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Research Article

On the Parametric Solution to the Second-Order Sylvester Matrix Equation $EVF^2 - AVF - CV = BW$

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This paper considers the solution to a class of the second-order Sylvester matrix equation $EVF^2 - AVF - CV = BW$. Under the controllability of the matrix triple (E,A,B), a complete, general, and explicit parametric solution to the second-order Sylvester matrix equation, with the matrix F in a diagonal form, is proposed. The results provide great convenience to the analysis of the solution to the second-order Sylvester matrix equation, and can perform important functions in many analysis and design problems in control systems theory. As a demonstration, an illustrative example is given to show the effectiveness of the proposed solution.

1. Introduction

In vibration and structure analysis, second-order linear systems capture the dynamic behaviour of many natural phenomena and have found wide applications, whose state equation can be given as follows:

$$E\ddot{q} - A\dot{q} - Cq = Bu, \tag{1.1}$$

where $q \in R^n$ is the generalized coordinate vector, $u \in R^r$ is the input vector, and E, A, B, C are matrices of appropriate dimensions. The matrices in (1.1) satisfy the following assumption.

Assumption 1.1. The matrix triple (E,A,B) is R-controllable, that is,

$$\operatorname{rank}[A - sE \quad B] = n, \quad \forall s \in C. \tag{1.2}$$

Assumption 1.2. The matrix B is of full rank, that is, $\operatorname{rank}(B) = r$.

Associated with the second-order linear system (1.1), a generalized second-order Sylvester matrix equation can be written in the form of

$$EVF^2 - AVF - CV = BW, \tag{1.3}$$

where $E, A, C \in R^{n\times n}$, $B \in R^{n\times r}$ and $F \in R^{p\times p}$ are known matrices; matrices $W \in R^{r\times p}$ and $V \in R^{n\times p}$ are to be determined.

When $E = 0_n$, $A = -I_n$, the generalized second-order Sylvester matrix equation (1.3) becomes the following generalized Sylvester matrix equation

$$VF - CV = BW. \tag{1.4}$$

When $E = 0_n$, $A = -B = -I_n$, $W = -Q$, the generalized second-order Sylvester matrix equation (1.3) becomes the following normal Sylvester matrix equation

$$CV - VF = Q. \tag{1.5}$$

Further, if we let $C = -F^T$ in (1.5), the above normal Sylvester matrix equation becomes the following well-known Lyapunov matrix equation

$$F^T V + VF = -Q. \tag{1.6}$$

Thus we can find that the second-order Sylvester equation (1.3) is more general than the first-order Sylvester equations (1.4) and (1.5). As we all know, the Sylvester equation is directly concerned with some control problems for linear systems such as eigenvalue assignment [1, 2], observer design [3], eigenstructure assignment design [4–6], constrained control [7], and so forth, and has been studied by many authors (see [3–6], and the references therein).

A solution to the normal Sylvester matrix equation (1.5) has been studied by several researchers. Jameson [8] and Souza and Bhattacharyya [9] gave solutions to this matrix equation in terms of the controllability and observability matrices of some matrix pairs, and Hartwig [10] gave a solution to this equation in terms of the inverse of the related Sylvester resultant, while Jones and Lew [11] presented a solution to this equation in terms of the principal idempotents and nilpotents of the coefficient matrices. Hearon [12] considered the case of Q as being a rank one matrix, and presented some conditions for the matrix V to be nonsingular. For the Lyapunov matrix equation (1.6) with the matrix C in companion form, many authors have considered the solution. Particularly, Sreeram and Agathoklis [13] presented an iterative method based on the Routh's table. Zhou and Duan [14] presented an explicit solution to the generalized Sylvester matrix equation (1.4), and a simple and effective approach for parametric pole assignment is proposed as a demonstration. Duan and Wang [15] proposed two analytical general solutions of (1.3), and by utilizing these results, they investigated many control problems such as the robust control problem [16], the eigenstructure assignment problem [17, 18], the model reference control problem [19], the reconfiguring control problem [20], and so on.

Different from the condition in [15], this paper provides another type of general parametric solutions to the second-order Sylvester matrix equation (1.3). In the next section, some preliminaries to be used in this paper are given. In Sections 3 and 4, the general solutions to the generalized Sylvester matrix equation (1.3) are presented in two cases: the undetermined diagonal matrix F and the determined diagonal matrix F. As a demonstration of the proposed solutions to the second-order Sylvester matrix equation, a numerical example is illustrated by utilizing the proposed two algorithms, and general complete parametric forms for the solution to the second-order Sylvester matrix equation are established in Section 5. In Section 6, concluding remarks are drawn.

