Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2007, Article ID 10120, 11 pages
doi:10.1155/2007/10120

Research Article

Variational Problems with Moving Boundaries Using Decomposition Method

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Received 12 December 2006; Revised 23 April 2007; Accepted 25 July 2007

Recommended by T. Zolezzi

The aim of this paper is to present a numerical method for solving variational problems with moving boundaries. We apply Adomian decomposition method on the Euler-Lagrange equation with boundary conditions that yield from transversality conditions and natural boundary conditions.

1. Introduction

In many problems arising in mathematics, physics, engineering, economics, and other sciences, it is necessary to minimize amounts of a certain functional. Because of the important role of this subject, considerable attention has been devoted to these kinds of problems. Such problems are called variational problems, see [1, 2].

The simplest form of a variational problem can be considered as

$$J[y] = \int_a^b f\left(x, y(x), y'(x)\right) dx, \tag{1.1}$$

where J is called a functional, and its minimum must be found. Functional J can be considered by two kinds of boundary conditions. The first kind, we refer to as fixed boundary problems, the admissible function $y(x)$ must satisfy the boundary conditions

$$y(a) = y_a, \qquad y(b) = y_b. \tag{1.2}$$

The second kind, we call problems with moving boundaries and concern ourselves with them, is variational problems for which at least one of the boundary points of the admissible function is movable along a boundary curve.

The variational problem where the beginning point and the endpoint are fixed is often referred to as point-point problem, and the problems with variable boundaries as point-curve, curve-point, and curve-curve problems.

A necessary condition for the admissible solutions of such problems is to satisfy the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \tag{1.3}$$

For the point-point problem, Euler-Lagrange equation must be considered by the boundary conditions (1.2). For the problems with variable boundaries, we have two cases.

Type 1: As the first case, we consider problems for which at least one of the boundary points move freely along a line parallel to the y-axis, in fact at this point $y(x)$ is not specified. In this case, admissible functions must fulfill the natural boundary conditions (2.3) and (2.4), (or only one of them).

Type 2: For the second case, we will let the beginning and endpoints (or only one of them) move freely on given curves $y = \psi(x)$, $y = \varphi(x)$. In this case, the admissible function $y(x)$ and points a and b (or one of them) must satisfy the necessary conditions (2.5) and (2.6) called transversality conditions.

The decomposition method proposed by G. Adomian is a power tool to investigate various models in applied science, engineering, and so forth, (see [3, 4]). The method provides the solution in a rapidly convergent series with components that can be computed recursively, see [5, 6] for convergence. Adomian decomposition is applied on various kinds of problems, for example, see [7–24] for some recent works. It produces an efficient solution with high accuracy and minimal calculation for finding the approximate solution of the linear and nonlinear problems. Dehghan and Tatari in [9] used Adomian decomposition to solve variational problems with fixed boundaries. In this work we solve variational problems with moving boundaries by this method. In fact, we apply Adomian decomposition method to solve the Euler-Lagrange equation considered by the boundary conditions which yield from natural boundary conditions and transversality conditions. In the last section, we use some examples to show the numerical treatment of the proposed method.

2. Statement of the problem

2.1. Boundary conditions. The first type of problem here is one for which at least at one of the endpoints, $y(x)$ is not specified. In this case, all admissible functions have the same domain of definition $[a,b]$ and satisfy the Euler-Lagrange equation in this interval. Furthermore, such function must satisfy conditions called natural boundary conditions prescribed in the following theorem.

Theorem 2.1. *Suppose the function $y = y_0(x) \in C^1[a,b]$ yield a relative minimum of the functional (1.1) for which*

$$y(a) = y_a \text{ is given}, \qquad y(b) \text{ is arbitrary} \quad (\text{free right endpoint}) \tag{2.1}$$

or

$$y(a),\ y(b)\ \text{are arbitrary} \quad (\text{free endpoints}). \tag{2.2}$$

Then $y_0(x)$ satisfies, respectively, the following natural boundary conditions:

$$\frac{\partial f}{\partial y'}(b, y_0(b), y_0'(b)) = 0 \quad (\text{free right endpoint}) \tag{2.3}$$

or

$$\frac{\partial f}{\partial y'}(a, y_0(a), y_0'(a)) = 0 = \frac{\partial f}{\partial y'}(b, y_0(b), y_0'(b)) \quad (\text{free endpoints}). \tag{2.4}$$

We now consider a generalized form of the problem. We let endpoints move freely on given curves.

