

CONDITIONS OF CRACKING OF A STRINGER

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In the limits of brittle failure of materials, we investigate the problem of possible cracking of an infinite stringer on the boundary of an elastic half-infinite plate. The plate is exposed to tensioning by uniformly distributed forces, and also to contact stresses due to application of forces on the stringer. The accurate solution of a contact problem of interaction of an infinitely continuous stringer is constructed. With the help of this solution, we formulate the condition of cracking of a stringer and the necessary restrictions for external loading, which provide the contacts of a broken stringer with a plate without propagation of a crack inside the stringer. The problem considered here is of interest for theory and practice. We firstly formulate the modified problem of E. Melan and with the help of Fourier integral transformation, we construct its acoustic solution and then formulate the condition of cracking of an infinite stringer. After that, reveal the necessary restrictions on external loadings, which provide the contact of the failed stringer with a plate without propagation of the crack inside the plate.

1. Introduction

One of the most important scientific and technical problems is the creation of structural materials with high rigidity, strength, and reliability. Such materials, for example, are age-hardenable alloys; alloy-treated steels; steels containing heterogeneous inclusions; and various types of composite materials in most cases undergoing brittle failure. Most number of modern structural composite materials fail mainly due to the propagation of cracks and this is confirmed by catastrophic consequences of increasing of the cracks in the coverings of welding vessels, planes, in reservoirs, large concrete, and various types of buildings.

Plates with stiffening ribs are widely used in engineering, and, in particular, in manufacturing of aircrafts. By stiffening ribs, we imply a curvilinear elastic rod of constant cross-section (generally speaking, made of elastic material) continuously attached to the plate along its edge, or along any other line, partially or wholly placed inside the physical region. The problem of reinforcing the plates becomes particularly complicated when in the area of reinforcing the base, the medium has such defects as openings, notches,

cracks, and so forth, and then, the definition of interaction of two opposite factors—its reinforcing and its weakening—takes on special significance.

Problems of contact interaction of thin-walled elements in the form of straight-line stringers with massive deformable bodies of different geometric forms, the results of the investigations which are important for application in the mechanics of composites, in metrical techniques, and in other fields of the applied mechanics were considered by many authors. In Melan's work [16], the problem of the contact of the infinite stringer with an elastic semi-infinite plate, where it is assumed that the rigidity in bending of the stringers is negligibly small, because of which the normal contact stresses are regarded to be equal to zero is first discussed and solved, and finally a model of one-dimensional elastic continuum is admitted for the stringer. Substantiation of this model with exact methods of the two-dimensional theory of elasticity is given in Buefler [6], and in Muki and Sternberg [17] later on. Numerous works forming a rather wide field of the theory of elasticity are dedicated to generalization and development of Melan's results. The basic achievements in this field with sufficient completeness are reflected in the works of Sternberg [19], Erdogan [10], Grigolyuk and Tolkachev [12], Cherepanov [8], Alexandrov and Mkhitarian [1] and in the book *The Development of the Theory of Contact Problems in the USSR* [10] edited by Galline. The problems of contact interaction of a discontinuous stringer or thin inclusions with elastic bodies and problems of stress state of massive elastic bodies with cracks, reinforced by stringers, are of great interest from the point of view of the fracture mechanics and are closely connected with the denoted circle of problems. Theoretical investigations of the questions of forming and propagating of cracks (cracking) in the composites are of great interest too. In this direction, we must mention the works of Muki and Sternberg [18], Greif and Sanders Jr. [11], Delale and Erdogan [9], Theocaris et al. [4, 20, 21], Bardzokas et al. [3], and Antipov et al. [2]. In the ideal and methodological aspects, the mentioned problems are highly close to the problems of the mechanics of the contact fracture which are reflected in the works of Lawn and Wilshaw [14], Kolesnikov and Morozov [13], and Cherepanov [7].

Consider the work of Delale and Erdogan [9] more thoroughly in which the question of initiation and propagation of a crack in an elastic half-plane, reinforced on its boundary by the stringer of finite length and uniformly tensioned at infinity, is discussed. On the basis of the analysis of the stress state of the elastic half-plane near the endpoints of the stringer by the criterion of maximum normal discontinuous stresses, the initial possible direction of the crack propagation is determined. It is shown that this direction actually coincides with the perpendicular direction to the half-plane boundary. Later, contact problems on interaction of the elastic half-plane with a finite stringer, containing a vertical inner crack or a vertical crack terminating at the boundary half-plane, are considered. An important particular case of an elastic half-plane, the known length of which is given beforehand, with two symmetrically situated finite stringer and a vertical edge crack, is discussed. In case of a vertical edge crack with the help of the defining system of singular integral equations of the problem and the functional and theoretical method of asymptotic behavior of the integral of Cauchy type near the tips of the integration interval, it is shown that the tangential contact stresses under the stringer and the displacement densities at the endpoint of the stringer and the crack do not have a singularity degree.