2. Preliminaries

If the matrix $F \in R^{p\times p}$ in the matrix equation (1.3) is not diagonal, there must exist a nonsingular matrix $M \in R^{p\times p}$ such that

$$MFM^{-1} = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_p) = \widetilde{F}, \tag{2.1}$$

where $\lambda_1,\lambda_2,\ldots,\lambda_p$ are the eigenvalues of F. Thus (1.3) can be changed into the following form:

$$EV(M^{-1}\widetilde{F}M)^2 - AVM^{-1}\widetilde{F}M - CV = BW \tag{2.2}$$

or

$$EVM^{-1}\widetilde{F}^2 - AVM^{-1}\widetilde{F} - CVM^{-1} = BWM^{-1}. \tag{2.3}$$

Denote

$$VM^{-1} = \hat{V}, \qquad WM^{-1} = \widehat{W}. \tag{2.4}$$

Then the above equation is changed into

$$E\hat{V}\widetilde{F}^2 - A\hat{V}\widetilde{F} - C\hat{V} = B\widehat{W}, \tag{2.5}$$

where $\widetilde{F} = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_p)$ is a diagonal Jordan form.

Therefore, without loss of generality, we assume that the matrix F in (1.3) is in a diagonal Jordan form. Let $\Gamma = \{s_i, s_i \in C,\ \operatorname{re}(s_i) < 0,\ i = 1,2,\ldots,p\}$, which is symmetric about the real axis and whose elements are distinct, be the set of eigenvalues of F. Thus the diagonal Jordan matrix F can be written in the following form:

$$F = \operatorname{diag}(s_1,s_2,\ldots,s_p). \tag{2.6}$$

Denote

$$\widetilde{V} = VF, \tag{2.7}$$

then (1.3) can be further changed into

$$E\widetilde{V}F - A\widetilde{V} - CV = BW. \tag{2.8}$$

The matrices V, $\widetilde{V}$, and W can be written by their columns as

$$\begin{aligned} V &= \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}, \\ \widetilde{V} &= \begin{bmatrix} \widetilde{v}_1 & \widetilde{v}_2 & \cdots & \widetilde{v}_p \end{bmatrix}, \\ W &= \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix}, \end{aligned} \tag{2.9}$$

then (2.8) can be decomposed into

$$E\widetilde{v}_i s_i - A\widetilde{v}_i - Cv_i = Bw_i, \quad i = 1,2,\ldots,p, \tag{2.10a}$$

or

$$\begin{bmatrix} A - s_iE & B \end{bmatrix} \begin{bmatrix} \widetilde{v}_i \\ w_i \end{bmatrix} = -Cv_i, \quad i = 1,2,\ldots,p; \tag{2.10b}$$

then the problem to solve the second-order Sylvester equation (1.3) can be changed into the following equivalent problem.

Problem SOS (second-order Sylvester): Given matrices E, A, B, C, and F with appropriate dimensions in the second-order Sylvester matrix equation (1.3), then find the solutions V and W to equations (2.7) and (2.8).

In the next two sections, we will consider the solution to the second-order Sylvester matrix equation (1.3) in two cases: one is the case of the undetermined diagonal matrix F, and the other is the case of the determined diagonal matrix F.

3. Case of the undetermined diagonal matrix F

If the matrix triple (E,A,B) is R-controllable, there exists a pair of unimodular matrices $P(s) \in R^{n\times n}[s]$ and $Q(s) \in R^{(n+r)\times(n+r)}[s]$, satisfying

$$P(s)\begin{bmatrix} A - sE & B \end{bmatrix} Q(s) = \begin{bmatrix} 0 & I_n \end{bmatrix}, \quad \forall s \in C. \tag{3.1}$$

Partition $Q(s)$ as follows:

$$Q(s) = \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ Q_{21}(s) & Q_{22}(s) \end{bmatrix}, \tag{3.2}$$

where $Q_{11}(s) \in R^{n\times r}[s]$, $Q_{12}(s) \in R^{n\times n}[s]$, $Q_{21}(s) \in R^{r\times r}[s]$, and $Q_{22}(s) \in R^{r\times n}[s]$. Then we have the following lemma, which gives the solutions to (2.8).

Lemma 3.1. *Assume that the matrix triple (E,A,B) is R-controllable, then all the solutions $\widetilde{V}$ and W in (2.8) can be given by their column vectors as*

$$\begin{bmatrix} \widetilde{v}_i \\ w_j \end{bmatrix} = \begin{bmatrix} Q_{11}(s_i) & Q_{12}(s) \\ Q_{21}(s_i) & Q_{22}(s) \end{bmatrix} \begin{bmatrix} f_i \\ -P(s_i)Cv_i \end{bmatrix}, \quad i = 1,2,\ldots,p, \tag{3.3a}$$

or

$$\begin{aligned}\tilde{v}_i &= Q_{11}(s_i)f_i - Q_{12}(s_i)P(s_i)Cv_i\\ w_i &= Q_{21}(s_i)f_i - Q_{22}(s_i)P(s_i)Cv_i,\end{aligned} \qquad i = 1,2,\dots,p, \tag{3.3b}$$

where matrices $Q_{11}(s) \in R^{n\times r}[s]$, $Q_{12}(s) \in R^{n\times n}[s]$, $Q_{21}(s) \in R^{r\times r}[s]$, and $Q_{22}(s) \in R^{r\times n}[s]$ satisfy (3.1) and (3.2); $f_i \in C^r$, $i = 1,2,\dots,p$, are a group of free parametric vectors.