In this case, we seek for a function $y(x)$, which emanates at some $x = a$ from the curve $y = \psi(x)$ and terminates for some $x = b$ on the curve $y = \varphi(x)$ and minimizes the functional (1.1). In this problem also the points a, b are not known, they must satisfy the necessary conditions called transversality conditions, prescribed in the following theorem.

Theorem 2.2. *If the function $y = y_0(x) \in C^1[a,b]$, which emanates at some $x = a$ from the curve $y = \psi(x) \in C^1(-\infty,\infty)$ and terminates for some $x = b$ on the curve $y = \varphi(x) \in C^1(-\infty,\infty)$, yields a relative minimum for functional (1.1), where $f \in C^1(R)$, R being a domain in the (x, y, y') space that contains all lineal elements of $y = y_0(x)$, then it is necessary that $y = y_0(x)$ must satisfy the Euler-Lagrange equation in the interval $[a,b]$ and that at the point of departure and the point of arrival, the transversality conditions*

$$\frac{\partial f}{\partial y'}(a, y_0(a), y_0'(a))\left(\psi'(a) - y_0'(a)\right) + f(a, y_0(a), y_0'(a)) = 0, \tag{2.5}$$

$$\frac{\partial f}{\partial y'}(b, y_0(b), y_0'(b))\left(\varphi'(b) - y_0'(b)\right) + f(b, y_0(b), y_0'(b)) = 0 \tag{2.6}$$

are satisfied. In the case that one of the points is fixed, then the transversality condition has to hold at the other point.

We see that the natural boundary conditions (2.3) and (2.4) may be viewed as a special case of the transversality conditions for $\psi'(a) = \infty$ and $\varphi'(b) = \infty$.

If the initial curve or the terminal curve (or both) is a horizontal line, that is, if $\psi'(x) = 0$ or $\varphi'(x) = 0$ (or both), then we obtain from (2.5) that

$$\frac{\partial f}{\partial y'}(a, y_0(a), y_0'(a))\, y_0'(a) - f(a, y_0(a), y_0'(a)) = 0, \tag{2.7}$$

and an analogous condition for the right endpoint holds.

One can consider transversality conditions for the problems with more than one unknown function.

For example, in the two-dimensional case we seek a vector function $\hat{y} = \hat{y}(x) = (y_1(x), y_2(x))$ such that it minimizes

$$J[y_1, y_2] = \int_a^b f(x, y_1(x), y_2(x), y_1'(x), y_2'(x))dx, \tag{2.8}$$

in which $y_1(a) = y_{1,a}$, $y_2(a) = y_{2,a}$ and the endpoint lies on a two-dimensional surface that is given by $x = u(y_1, y_2)$. Here, the necessary condition for the extremum of the functional (2.8) is to satisfy the following system of second-order differential equations:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \quad i = 1, 2; \tag{2.9}$$

and the terminal point $x = b$ must satisfy the following transversality conditions:

$$\begin{aligned} &\left(\frac{\partial u}{\partial y_1} f + \left(1 - \frac{\partial u}{\partial y_1} y_1^{0'} - \frac{\partial u}{\partial y_2} y_2^{0'}\right)\frac{\partial f}{\partial y'_1}\right)\bigg|_{x=b} = 0, \\ &\left(\frac{\partial u}{\partial y_2} f + \left(1 - \frac{\partial u}{\partial y_1} y_1^{0'} - \frac{\partial u}{\partial y_2} y_2^{0'}\right)\frac{\partial f}{\partial y'_2}\right)\bigg|_{x=b} = 0, \end{aligned} \tag{2.10}$$

in which $\hat{y} = \hat{y}_0(x) = (y_1^0(x), y_2^0(x))$ is an admissible vector function. For further information on transversality conditions, specially for the proofs of Theorems 2.1 and 2.2 and condition (2.10), see [2].

