejection, we consider again the system of two modes we considered in the previous example and solve the optimization problem P. The resolution of such system gives

$$X = \begin{bmatrix} 0.0403 & -0.0001 \\ -0.0001 & 0.0333 \end{bmatrix}, \qquad Y = \begin{bmatrix} -1.4035 & 0.0007 \\ 0.0007 & -1.3327 \end{bmatrix}, \tag{4.6}$$

which gives the following gains:

$$\mathscr{K} = \begin{bmatrix} -34.7838 & -0.0520 \\ -0.0520 & -40.0550 \end{bmatrix}. \tag{4.7}$$

Using the results of Theorem 3.2, it results that the system of this example is stochastically stable under the state feedback controller with the computed constant gain, and assures the disturbance rejection of level $\gamma = 1.0$.

5. Conclusion

This paper dealt with the class of hybrid stochastic systems with multiplicative Wiener process. Both the stability and stabilizability problems were treated. A state feedback controller with constant gain was proposed to stochastically stabilize the class of stochastic systems and, at the same time, reject the disturbance with a desired level $\gamma > 0$. The conditions we developed are in LMI form which makes the resolution easier using the existing tools.

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