Proof. Substitute (3.2) into (3.1), we can obtain

$$[A - sE \quad B]\begin{bmatrix} Q_{11}(s)\\ Q_{21}(s)\end{bmatrix} = 0, \tag{3.4}$$

$$P(s)[A - sE \quad B]\begin{bmatrix} Q_{12}(s)\\ Q_{22}(s)\end{bmatrix} = I_n. \tag{3.5}$$

It is clear to see that (3.5) is equivalent with the following equation

$$[A - sE \quad B]\begin{bmatrix} Q_{12}(s)\\ Q_{22}(s)\end{bmatrix}P(s) = I_n. \tag{3.6}$$

By utilizing (3.3a), we can obtain

$$\begin{aligned}(A - s_iE)\tilde{v}_i + Bw_i &= [A - s_iE \quad B]\begin{bmatrix}\tilde{v}_i\\ w_i\end{bmatrix}\\ &= [A - s_iE \quad B]\begin{bmatrix} Q_{11}(s_i) & Q_{12}(s_i)\\ Q_{21}(s_i) & Q_{22}(s_i)\end{bmatrix}\begin{bmatrix} f_i\\ -P(s_i)Cv_i\end{bmatrix} = -Cv_i,\end{aligned} \tag{3.7}$$

thus (3.3) are the solutions to (2.8). □

Now let us show that the vectors $\tilde{v}_i$ and w_i satisfying (2.10a) can be expressed in the form of (3.3a). Premultiplying by $P(s_i)$ both sides of (2.10b) and using (3.1) yield

$$[0 \quad I_n]Q^{-1}(s_i)\begin{bmatrix}\tilde{v}_i\\ w_i\end{bmatrix} = -P(s_i)Cv_i, \quad i = 1,2,\dots,p. \tag{3.8}$$

Let

$$\begin{bmatrix} f_i\\ e_i\end{bmatrix} = Q^{-1}(s_i)\begin{bmatrix}\tilde{v}_i\\ w_i\end{bmatrix}, \quad i = 1,2,\dots,p, \tag{3.9}$$

then (3.8) becomes

$$[0 \quad I_n]\begin{bmatrix} f_i\\ e_i\end{bmatrix} = -P(s_i)Cv_i, \quad i = 1,2,\dots,p, \tag{3.10}$$

which produces

$$e_i = -P(s_i)Cv_i, \quad i = 1,2,\dots,p. \tag{3.11}$$

Substituting (3.11) into (3.9), we can get (3.3a).

Finally, noticing that the equivalence between (3.3a) and (3.3b) is obvious, the proof is completed.

For control applications, the second-order linear system (1.1) is usually transformed into the following first-order linear system

$$E'\dot{x} = A'x + B'u, \tag{3.12}$$

where

$$E' = \begin{bmatrix} I_n & 0 \\ 0 & E \end{bmatrix}, \qquad A' = \begin{bmatrix} 0 & I_n \\ C & A \end{bmatrix}, \qquad B' = \begin{bmatrix} 0 \\ B \end{bmatrix}, \qquad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}. \tag{3.13}$$

Then the following lemma gives the relations of the controllability of the matrix triples (E,A,B) and (E',A',B'), which offers more convenience to solve Problem SOS.

Lemma 3.2. *Let the matrix triple (E,A,B) be R-controllable, then the matrix triple (E',A',B') is also R-controllable if and only if*

$$\operatorname{rank}\left[Q_{12}(s)P(s)C + sI_n - Q_{11}(s)\right] = n, \quad \forall s \in C, \tag{3.14}$$

where matrices $Q_{11}(s) \in R^{n\times r}[s]$, $Q_{12}(s) \in R^{n\times n}[s]$, and $P(s) \in R^{n\times n}[s]$ satisfy (3.1) and (3.2).

Proof. Since the matrix triple (E,A,B) is R-controllable, (3.1) holds for some unimodular matrices $P(s) \in R^{n\times n}[s]$ and $Q(s) \in R^{(n+r)\times(n+r)}[s]$. Using the controllability of the matrix triple (E',A',B') and the structure of matrices E', A', and B', we can obtain

$$\begin{aligned}
&\begin{bmatrix} I_n & 0 \\ 0 & P(s) \end{bmatrix} \begin{bmatrix} A' - sE' & B' \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Q(s) \end{bmatrix} \\
&\quad = \begin{bmatrix} I_n & 0 \\ 0 & P(s) \end{bmatrix} \begin{bmatrix} -sI_n & I_n & 0 \\ C & A - sE & B \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Q(s) \end{bmatrix} \\
&\quad = \begin{bmatrix} -sI_n & I_n & 0 \\ P(s)C & P(s)[A - sE & B] \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Q(s) \end{bmatrix} \\
&\quad = \begin{bmatrix} -sI_n & [I_n \quad 0]Q(s) \\ P(s)C & P(s)[A - sE \quad B]Q(s) \end{bmatrix} \\
&\quad = \begin{bmatrix} -sI_n & Q_{11}(s) & Q_{12}(s) \\ P(s)C & 0 & I_n \end{bmatrix}.
\end{aligned} \tag{3.15}$$