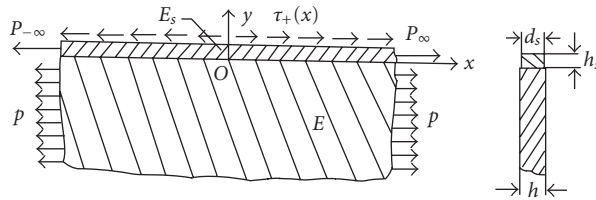


Figure 2.1. Geometry and loading of an elastic semi-infinite plate reinforced on its boundary by the infinite continuous stringer ($P_{-\infty} = P_{\infty}$).

In the given work, firstly, we formulate the modified problem of Melan and with the help of Fourier integral transformation, we construct its acoustic solution and then formulate the condition of cracking of an infinite stringer. After that, reveal the necessary restrictions on external loadings, which provide the contact of the failed infinite stringer with a plate without propagation of the crack inside the plate.

2. Formulation of Melan’s modified problem and its solution

Let an elastic semi-infinite plate related to the right rectangular coordinate Oxy (Figure 2.1) with elasticity module E and thickness h on its boundary be reinforced by an infinite stringer with elasticity module E_s ($E_s > E$) and area A_s of the cross-section, when $A_s = d_s h_s$, where d_s is the width and h_s is the height of the stringer. Let then the plate at infinity be tensioned by uniformly distributed forces of intensity p and let the stringer at infinity be stretched by two concentrated forces $P_{\pm\infty}$ ($P_{-\infty} = P_{+\infty}$). Besides, take into account that the tangential forces of intensity $\tau_+(x)$ ($-\infty < x < \infty$) act along the middle line of the top edge of the stringer, when $\tau_+(-x) = -\tau_+(x)$. It is required to determine the law of distribution of the tangential contact stresses $\tau_{xy}(x, 0)$ and the axial stresses $\sigma_x^s(x)$ in the stringer.

Assume that the plate is in the generalized plane stress state and the stringer according to the model of one-dimensional elastic continuum is in a uniaxial stress state. Going into the case of the plane deformation is realized by the well-known means.

Introduce the concept of effective width $d = d_{\text{eff}}$ of the contact zone, along which the stringer is actually fastened with the plate. In the particular case when the stringer width is connected with the plate, we have $d = \min(d_s, h)$. When there is a partial connection, $d < d_s, h$. In the assumptions made, derive the determining equation of the problem. For this, at first consider the equilibrium of the element, of length dx , of the stringer (Figure 2.2).

According to the admitted model of the stringer, the equation of the equilibrium of this element has the form

$$\frac{d\sigma_x^s}{dx} = \frac{d \cdot \tau_-(x) - \tau_+(x)}{A_s} \quad (-\infty < x < \infty), \tag{2.1}$$

where $\tau_-(x)$ denotes the unknown tangential contact stress related to the unit of the effective width d of the contact zone, when $\tau_-(-x) = -\tau_-(x)$. Then, $\tau_{xy}(x, 0) = d \cdot \tau_-(x)/h$.

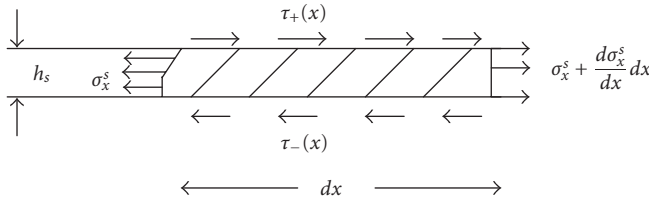


Figure 2.2. Equilibrium of the element of the stringer.