2.2. Solution of problem using Adomian decomposition method. The Euler-Lagrange (1.3) can be considered in an operator form

$$L(y) - N(y) = g \tag{2.11}$$

for $a \le x \le b$, where $L = d^2/dx^2$ is the second-order derivative operator, N is an operator (which may be nonlinear), and g is a given function. We consider the inverse operator in the following form:

$$L^{-1}(h(x)) = \int_a^x \int_a^{x_2} h(x_1)dx_1\, dx_2. \tag{2.12}$$

Applying operator L^{-1} to both sides of (2.11), we have

$$y(x) - y(a) - y'(a)x + y'(a)a = L^{-1}N(y) + L^{-1}g. \tag{2.13}$$

So that,

$$y(x) = \alpha + Ax - Aa + L^{-1}g + L^{-1}N(y). \tag{2.14}$$

Now according to the decomposition procedure of Adomian, we construct the unknown function $y(x)$ by a sum of components defined by the following series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{2.15}$$

and consider the nonlinear expression $N(y)$ by the infinite series of the polynomials given by $N(y) = \sum_{n=0}^{\infty} N_n$ where components N_n have the following form:

$$N_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left[\sum_{k=0}^{\infty}\lambda^k y_k\right]\right]_{\lambda=0}, \quad n \geq 0. \tag{2.16}$$

Notice that if N be a linear operator then we have $N_n = y_n$. Now we have the following recursive relations:

$$\begin{aligned} y_0(x) &= \alpha + Ax + L^{-1}g(x), \\ y_{n+1}(x) &= L^{-1}N_n, \quad n \geq 0. \end{aligned} \tag{2.17}$$

Based on the Adomian decomposition method, we constructed the solution $y(x)$ as

$$y = \lim_{n\to\infty}\phi_n, \tag{2.18}$$

where the $n+1$ term approximation of the solution is defined in the following form:

$$\phi_n = \sum_{k=0}^{n} y_k(x), \quad n > 0. \tag{2.19}$$

The solution here is given in a series form that converges rapidly in physical problems.

Applying the decomposition procedure of Adomian, we find the series solution of $y(x)$ follows with constants α, A which are unknown. For the determination of these constants, we have two cases.

Case (1): For the variational problems of type 1, we impose the boundary conditions that yield from natural boundary conditions (2.3) and (2.4) to the obtained approximation of the solution (2.19) which results equations in α, A. Remark that for the point-curve and curve-point problems of this type only constant A is undetermined, also we have one equation in A. However, by solving these equations we find α, A.

Case (2): Consider variational problems of type 2, for example, a point-curve problem. In such problem, the point a and constant α are known but the point b and the constant A are unknown. In this case, we have two equations, one equation yields from transversality condition at $x = b$ and another equation yields from the intersection of $y(x)$ and the terminal curve $y = \varphi(x)$ at $x = b$. By solving this system of equations, which is usually nonlinear, we find both b and A and the solution of the Euler-Lagrange equation follows immediately.

Note that for a curve-curve problem of this type, we have four unknown a, b, α, A. In this case, we have four equations. Two equations yield from transversality conditions at $x = a, b$ and two equations obtained from the intersection of $y(x)$ by $y = \psi(x)$ and $y = \varphi(x)$ at $x = a$ and $x = b$, respectively. Solution of this system of equations determines these unknown.

3. Numerical tests

To give an obvious overview of the Adomian decomposition method on the variational problems with moving boundaries, we present some examples.

3.1. Example 1 (Ramsey growth model). Consider the following variational problem:

$$J[y] = \int_0^T a\left(by(t) - y'(t) - c^*\right)^2 dt, \tag{3.1}$$

in which $a, b > 0$, $C^* > 0$ and $y(t)$ is the amount of a capital at time t (see [1] for further information on this economic model).

Here, the capital stock $y(0)$ at the initial time $t = 0$ of the planing period is assumed to be known: $y(0) = y_0$; on the other hand, the planner will not want to prescribe how large the capital will be at time $t = T$. We have therefore a variational problem with free right endpoint.

For this numerical example, we let $a = b = c^* = 1$, $T = 1$, and $y_0 = 2$ which has the analytical solution $y(t) = 1 + e^t$.