The above matrix can easily be transformed, by elementary transformations, into the following form:

$$\begin{bmatrix} Q_{12}(s)P(s)C+sI_n & -Q_{11}(s) & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{3.16}$$

Thus we have

$$\operatorname{rank}[A'-sE' \quad B'] = \operatorname{rank}\left[Q_{12}(s)P(s)C+sI_n \quad -Q_{11}(s)\right]. \tag{3.17}$$

From this, we can conclude that the matrix triple (E',A',B') is R-controllable if and only if (3.14) is valid. □

If (3.14) holds, there exists a pair of unimodular matrices: $H(s) \in R^{n\times n}[s]$ and $L(s) \in R^{(n+r)\times(n+r)}[s]$, satisfying the following condition:

$$H(s)[Q_{12}(s)P(s)C+sI_n \quad -Q_{11}(s)]L(s) = [0 \quad I_n], \quad \forall s \in C. \tag{3.18}$$

Partition $L(s)$ as follows:

$$L(s) = \begin{bmatrix} L_{11}(s) & L_{12}(s) \\ L_{21}(s) & L_{22}(s) \end{bmatrix}, \tag{3.19}$$

where $L_{11}(s) \in R^{n\times r}[s]$, $L_{12}(s) \in R^{n\times n}[s]$, $L_{21}(s) \in R^{r\times r}[s]$, and $L_{22}(s) \in R^{r\times n}[s]$. Then we have the following lemma.

Lemma 3.3. *Suppose that (3.18) holds for some pair of unimodular matrices $H(s) \in R^{n\times n}[s]$ and $L(s) \in R^{(n+r)\times(n+r)}[s]$. Then all the vectors y and z satisfying*

$$(Q_{12}(s)P(s)C+sI_n)y - Q_{11}(s)z = 0 \tag{3.20}$$

are given as

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} L_{11}(s) \\ L_{21}(s) \end{bmatrix} g \tag{3.21a}$$

or

$$y = L_{11}(s)g, \qquad z = L_{21}(s)g \tag{3.21b}$$

with $g \in C^r$ being an arbitrary parameter vector.

Proof. Rewriting (3.20) as

$$\left[Q_{12}(s)P(s)C+sI_n \quad -Q_{11}(s)\right]\begin{bmatrix} y \\ z \end{bmatrix} = 0 \tag{3.22}$$

and premultiplying by $H(s)$ both sides of (3.22) give

$$H(s)\left[Q_{12}(s)P(s)C+sI_n \quad -Q_{11}(s)\right]L(s)L^{-1}(s)\begin{bmatrix} y \\ z \end{bmatrix} = 0, \tag{3.23}$$

which reduces, in view of (3.18), to

$$[0 \quad I_n]L^{-1}(s)\begin{bmatrix} y \\ z \end{bmatrix} = 0. \tag{3.24}$$

Letting

$$\begin{bmatrix} g \\ h \end{bmatrix} = L^{-1}(s)\begin{bmatrix} y \\ z \end{bmatrix} \tag{3.25}$$

and substituting (3.25) into (3.24), we can obtain $h = 0$. Thus it can be obtained from (3.25) that

$$\begin{bmatrix} y \\ z \end{bmatrix} = L(s)\begin{bmatrix} g \\ 0 \end{bmatrix}, \tag{3.26}$$

which is equivalent to (3.21a) or (3.21b). □

From Lemma 3.1, we can get that

$$\widetilde{v}_i = -Q_{12}(s_i)P(s_i)Cv_{1i} + Q_{11}(s_i)f_i, \quad i = 1,2,\dots,p. \tag{3.27}$$

Equation (2.7) can be equivalently written in the following vector form:

$$\widetilde{v}_i = s_i v_i, \quad i = 1,2,\dots,p. \tag{3.28}$$

Combining (3.27) and (3.28) yields

$$[Q_{12}(s_i)P(s_i)C + s_i I_n]v_i - Q_{11}(s_i)f_i = 0, \quad i = 1,2,\dots,p, \tag{3.29}$$

which is clearly in the form of (3.20). Applying Lemmas 3.2 and 3.3 to (3.29) gets

$$v_i = L_{11}(s_i)g_i, \quad i = 1,2,\dots,p, \tag{3.30}$$

$$f_i = L_{21}(s_i)g_i, \quad i = 1,2,\dots,p, \tag{3.31}$$

where $g_i \in C^r$, $i = 1,2,\dots,p$ are a group of parametric vectors.

Substituting (3.30) and (3.31) into (3.3b) produces

$$w_i = [Q_{21}(s_i)L_{21}(s_i) - Q_{22}(s_i)P(s_i)CL_{11}(s_i)]g_i, \quad i = 1,2,\dots,p. \tag{3.32}$$

To sum up, we now have the following theorem for solutions to Problem SOS proposed in Section 2.