Since

$$A_s \sigma_x^s(x)|_{x=\pm\infty} = P_{\pm\infty} = P_{\infty}, \tag{2.2}$$

with the help of the integrity from (2.1), we obtain

$$\sigma_x^s(x) = A_s^{-1} \left[P_{\infty} - d \int_x^{\infty} \tau_-(u) du + \int_x^{\infty} \tau_+(u) du \right] \quad (-\infty < x < \infty). \tag{2.3}$$

Here by Hook's law for axial deformations $\epsilon_x^s(x)$ of the stringer, we find

$$\epsilon_x^s(x) = E_s^{-1} \sigma_x^s(x) = (E_s A_s)^{-1} \left[P_{\infty} - d \int_x^{\infty} \tau_-(u) du + \int_x^{\infty} \tau_+(u) du \right] \quad (-\infty < x < \infty). \tag{2.4}$$

On the other hand, the horizontal strain component $\epsilon_x(x, 0)$ of boundary points of the elastic semi-infinite plate, when horizontal forces of intensity $\tau_-(x)$ are applied on its boundary, normal forces are absent, and uniformly tensile forces p are applied at infinity, is given by the formula derived in the works of Alexandrov and Mkhitarayan [1], and Timoshenko and Goodier [22]:

$$\epsilon_x(x, 0) = \frac{2d}{\pi E h} \int_{-\infty}^{\infty} \frac{\tau_-(u) du}{u - x} + \frac{p}{E} \quad (-\infty < x < \infty), \tag{2.5}$$

where the integral is understood in the sense of Cauchy principal value.

Now the contact condition $\epsilon_x(x, 0) = \epsilon_x^s(x)$ ($-\infty < x < \infty$), taking into account (2.4) and (2.5), the solution of the produced problem is reduced to the solution of the following integral equation of a convolution type referring to $\tau_-(x)$:

$$\frac{2d}{\pi E h} \int_{-\infty}^{\infty} \frac{\tau_-(u) du}{u - x} = \frac{1}{E_s A_s} \left[-d \int_x^{\infty} \tau_-(u) du + \int_x^{\infty} \tau_+(u) du \right] \quad (-\infty < x < \infty). \tag{2.6}$$

Here the following relation is taken into account:

$$\frac{p}{E} = \frac{P_{\infty}}{E_s A_s}, \tag{2.7}$$

following from the condition of compatibility of horizontal components of the stringer deformation and plate at infinity, according to (2.4) and (2.5).

Furthermore in (2.6), we pass to new values, using

$$\begin{aligned}
 x &= \frac{E_s A_s}{Ed} \xi, & u &= \frac{E_s A_s}{Ed} \eta, & h_0 &= \frac{h}{d}, \\
 \tau_-(x) &= \tau_-\left(\frac{E_s A_s}{Ed} \xi\right) = \tau_-^*(\xi), & \tau_+(x) &= \tau_+\left(\frac{E_s A_s}{Ed} \xi\right) = \tau_+^*(\xi).
 \end{aligned}
 \tag{2.8}$$

As a result, (2.6) is transformed into the following integro-differential equation of a convolution type of a simpler form:

$$\frac{2}{\pi} \frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{\tau_-^*(\eta) d\eta}{\eta - \xi} = h_0 \tau_-^*(\xi) - h_0 d^{-1} \tau_+^*(\xi) \quad (-\infty < x < \infty).
 \tag{2.9}$$

Equation (2.9) is solved by a standard method of Fourier integral transform. Introducing Fourier transform

$$T_{\pm}^*(\lambda) = \int_{-\infty}^{\infty} \tau_{\pm}^*(\xi) \exp(i\lambda\xi) d\xi,
 \tag{2.10}$$

from this equation, we easily find the unknown transform

$$T_-^*(\lambda) = \frac{h_0}{d} \frac{T_+^*(\lambda)}{h_0 + 2|\lambda|} \quad (-\infty < \lambda < \infty).
 \tag{2.11}$$

Since $\tau_+^*(\xi)$ and $T_+^*(\lambda)$ are odd functions, from here by Fourier formula of reverse transform, we obtain

$$\tau_-^*(\xi) = \frac{h_0}{d} \int_{-\infty}^{\infty} K(h_0|\xi - \eta|) \tau_+^*(\eta) d\eta \quad (-\infty < \xi < \infty),
 \tag{2.12}$$

where

$$\begin{aligned}
 K(\xi) &= \frac{1}{\pi} \int_0^{\infty} \frac{\cos(\lambda\xi) d\lambda}{1 + 2\lambda} \\
 &= -\frac{2}{2\pi} \left[\sin\left(\frac{\xi}{2}\right) \text{Si}\left(\frac{\xi}{2}\right) + \cos\left(\frac{\xi}{2}\right) \text{Ci}\left(\frac{\xi}{2}\right) \right] \quad (-\infty < \xi < \infty).
 \end{aligned}
 \tag{2.13}$$

Here Si(ξ) and Ci(ξ) are well-known integral sine and cosine functions, see Bateman and Erdélyi [5],

$$\text{Si}(\xi) = -\int_{\xi}^{\infty} \frac{\sin u}{u} du, \quad \text{Ci}(\xi) = -\int_{\xi}^{\infty} \frac{\cos u}{u} du.
 \tag{2.14}$$