The corresponding Euler-Lagrange equation is

$$y''(t) - y(t) + 1 = 0. \tag{3.2}$$

Now we consider the natural boundary condition (2.3) at $t = 1$:

$$\frac{\partial f}{\partial y'}\left(1, y(1), y'(1)\right) = -2\left(y(t) - y'(t) - 1\right)\big|_{t=1} = 0. \tag{3.3}$$

Therefore, we have the following boundary conditions for (3.2):

$$y(0) = 2, \qquad y(1) - y'(1) - 1 = 0. \tag{3.4}$$

Using the operator form of (3.2), we have

$$\begin{gathered} Ly = y - 1, \\ y(x) = 2 + Ax - L^{-1}(1) + L^{-1}\left(y(x)\right). \end{gathered} \tag{3.5}$$

Applying the Adomian decomposition, we have

$$\sum_{n=0}^{\infty} y_n(x) = 2 + Ax - \frac{x^2}{2} + L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right). \tag{3.6}$$

Therefore, we use the reccursive relations

$$\begin{gathered} y_0(x) = 2 + Ax - \frac{x^2}{2}, \\ y_{n+1}(x) = L^{-1}\left(y_n(x)\right), \quad n \geq 0. \end{gathered} \tag{3.7}$$

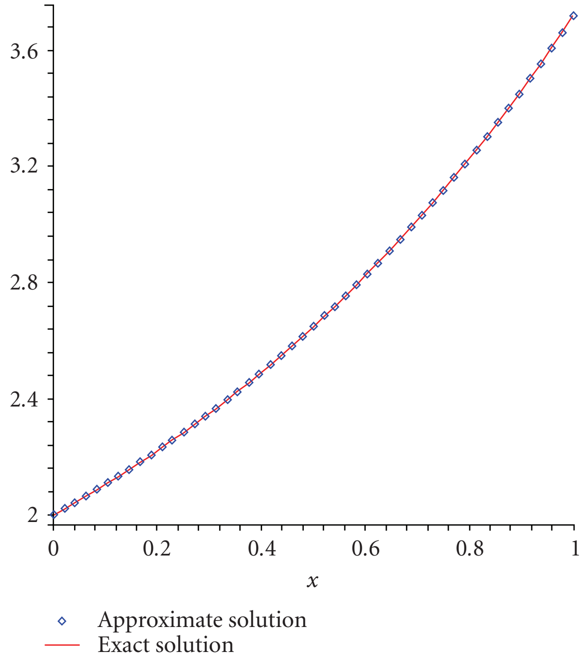

FIGURE 3.1. Plot of $y(x)$ and $\phi_3(x)$ in example 1.

Now we use $\phi_n(x)$ as the approximation of $y(x)$, for example, for $n = 3$ we have

$$\begin{aligned}\phi_3(x) = 2 + Ax + \frac{1}{2}x^2 + \frac{1}{6}Ax^3 + \frac{1}{24}x^4 + \frac{1}{120}Ax^5 \\ + \frac{1}{720}x^6 + \frac{1}{5040}Ax^7 - \frac{1}{40320}x^8.\end{aligned} \tag{3.8}$$

For the determination of A, we solve the equation $\phi_3(1) - \phi_3'(1) - 1 = 0$. Figure 3.1 shows $y(x)$ and $\phi_3(x)$ and Table 3.1 shows the error of ϕ_3.

3.2. Example 2 (Fermat's principle). We consider in this example the problem of a light ray emanating from a fixed point P_a, propagating in a plane through a medium with $n(x, y) > 0$ as an index of refraction, and terminating at some point $x = b$ on the curve $y = \varphi(x)$.

By Fermat's principle, the light ray will take the path that minimizes the total traveling time from P_a to $x = b$ on $y = (x)$, that is,

$$J[y] = \int_a^b n(x, y(x))\sqrt{1 + y'^2(x)}dx. \tag{3.9}$$

Observe that the Brachistocherone problem, the minimal surfaces of revolution and the shortest path problem are special cases of Fermat's principle for $n(x, y) = 1/\sqrt{y}, n(x, y) = y$, and $n(x, y) = 1$, respectively.

TABLE 3.1. Error ($e(x) = |y(x) - \phi_3(x)|$) in Example 1.

x	0	0.2	0.4	0.6	0.8	1
$e(x)$	0	.203617e-3	.415371e-3	.643003e-3	.889442e-3	.1135848e-2

The transversality condition in this case is of the form

$$\left(n(x,y)\sqrt{1+y'^2} + \frac{n(x,y)y'}{\sqrt{1+y'^2}}(\varphi' - y')\right)\Bigg|_{x=b} = 0 \tag{3.10}$$

or

$$\left(\frac{n(x,y)(1+\phi' y')}{\sqrt{1+y'^2}}\right)\Bigg|_{x=b} = 0. \tag{3.11}$$

Finally, we have $y'(b) = -1/\varphi'(b)$, that is, the transversality condition is in this case reduced to the orthogonality condition.