Theorem 3.4. *Let the matrix triples (E,A,B) and (E',A',B') both be R-controllable, then all the solutions V and W to Problem SOS are given by (3.30) and (3.32) in their column vectors, respectively.*

Algorithm [SOS1]. (1) Solve the pair of unimodular matrices $P(s)$ and $Q(s)$ satisfying (3.1) by applying elementary transformations to matrix $[\,A-sE \;\; B\,]$ and partition matrix $Q(s)$ as in (3.2).

(2) Solve the pair of unimodular matrices $H(s)$ and $L(s)$ satisfying (3.18) by applying elementary transformations to the matrix $[\,Q_{12}(s)P(s)C+sI \;\; -Q_{11}(s)\,]$ and partition matrix $L(s)$ as in (3.19).

(3) Establish the general parametric forms for the matrices V and W according to (3.30) and (3.32), respectively.

Remark 3.5. The advantage of the proposed solutions to the second-order Sylvester equation is that it can provide the general explicit parametric solution. In the case of the undetermined eigenvalues of the matrix F, the free parameters in the solutions are composed of two parts: one is clearly the group of parameter vectors $g_i \in C^r$, $i = 1,2,\dots,p$, while the other is the group of eigenvalues s_i, $i = 1,2,\dots,p$ of the matrix F. Both the two parts of degrees of freedom may be selected to design some control problems such as the eigenstructure assignment problem, the model tracking problem, the observer design problem, and so forth.

4. Case of the determined diagonal matrix F

When the eigenvalues s_i, $i = 1,2,\dots,p$ of the matrix F are prescribed, the solutions to the generalized Sylvester matrix equation (1.3) are actually dependent on the constant matrices

$$\widetilde{P}_i = P(s_i), \quad \widetilde{Q}_i = Q(s_i), \quad i = 1,2,\dots,p, \tag{4.1}$$

which satisfy the following:

$$\widetilde{P}_i[A - s_iE \quad B]\widetilde{Q}_i = [0 \quad I_n], \quad i = 1,2,\dots,p. \tag{4.2}$$

Therefore, we can, instead of seeking the polynomial matrices $P(s)$ and $Q(s)$ satisfying the relation (3.1), find these constant matrices directly so as to avoid polynomial matrix manipulations.

The constant matrices $P_i = P(s_i) \in C^{n\times n}$ and $Q_i = Q(s_i) \in C^{(n+r)\times(n+r)}$, $i = 1,2,\dots,p$ can be easily obtained by singular value decompositions. In fact, by applying singular value decomposition to the matrix $[\,A-s_iE \;\; B\,]$, we obtain two orthogonal matrices $P_i \in C^{n\times n}$ and $Q_i \in C^{(n+r)\times(n+r)}$ satisfying the following equations:

$$P_i[A - s_iE \quad B]Q_i = [0 \quad \Sigma_i], \quad i = 1,2,\dots,p, \tag{4.3}$$

where Σ_i, $i = 1,2,\dots,p$ are diagonal matrices with positive diagonal elements. By rearranging (4.3) in the following form:

$$\Sigma^{-1}P_i[A - s_iE \quad B]Q_i = [0 \quad I], \quad i = 1,2,\dots,p, \tag{4.4}$$

we can obtain the constant matrices $\widetilde{P}_i$, and $\widetilde{Q}_i$, $i = 1,2,\dots,p$ satisfying (4.2) as

$$\widetilde{P}_i = \Sigma^{-1}P_i, \quad \widetilde{Q}_i = Q_i, \quad i = 1,2,\dots,p. \tag{4.5}$$

Partitioning the matrix Q_i, $i = 1, 2, \ldots, p$ is as follows:

$$Q_i = \begin{bmatrix} Q_{11}^i & Q_{12}^i \\ Q_{21}^i & Q_{22}^i \end{bmatrix}, \tag{4.6}$$

where $Q_{11}^i \in C^{n\times r}$, $Q_{12}^i \in C^{n\times n}$, $Q_{21}^i \in C^{r\times r}$, and $Q_{22}^i \in C^{r\times n}$. Then we can give a corollary of Lemma 3.1 as follows.

Corollary 4.1. *Let the matrix triple (E, A, B) be R-controllable, and let the diagonal matrix F be defined as in (2.6) with s_i, $i = 1, 2, \ldots, p$ known. Then all the solutions $\widetilde{V}$ and W in (2.8) can be given by their columns as*

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = Q_i \begin{bmatrix} f_i \\ -\Sigma_i^{-1} P_i C v_i \end{bmatrix}, \quad i = 1, 2, \ldots, p, \tag{4.7}$$

or, equivalently, as

$$\begin{aligned} v_i &= Q_{11}^i f_i - Q_{12}^i \Sigma_i^{-1} P_i C v_i \\ w_i &= Q_{21}^i f_i - Q_{22}^i \Sigma_i^{-1} P_i C v_i, \end{aligned} \quad i = 1, 2, \ldots, p, \tag{4.8}$$

where $f_i \in C^r$, $i = 1, 2, \ldots, p$, are a group of arbitrary parameter vectors.

Similarly, we can derive a corollary of Lemma 3.3.