Now in formula (2.12), return to the previous values by (2.8). As a result, we have

$$\tau_-(x) = \frac{h_0}{cd} \int_{-\infty}^{\infty} K\left(\frac{h_0}{c} |x - u|\right) \tau_+(u) du \quad (-\infty < x < \infty) \quad \left(c = \frac{E_s A_s}{Ed}\right).
 \tag{2.15}$$

Further consider a particular case of the external loading $\tau_+(x)$ when at points $x = \pm a$ of the top edge of the stringer, there are two opposite directed horizontal concentrated

forces P , that is when

$$\tau_+(x) = Pd[\delta(x - a) - \delta(x + a)] \quad (0 < a < \infty), \tag{2.16}$$

where $\delta(x)$ is Dirac's well-known Delta function. Admit also that $d_s = d = h$. Then $h_0 = 1$ and formula (2.15) will have the following form:

$$\tau_-(x) = \frac{P}{c} \left[K\left(\frac{|x - a|}{c}\right) - K\left(\frac{|x + a|}{c}\right) \right] \quad (-\infty < x < \infty). \tag{2.17}$$

Now using (2.16) and (2.17) in formula (2.3) and taking into account relation (2.7) for the axial stresses in the stringer, we get

$$\begin{aligned} \sigma_x^s(x) = xp - \frac{P}{\chi h_s^2} \int_x^\infty \left[K\left(\frac{|u - a|}{c}\right) - K\left(\frac{|u + a|}{c}\right) \right] du \\ - \frac{P}{h_s} [H(x - a) - H(x + a)], \quad -\infty < x < \infty; c = \chi h_s, \chi = \frac{E_s}{E}, \end{aligned} \tag{2.18}$$

where $H(x)$ is Heaviside's function. Since the maximum tensile stresses are reached in the stringer in its section, $x = 0$ according to (2.18), we find

$$\max \sigma_x^s = \sigma_x^s(0) = \chi p - \frac{P}{\chi h_s^2} \int_x^\infty \left[K\left(\frac{|u - a|}{c}\right) - K\left(\frac{|u + a|}{c}\right) \right] du + \frac{P}{h_s}. \tag{2.19}$$

The integral entering this formula with the help of the integral presentation of function $K(\xi)$ from (2.13) will obtain the following form:

$$\begin{aligned} I(a) &= \int_0^\infty \left[K\left(\frac{|u - a|}{c}\right) - K\left(\frac{|u + a|}{c}\right) \right] du \\ &= \frac{1}{\pi} \int_0^\infty du \int_0^\infty \frac{\cos[\lambda(\eta - \alpha)]}{1 + 2\lambda} d\lambda - \frac{1}{\pi} \int_0^\infty du \int_0^\infty \frac{\cos[\lambda(\eta + \alpha)]}{1 + 2\lambda} d\lambda \\ &= \frac{2c}{\pi} \lim_{N \rightarrow \infty} \int_0^N d\eta \int_0^\infty \frac{\sin(\lambda\eta) \sin(\lambda\xi)}{1 + 2\lambda} d\lambda \quad (\eta = \frac{u}{c}, \alpha = \frac{a}{c}). \end{aligned} \tag{2.20}$$

Changing the order of the integration and using the Riemann-Lebesgue theorem (see Titchmarsh [23]), we have

$$I(a) = c[1 + 2\omega(\alpha)] \quad \left(\alpha = \frac{a}{c} = \frac{a}{\chi h_s}\right), \tag{2.21a}$$

$$\omega(\alpha) = -\frac{2}{\pi} \int_0^\infty \frac{\sin(\lambda\alpha)}{1 + 2\lambda} d\lambda = \frac{1}{\pi} \left[\cos\left(\frac{\alpha}{2}\right) \text{Si}\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \text{Ci}\left(\frac{\alpha}{2}\right) \right]. \tag{2.21b}$$

Therefore, formula (2.19) with the help of (2.21) is transformed into the form

$$\max \sigma_x^s = \sigma_x^s(0) = \chi p - 2Ph_s^{-1} \omega(\alpha). \tag{2.22}$$

On the base of this equality, we can write the cracking condition of the stringer

$$\chi p - 2Ph_s^{-1} \omega(\alpha) = \sigma_s, \tag{2.23}$$

where σ_s is the brittle strength limit of the material of the stringer under the tension, and the function $\omega(\alpha)$ is given in (2.21). This function by the well-known theorem of Fourier integral theory (see Titchmarsh [23]) is not positive, but the right-hand side of (2.22) or the left-hand side of (2.23) is always positive.