For this numerical example, we let $n(x,y) = 1 + y$, $P_a = (0,0)$, and $y = \varphi(x) = x+1$ which has the analytical solution $y(x) = -1 + d\cosh(x/d)$, in which d, b must satisfy transversality condition at $x = b$. The corresponding Euler-Lagrange equation for this problem is

$$y''(x) = \frac{1+y'^2(x)}{1+y(x)}. \tag{3.12}$$

By using (2.16), we obtain the following Adomian polynomials for $N(y) = (1+y'^2)/(1+y)$:

$$\begin{gathered} N_0(y_0) = \frac{1+y_0'^2}{1+y_0}, \qquad N_1(y_0,y_1) = \frac{2y_1'y_0'(1+y_0) - y_1(1+y_0'^2)}{(1+y_0)^2}, \\ N_2(y_0,y_1,y_2) = \frac{B}{2(1+y_0)^4}, \end{gathered} \tag{3.13}$$

in which

$$\begin{aligned} B = {} & (1+y_0)^2\left(2y_1'^2(1+y_0) + 4y_0'y_2'(1+y_0) + 2y_0'y_1'y_1 - 2y_2(1+y_0'^2) - 2y_1y_1'y_0'\right) \\ & -2y_1(1+y_0)\left(2y_0'y_1'(1+y_0) - y_1(1+y_0'^2)\right). \end{aligned} \tag{3.14}$$

Now we have the following recursive relations:

$$y_0(x) = Ax, \qquad y_{n+1}(x) = L^{-1}\left(N_n(y_0,\dots,y_n)\right), \quad n \ge 0. \tag{3.15}$$

In this example, we use 2-term approximation of the solution, $\phi_2(x)$. For the determination of A and b, we solve the system of two equations $\phi_2(b) = b+1$ and $\phi_2'(b) = -1$.

Figure 3.2 shows $y(x)$ and $\phi_2(x)$ and Table 3.2 shows the error of ϕ_2.

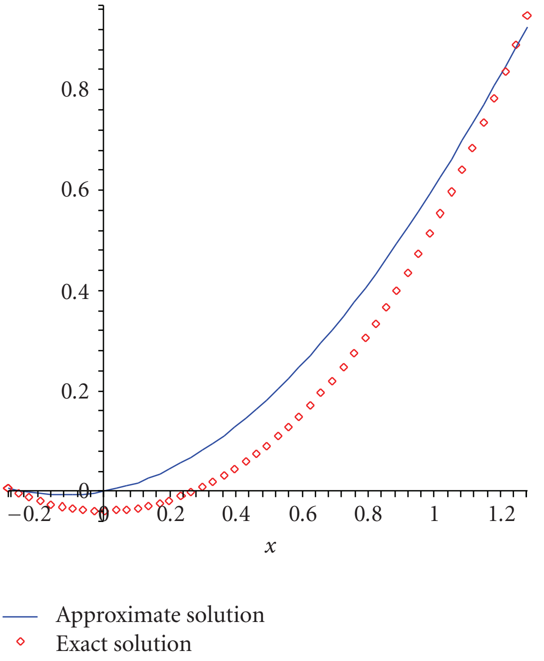

FIGURE 3.2. Plot of $y(x)$ and $\phi_2(x)$ in example 2.

TABLE 3.2. Error $(e(x) = |y(x) - \phi_2(x)|)$ in Example 2.

x	-0.3	0	0.3	0.6	1.25
$e(x)$	.68239187e-3	.38763200e-1	.72982030e-1	.9805628e-1	.6617189e-2

4. Conclusion

In this work, the Adomian decomposition method was successfully used to solve variational problems with moving boundaries. We solve Euler-Lagrange equation with the boundary conditions that yield from natural boundary conditions and transversality conditions, using Adomian method. It is important to note that this method does not require any disceretization or linearization and is not affected by computation roundoff errors.

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