By applying singular value decomposition to the matrix $[Q_{12}^i P_i C + s_i I - Q_{11}^i]$, we obtain two orthogonal matrices $H_i \in R^{n\times n}$ and $L_i \in R^{(n+r)\times(n+r)}$ satisfying the following equation:

$$H_i \begin{bmatrix} Q_{12}^i \Sigma_i^{-1} P_i C + s_i I_n & -Q_{11}^i \end{bmatrix} L_i = \begin{bmatrix} 0 & \Xi_i \end{bmatrix}, \quad i = 1, 2, \ldots, p, \tag{4.9}$$

where Ξ_i, $i = 1, 2, \ldots, p$, are diagonal matrices with positive diagonal elements. Note that (4.9) can be arranged in the following form:

$$\Xi_i^{-1} H_i \begin{bmatrix} Q_{12}^i \Sigma_i^{-1} P_i C + s_i I_n & -Q_{11}^i \end{bmatrix} L_i = \begin{bmatrix} 0 & I_n \end{bmatrix}, \quad i = 1, 2, \ldots, p. \tag{4.10}$$

Matrices $H(s_i)$ and $L(s_i)$, $i = 1, 2, \ldots, p$, in Lemma 3.3 can thus be substituted by the matrices $\Xi_i^{-1} H_i$ and L_i, $i = 1, 2, \ldots, p$. Further, partition the matrix L_i as

$$L_i = \begin{bmatrix} L_{11}^i & L_{12}^i \\ L_{21}^i & L_{22}^i \end{bmatrix}, \quad L_{11}^i \in R^{n\times r}, \tag{4.11}$$

where $L_{11}^i \in C^{n\times r}$, $L_{12}^i \in C^{n\times n}$, $L_{21}^i \in C^{r\times r}$, and $L_{22}^i \in C^{r\times n}$. Then a corollary of Lemma 3.3 can be obtained as follows.

Corollary 4.2. *Let P_i, Q_{11}^i, Q_{12}^i, and Σ_i be given by (4.3) and (4.6). Then all the vectors y and z satisfying*

$$\left(Q_{12}^i \Sigma_i^{-1} P_i C + s_i I\right) y - Q_{11}^i z = 0 \tag{4.12}$$

are given as

$$y = L_{11}^i g, \qquad z = L_{21}^i g \tag{4.13}$$

with $g \in C^r$ being an arbitrary parameter vector.

With the help of the above two corollaries, via a similar development as in Section 3, the following theorem can be obtained for solution to Problem SOS in the case that the eigenvalues of the matrix F are given a priori.

Theorem 4.3. *Let the matrix triples (E,A,B) and (E',A',B') both be R-controllable, and let the desired closed-loop eigenvalues s_i, $i = 1,2,\ldots,p$ be given a priori. Then all the solutions V and W to Problem SOS can be given by their column vectors, respectively:*

$$v_i = L_{11}^i g_i, \quad i = 1,2,\ldots,p, \tag{4.14}$$

$$w_i = \left[Q_{21}^i L_{21}^i - Q_{22}^i \Sigma_i^{-1} P_i C L_{11}^i\right] g_i, \quad i = 1,2,\ldots,p, \tag{4.15}$$

where $g_i \in C^r$, $i = 1,2,\ldots,p$, are a group of parameter vectors.

Based on the above theorem, an algorithm for solution to Problem SOS in the case of the determined eigenvalues of the matrix F can be given as follows.

Algorithm [SOS2]. (1) Solve the matrices P_i, Q_i, and Σ_i, $i = 1,2,\ldots,p$, satisfying singular value decompositions in (4.3), and partition the matrix Q_i, $i = 1,2,\ldots,p$, as in (4.6).

(2) Solve the matrices H_i, L_i, and Ξ_i, $i = 1,2,\ldots,p$, satisfying singular value decompositions in (4.10), and partition the matrix L_i, $i = 1,2,\ldots,p$, as in (4.11).

(3) Establish the general parametric forms for the matrices V and W according to (4.14) and (4.15), respectively.

Remark 4.4. In the case that the eigenvalues of the matrix F are given a priori, the proposed solution to the second-order Sylvester equation also provides the general explicit parametric solution. The free parameters are composed of the group of parameter vectors $g_i \in C^r$, $i = 1,2,\ldots,p$, which can also be selected to design some control problems.

5. An illustrative example

Consider a second-order Sylvester equation in the form of (1.1) with the following parameters:

$$\begin{gathered} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} -2.5 & 0.5 & 0 \\ 0.5 & -2.5 & 2 \\ 0 & 2 & -2 \end{bmatrix}, \\ C = \begin{bmatrix} -10 & 5 & 0 \\ 5 & -25 & 20 \\ 0 & 20 & -20 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{gathered} \tag{5.1}$$

It can be easily verified that the matrix triples (E,A,B) and (E',A',B') are both R-controllable.

5.1. Case of the undetermined diagonal matrix F**.** We assume that the matrix $F = \operatorname{diag}(s_1, s_2, s_3, s_4, s_5, s_6)$. In the following, we solve the second-order Sylvester equation using Algorithm SOS1.