The condition (2.23) can be written in the nondimensional form

$$P_0 = -\frac{k - \chi}{2\omega(\alpha)} \quad \left(P_0 = \frac{P}{h_s p}, k = \frac{\sigma_s}{p} \right). \tag{2.24}$$

Thus, when the external loadings are subjected to condition (2.23) or (2.24), a brittle rupture of the stringer takes place.

Note that if $\tau_+(x) \equiv 0$, the homogeneous equation (2.6) or (2.9) has only the trivial zero solution $\tau_-(x) = \tau_-^*(\xi) \equiv 0$ and on the bases of (2.3) and (2.7), we get

$$\sigma_x^s(x) = \frac{P_\infty}{A_s} = p\chi. \tag{2.25}$$

Then the condition (2.23) goes over to the condition $\chi p = \sigma_s$.

3. Contact of a ruptured infinite stringer with an elastic semi-infinite plate

Consider the following stage of development of the above-described contact problem, when the acting external forces satisfy condition (2.23) or the analogous condition in case of general loading $\tau_+(x)$. In this way, we come to the problem of a contact interaction of a ruptured infinite stringer with an elastic semi-infinite plate (Figure 3.1). In the same statement, which was above, we can derive the determining equation of the problem.

Since in the discussed case $\sigma_x^s(0) = 0$, from (2.1), we get

$$\sigma_x^s(x) = A_s^{-1} \left[d \int_0^x \tau_-(u) du - \int_0^x \tau_+(u) du \right] \quad (-\infty < x < \infty). \tag{3.1}$$

The equality (3.1) can also be obtained immediately considering the equilibrium of portion $[0, x]$ of the right part of the stringer and taking into account the oddness of function $\tau_\pm(x)$. It is evident that $\sigma_x^s(x)$ is the even function. On the other hand, since the stringer at infinity is tensioned by concentrated forces $P_{\pm\infty} = P_\infty$, fulfilling in (3.1) the limiting transition $x \rightarrow \infty$, we will come to the following equality:

$$d \int_0^\infty \tau_-(u) du - \int_0^\infty \tau_+(u) du = P_\infty, \tag{3.2}$$

and here relation (2.7) again takes place.

Thus, in the discussed problem, unknown tangential contact stresses $\tau_-(x)$ must be subjected to condition (3.2).

Further by Hook's law instead of (2.4), we will have

$$\varepsilon_x^s(x) = E_s^{-1} \sigma_x^s(x) = (E_s A_s)^{-1} \left[d \int_0^x \tau_-(u) du - \int_0^x \tau_+(u) du \right] \quad (-\infty < x < \infty), \tag{3.3}$$

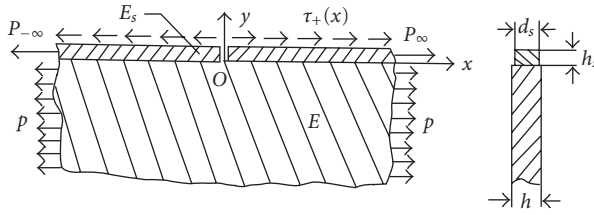


Figure 3.1. Geometry and loading of an elastic semi-infinite plate reinforced on its boundary by the infinite discontinuous stringer ($P_{-\infty} = P_{\infty}$).

and $\epsilon_x(x, 0)$ of the plate is again expressed by the formula (2.5). The condition of contact $\epsilon_x(x, 0) = \epsilon_x^s(x)$ ($-\infty < x < \infty$) reduces to the following integral equation in the given case

$$\frac{2d}{\pi E h} \int_{-\infty}^{\infty} \frac{\tau_-(u) du}{u - x} + \frac{p}{E} = \frac{1}{E_s A_s} \left[d \int_0^x \tau_-(u) du - \int_0^x \tau_+(u) du \right] \quad (-\infty < x < \infty), \quad (3.4)$$

the solution of which must satisfy condition (3.2). Using this condition and (2.7), it is easy to show that (2.6) and (3.4) are identical and, therefore, have the same solution (2.15).

However, taking into account the peculiarity of the given problem determining the integral equation, we will formulate the determinative integral equation on semiaxis $(0, \infty)$ and apply to it the means of Fourier integral sine transform. Such a way of solving of the equation is necessary for substantiating some facts for their use in the future.

For this purpose at first in (3.4), we will go over to new variables by formula (2.8). Then this equation will be transformed to the form

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\tau_-^*(\eta) d\eta}{\eta - \xi} = h_0 \int_0^{\xi} \tau_-^*(\eta) d\eta - h_0 d^{-1} \int_0^{\xi} \tau_+^*(\eta) d\eta - h_0 p \quad (-\infty < \xi < \infty), \quad (3.5)$$

and condition (3.2) allowing for (2.7) will be transformed to the form

$$\int_0^{\infty} \tau_-^*(\eta) d\eta - d^{-1} \int_0^{\infty} \tau_+^*(\eta) d\eta = p. \quad (3.6)$$

Then putting

$$\begin{aligned} X(\xi) &= h_0 \int_0^{\xi} \tau_-^*(\eta) d\eta - h_0 d^{-1} \int_0^{\xi} \tau_+^*(\eta) d\eta - h_0 p \\ &= -h_0 \int_{\xi}^{\infty} \tau_-^*(\eta) d\eta + h_0 d^{-1} \int_{\xi}^{\infty} \tau_+^*(\eta) d\eta \quad (-\infty < \xi < \infty), \end{aligned} \quad (3.7)$$

we write (3.5) in the form

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{X'(\eta) d\eta}{\eta - \xi} = h_0 X(\xi) - \frac{2h_0}{\pi d} \int_{-\infty}^{\infty} \frac{\tau_+^*(\eta) d\eta}{\eta - \xi} \quad (-\infty < \xi < \infty). \quad (3.8)$$

Hence by the formula of Hilbert reverse transform, we find

$$X'(\xi) = -\frac{h_0}{2\pi} \int_{-\infty}^{\infty} \frac{X(\eta) d\eta}{\eta - \xi} - \frac{h_0}{d} \tau_+^*(\xi) \quad (-\infty < \xi < \infty). \quad (3.9)$$

If now we make integration by parts in the last integral with Cauchy kernel and take into account the property of oddness of functions $\tau_{\pm}^*(\xi)$, we will arrive at the following integral equation on the positive semiaxis with respect to function $\tau_{-}^*(\xi)$:

$$\tau_{-}^*(\xi) + \frac{h_0}{2\pi} \int_0^{\infty} \ln \frac{\xi + \eta}{|\xi - \eta|} \tau_{-}^*(\eta) d\eta = \frac{h_0}{2\pi d} \int_0^{\infty} \ln \frac{\xi + \eta}{|\xi - \eta|} \tau_{+}^*(\eta) d\eta \quad (0 < \xi < \infty). \quad (3.10)$$

Here integrating by parts, it is taken into account that according to (3.7), at least $X(\xi) = O(|\xi|^{-\epsilon})$ ($\xi \rightarrow \infty, \epsilon > 0$).

For solving integral equation (3.10), we will use the spectral relation

$$\frac{1}{\pi} \int_0^{\infty} \ln \frac{\xi + \eta}{|\xi - \eta|} \sin \lambda \eta d\eta = \frac{\sin \lambda \xi}{\lambda} \quad (\xi > 0, \lambda > 0), \quad (3.11)$$

following from the well-known formula given in the book by Bateman and Erdélyi [5, page 77, formula 2.6(1)] and, besides, for the further calculations, we introduce

$$\tau_{\pm}^*(\xi) = \int_0^{\infty} T_{\pm}(\lambda) \sin \lambda \xi d\lambda, \quad (3.12)$$

where $T_{+}(\lambda)$ is the known function and $T_{-}(\lambda)$ is the unknown function. Later we introduce the expressions $\tau_{\pm}^*(\xi)$ from (3.12) into (3.10) and change the order of the integration taking into account relation (3.11). In the result, we find

$$T_{-}(\lambda) = \frac{h_0}{d} \frac{T_{+}(\lambda)}{h_0 + 2\lambda} \quad (0 < \lambda < \infty) \quad (3.13)$$

and therefore, by (3.12), we will have

$$\tau_{-}^*(\xi) = \frac{h_0}{d} \int_0^{\infty} \{K[h_0|\xi - \eta|] - K[h_0(\xi + \eta)]\} \tau_{+}^*(\eta) d\eta \quad (0 < \xi < \infty), \quad (3.14)$$

where function $K(\xi)$ is given by formula (2.13). The obtained solution (3.14) of the integral equation (3.10) coincides with (2.12) obtained above.

Taking into account solution (3.14), condition (3.6) transforms into

$$d^{-1} \int_0^{\infty} \left\{ h_0 \int_0^{\infty} [K(h_0|\xi - \eta|) - K(h_0(\xi + \eta))] d\xi - 1 \right\} \tau_{+}^*(\eta) d\eta = p. \quad (3.15)$$

Thus the solution of problem (3.14) must be subjected to condition (3.15).