(1) By applying some elementary transformations to matrix $[\,A-sE \;\; B\,]$, a pair of unimodular matrices $P(s)$ and $Q(s)$ satisfying (3.1) can be obtained as

$$P(s) = \operatorname{diag}(1,2,1), \tag{5.2}$$

$$Q(s) = \begin{bmatrix} 2s+5 & -4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2(s+2.5)^2-0.5 & -4s-10 & 1 & s+2.5 & 0 \\ -2 & 2-s & 0 & 0 & 1 \end{bmatrix}. \tag{5.3}$$

(2) Note that the matrix

$$\begin{bmatrix} Q_{12}(s)P(s)C+sI_3 & -Q_{11}(s) \end{bmatrix} = \begin{bmatrix} 10+s & -50 & 40 & -(2s+5) & 4 \\ 0 & s & 0 & -1 & 0 \\ 0 & 0 & s & 0 & -1 \end{bmatrix} \tag{5.4}$$

is controllable. By applying elementary transformations to the above matrix, unimodular matrices $H(s)$ and $L(s)$ satisfying (3.18) are obtained as

$$H(s) = \begin{bmatrix} -0.005 & 0.005(2s+5) & -0.02 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{5.5}$$

$$L(s) = \begin{bmatrix} 2s^2+5s+50 & -4 & 2s-15 & 0 & 0 \\ s+10 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ s(s+10) & 0 & s & 1 & 0 \\ 0 & s & 0 & 0 & 1 \end{bmatrix}. \tag{5.6}$$

(3) Let

$$g_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}, \quad i = 1 \sim 6. \tag{5.7}$$

According to (3.30), the matrix V in (1.3) is given by its column vectors as

$$v_i = \begin{bmatrix} (2s_i^2+5s_i+50)\alpha_i - 4\beta_i \\ (s_i+10)\alpha_i \\ \beta_i \end{bmatrix}, \quad i = 1 \sim 6. \tag{5.8}$$

Further, we have

$$Q_{21}(s)L_{21}(s) - Q_{22}(s)P(s)CL_{11}(s) = \begin{bmatrix} 2s^4+10s^3+82s^2+165s+450 & -4s^2-10s-40 \\ -2s^2-40s-200 & 2s+20-s^2 \end{bmatrix}. \tag{5.9}$$

Thus from (3.32), the matrix W in (1.3) is given by its column vectors as

$$w_i = \begin{bmatrix} (2s_i^4 + 10s_i^3 + 82s_i^2 + 165s_i + 450)\alpha_i - (4s_i^2 + 10s_i + 40)\beta_i \\ -2(s_i^2 + 20s_i + 100)\alpha_i + (2s_i + 20 - s_i^2)\beta_i \end{bmatrix}, \quad i = 1 \sim 6. \tag{5.10}$$

5.2. Case of the determined diagonal matrix F. We assume that the matrix $F = \operatorname{diag}(s_1, s_2, s_3)$, where $s_{1,2} = -1 \pm 2j$, $s_3 = -3$. In the following, we solve the second-order Sylvester equation using Algorithm SOS2.

(1) By utilizing SVD to the matrices $[\,A - s_iE \;\; B\,]$, we can obtain the matrices P_i, Q_i, and Σ_i, $i = 1 \sim 3$, as follows:

$$P_1 = \overline{P}_2 = \begin{bmatrix} -0.0811 & 0.5580 - 0.1985i & -0.7443 - 0.2978i \\ -0.9862 & 0.0587 + 0.0695i & 0.0912 + 0.1043i \\ -0.1446 & -0.7136 - 0.3629i & -0.2047 - 0.5443i \end{bmatrix},$$

$Q_1 = \overline{Q}_2 =$

$$\begin{bmatrix} 0.2128-0.2885i & -0.0089-0.0613i & 0.0771-0.0121i & 0.5510-0.7330i & -0.1138-0.0876i \\ -0.0560-0.0942i & 0.2438-0.1864i & -0.5319+0.2721i & -0.0948+0.0048i & -0.1114-0.7179i \\ -0.0010-0.0545i & 0.3715+0.1194i & 0.7592+0.1910i & -0.1333-0.0032i & 0.2240-0.4048i \\ 0.9242+0.0398i & -0.0127-0.0165i & -0.0156-0.0000i & -0.3602-0.0000i & -0.1176-0.0000i \\ -0.0000+0.0268i & 0.8655-0.0121i & -0.1433+0.0573i & 0.0333-0.0381i & -0.1664+0.4426i \end{bmatrix},$$

$$P_3 = \begin{bmatrix} -0.0256 & 0.3018 & -0.9530 \\ -0.6688 & -0.7137 & -0.2081 \\ -0.7430 & 0.6321 & 0.2201 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -0.8896 & 0.0154 & 0.0241 & -0.4514 & -0.0628 \\ 0.3227 & -0.2958 & -0.3083 & -0.7232 & 0.4361 \\ 0.1417 & 0.0701 & 0.9363 & -0.2527 & 0.1855 \\ 0.2834 & 0.1402 & -0.0045 & -0.4368 & -0.8421 \\ 0.0631 & 0.9422 & -0.1662 & -0.1359 & 0.2495 \end{bmatrix},$$

$$\Sigma_1 = \Sigma_2 = \operatorname{diag}(5.1953, 2.7380, 1.2297), \qquad \Sigma_3 = \operatorname{diag}(5.7338, 1.5312, 0.8822), \tag{5.11}$$

and partition the matrices Q_i, $i = 1 \sim 3$, as in (4.6).

(2) Note that the matrix

$$\begin{bmatrix} Q_{12}(s)P(s)C + sI & -Q_{11}(s) \end{bmatrix} = \begin{bmatrix} 10+s & -50 & 40 & -(2s+5) & 4 \\ 0 & s & 0 & -1 & 0 \\ 0 & 0 & s & 0 & -1 \end{bmatrix} \tag{5.12}$$

is controllable. By utilizing SVD to the matrices $[\,Q_{12}(s_i)P(s_i)C+s_iI_3 \;\; -Q_{11}(s_i)\,]$, we can obtain the matrices H_i, L_i, and Ξ_i, $i = 1 \sim 3$, satisfying singular value decompositions in (4.10) as follows:

$$H_1 = \overline{H}_2 = \begin{bmatrix} -0.1763 & 0.5323+0.3247i & -0.4581+0.6085i \\ -0.8716 & 0.2685-0.1202i & 0.0449-0.3895i \\ 0.4573 & 0.7169-0.1039i & -0.0911-0.5078i \end{bmatrix},$$

$$L_1 = \overline{L}_2 =$$

$$\begin{bmatrix} -0.0283-0.0635i & -0.1145-0.1832i & -0.1032-0.0360i & -0.6041-0.6691i & 0.2377-0.2597i \\ -0.1224-0.0827i & -0.1849-0.2605i & 0.6900+0.2510i & 0.1722+0.1214i & 0.2945-0.4538i \\ -0.1340-0.0724i & -0.1962-0.2487i & -0.6420-0.1921i & 0.2993+0.2265i & 0.2534-0.4716i \\ 0.9726-0.0042i & -0.0494-0.0311i & 0.0001+0.0002i & 0.0282+0.0279i & -0.0231-0.2201i \\ -0.0638+0.0070i & 0.8634-0.0447i & 0.0012+0.0039i & -0.0138-0.0200i & -0.2084-0.4521i \end{bmatrix},$$

$$H_3 = \begin{bmatrix} -0.1473 & -0.2209 & -0.9641 \\ 0.9238 & -0.3790 & -0.0543 \\ -0.3534 & -0.8986 & 0.2600 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0.1589 & 0.0165 & -0.0587 & -0.9266 & 0.3353 \\ -0.0141 & 0.0667 & 0.7851 & 0.1638 & 0.5934 \\ -0.0301 & 0.0836 & -0.6165 & 0.2972 & 0.7236 \\ 0.9867 & 0.0002 & 0.0019 & 0.1606 & -0.0233 \\ 0.0006 & 0.9941 & 0.0001 & -0.0207 & -0.1062 \end{bmatrix},$$

$$\Xi_1 = \Xi_2 = \mathrm{diag}(41.4945, 8.2624, 0.8549), \qquad \Xi_3 = \mathrm{diag}(40.6893, 5.9256, 2.6234), \tag{5.13}$$

and partition the matrices L_i, $i = 1 \sim 3$, as in (4.11).

(3) Let

$$g_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}, \quad i = 1 \sim 3. \tag{5.14}$$

According to (4.14), the matrix V in (1.3) is given by its column vectors as

$$v_1 = \begin{bmatrix} (39+2j)\alpha_1 - 4\beta_1 \\ (9+2j)\alpha_1 \\ \beta_1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} (39-2j)\alpha_2 - 4\beta_2 \\ (9-2j)\alpha_2 \\ \beta_2 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 53\alpha_3 - 4\beta_3 \\ 7\alpha_3 \\ \beta_3 \end{bmatrix}. \tag{5.15}$$

Thus from (4.15), the matrix W in (1.3) is given by its column vectors as

$$w_1 = \begin{bmatrix} (135+30j)\alpha_1 - (18+4j)\beta_1 \\ (21+8j)\beta_1 - (154-72j)\alpha_1 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} (135-30j)\alpha_2 + (4j-18)\beta_2 \\ (21-8j)\beta_2 + (72j-154)\alpha_2 \end{bmatrix},$$
$$w_3 = \begin{bmatrix} 585\alpha_3 - 46\beta_3 \\ 5\beta_3 - 98\alpha_3 \end{bmatrix}. \tag{5.16}$$

6. Conclusions

This paper considers the solutions to a second-order Sylvester matrix equation. Under the controllability of some matrix triples, complete, general, and explicit parametric solutions to the generalized second-order Sylvester matrix equation are proposed in two cases of the undetermined eigenvalues of the matrix F and the determined eigenvalues of the matrix F. The general explicit parametric solutions to the second-order Sylvester matrix equation are presented. These results provide great convenience to the analysis of the solution to the equation, and can perform important functions in many analysis and design problems in control systems theory. As a demonstration, a numerical example is offered to show the effectiveness of the proposed approaches.

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