Now we must note that according to (3.11), the spectrum introduced into (3.10) integral operator considered in space $L_1(0, \infty)$ continuously fills positive semiaxis $(0, \infty)$ and that is why, when $\tau_{+}(x) = \tau_{+}^*(\xi) \equiv 0$, this homogeneous integral equation has a unique zero solution, which, of course, does not satisfy condition (3.6) or (3.15). Therefore, the problem of the contact of a discontinuous infinite stringer with an elastic semi-infinite plate, when only uniformly tensioned at infinity forces affect the system of stringer-plate, does not have a solution unlike the case of continuous stringer.

Further, the question of existence of the solution will be discussed in case of particular load (2.16) and for the simplicity, we will have $h_0 = 1$, as above. Then after returning

to the previous variables by (2.8), formula (3.14) will give solution (2.17), and solution (3.15) will turn into the following:

$$P \int_0^\infty \left[K\left(\frac{|x-a|}{c}\right) - K\left(\frac{x+a}{c}\right) \right] dx = c(P + cp) \quad \left(c = \chi h_s = \frac{h_s E_s}{E} \right). \quad (3.16)$$

Later transforming this relation with the help of (2.21a) and (2.21b), we will get

$$2P_0 \omega(\alpha) = \chi \quad \left(\chi = \frac{E_s}{E}, P_0 = \frac{P}{ph_s} \right). \quad (3.17)$$

Thus, in case of the discussed particular load, the solution of problem (2.17) must satisfy condition (3.17). But it is evident that following the dispositiveness of the function $\omega(\alpha)$, this solution cannot be satisfied and, therefore, the problems again do not have solutions.

But, nevertheless, in order to get the fulfillment of condition (3.17) to available load (2.16), we will add another similar pair of horizontal concentrated forces P_1 placed at points $x = \pm a_1$ on the top edge of the stringer with opposite direction to the one of the initial pair of forces P , therefore instead of (2.16), we will consider the following load:

$$\tau_+(x) = Pd[\delta(x-a) - \delta(x+a)] - P_1 d[\delta(x-a_1) - \delta(x+a_1)]. \quad (3.18)$$

Then instead of problem (2.17), we will have the following solution ($-\infty < x < \infty$):

$$\tau_-(x) = \frac{P}{c} \left[K\left(\frac{|x-a|}{c}\right) - K\left(\frac{x+a}{c}\right) \right] - \frac{P_1}{c} \left[K\left(\frac{|x-a_1|}{c}\right) - K\left(\frac{x+a_1}{c}\right) \right], \quad (3.19)$$

and instead of solution (3.17), we will have the following condition

$$2[P_0 \omega(\alpha) - P_1^* \omega(\alpha_1)] = \chi \quad \left(\chi = \frac{E_s}{E} \right), \quad (3.20)$$

where

$$\alpha = \frac{a}{c} = \frac{a}{\chi h_s}, \quad \alpha_1 = \frac{a_1}{c} = \frac{a_1}{\chi h_s}, \quad P_0 = \frac{P}{ph_s}, \quad P_1^* = \frac{P_1}{ph_s}, \quad (3.21)$$

and function $\omega(\alpha)$ is given by formula (2.21b). It is evident, that by a suitable choice of the parameters P_0 , P_1^* , α , and α_1 , it is possible to obtain the satisfaction of condition (2.4). Note that $\omega(\alpha) \leq 0$ when ($0 < \alpha < \infty$) and $\omega(\alpha) \rightarrow 0$ when $\alpha \rightarrow 0$.

Performing (3.20), we reveal the asymptotic behavior of contact stresses $\tau_-(x)$ from (3.19) in the vicinity of point $x = 0$. With this purpose, we will observe that with the help of (2.13) we can represent formula (3.19), in the form of

$$\tau_-(x) = \frac{2}{\pi c} \left[P \int_0^\infty \frac{\sin(\lambda x/c) \sin(\lambda \alpha)}{1 + 2\lambda} d\lambda - P_1 \int_0^\infty \frac{\sin(\lambda x/c) \sin(\lambda \alpha_1)}{1 + 2\lambda} d\lambda \right] \quad (3.22)$$

$$\left(0 < x < \infty; c = \chi h_s = \frac{h_s E_s}{E} \right).$$

Since

$$\begin{aligned} \frac{\sin(\lambda\alpha)}{1+2\lambda} &= \sin(\lambda\alpha)(1+2\lambda)^{-1} = \frac{\sin(\lambda\alpha)}{2\lambda} \left(1 - \frac{1}{2\lambda} + \frac{1}{4\lambda^2} - \dots \right) \\ &\approx \frac{\sin(\lambda\alpha)}{2\lambda} (\lambda \rightarrow \infty), \end{aligned} \quad (3.23)$$

by the well-known results reflected in the works of Lighthill [15] and according to (3.11), we will get

$$\begin{aligned} \tau_-(x) &\approx \frac{P}{2\pi c} \ln \frac{\alpha+x/c}{|\alpha-x/c|} - \frac{P_1}{2\pi c} \ln \frac{\alpha_1+x/c}{|\alpha_1-x/c|} \\ &= \frac{P}{2\pi c} \ln \frac{1+x/a}{|1-x/a|} - \frac{P_1}{2\pi c} \ln \frac{1+a_1}{|1-x/a_1|} \quad (x \rightarrow 0). \end{aligned} \quad (3.24)$$

Then taking into consideration the known MacLaurin series for function $\ln(1+x)$ ($|x| < 1$), we finally find

$$\tau_-(x) \approx \left(\frac{P}{a} - \frac{P_1}{a_1} \right) \frac{x}{\pi c} \quad (x \rightarrow 0), \quad (3.25)$$

from which it follows that when observing condition (3.20), the contact stresses at the point of the stringer break are equal to zero.

On the basis of this statement, it may be confirmed that the solution of the problem of the contact of an infinite discontinuous stringer with an elastic semi-infinite plate, generally, does not exist. It exists only at observance of condition (3.15) or in particular, of condition (3.20) and then the problem has the solution in form of (3.14) or (2.15) or (3.19), correspondingly, which coincides with the corresponding solution in case of the infinite continuous stringer. It means that at restriction of type (3.15) or (3.20) on the external forces, the contact of the discontinuous infinite stringer with the plate is possible without propagation of the crack into the inner part of the plate. In the opposite case, when these restrictions are not maintained after the stringer cracking, the crack is sure to be propagated into the inner part of the plate. In that case, a new problem arises, which may be of interest for further investigations.

It must be noticed that the indicated difficulties, connected with the question of the existence of the problem solution, are removed if we define the model of one-dimensional elastic continuum of the stringer and consider the discontinuous stringer within the limits of two-dimensional theory of elasticity as two half-planes.

4. Conclusions

In case of particular load (2.16) for calculation of nondimensional force P_0 by formula (2.24), it is necessary to admit that $k = 10$ for a brittle cracking of the continuous stringer. The results of the calculations by this formula for different values of parameters χ and h_s/a are given in Table 4.1. Parameter χ characterizes the relative rigidity of the stringer-plate system and parameter h_s/a is the relative distance of right-concentrated force P from the origin of coordinates. The analysis of the data of Table 4.1 shows that by fixed

Table 4.1. Variation of the undimensional force $P_0 = -((k - \chi)/2\omega(\alpha))$ ($P_0 = P/h_s p$, $k = \sigma_s/p$, $\alpha = a/\chi h_s$) necessary for the brittle cracking of the continuous infinite stringer depending on related rigidity of the stringer-plate system $\chi = E_s/E$ and on the geometric parameter h_s/a when at the points $x = \pm a$ of the top edge of the stringer, there act two equal and opposite directed horizontal concentrated forces P (formula (2.24)).

h_s/a	χ							
	1	2	3	4	5	6	7	8
0.03	474.573	214.850	128.615	85.218	58.684	40.452	26.895	16.222
0.05	287.993	133.582	82.157	55.850	39.366	27.707	18.766	11.509
0.07	208.822	99.504	62.755	43.583	31.277	22.351	15.337	9.512
0.1	150.280	74.467	48.487	34.526	25.276	18.358	12.769	8.010
0.3	62.340	36.717	26.662	20.469	15.840	12.010	8.645	5.579

quantity χ with decreasing of parameter h_s/a , that is, with increasing of distance a , value P_0 increases. It means that with the removal of concentrated tensioned forces P from the origin of coordinates for the stringer breaking, more forces will be required. On the contrary, by fixed value h_s/a with the increase of the parameter χ , a decrease of the value P_0 takes place.

In conclusion, we should underline that in the problem considered here, we simulate the problem of cracking of a two-layered composite of a rectangular form. The problem of such type with mixed boundary conditions theoretically brings us to new statements and extends the class of the boundary problems of the mathematical theory of elasticity and of adjacent branches of the mechanics of solid media. As to the applied sciences, such problems may be encountered in various fields of the applied mechanics, for example, in the mechanics of composites, in structural mechanics, and so forth. The investigations of the analogous problems are of particular interest theoretically as well as practically, and can be carried out by the methods similar to those that are used in the given work.

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