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Research Article

Quadratic Stabilization of LPV System by an LTI Controller Based on ILMI Algorithm

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A linear time-invariant (LTI) output feedback controller is designed for a linear parameter-varying (LPV) control system to achieve quadratic stability. The LPV system includes immeasurable dependent parameters that are assumed to vary in a polytopic space. To solve this control problem, a heuristic algorithm is proposed in the form of an iterative linear matrix inequality (ILMI) formulation. Furthermore, an effective method of setting an initial value of the ILMI algorithm is also proposed to increase the probability of getting an admissible solution for the controller design problem.

1. Introduction

A linear parameter-varying (LPV) system is formalized as a certain type of nonlinear system, and is successfully applied in developing a control strategy which is based on classical gain-scheduled methodology [1]. Several tutorial papers and special publications concerning the gain-scheduled method of LPV control system are [2–7]. These gain-scheduled LPV controller design approaches are applicable under the assumption that the dependent parameters can be measured online. In practical application, it is often difficult to satisfy this requirement. Therefore, it is crucial to design an effective LTI controller to get robust stability for an LPV plant with immeasurable dependent parameters. Here, these dependent parameters are assumed to vary in a polytopic space. In robust control framework of LPV system, a necessary and sufficient condition of quadratic stability for polytopic LPV system is formulated in terms of a finite LMIs optimization problem [8]. The underlying quadratic Lyapunov functions are also used to derive bounds on robust performance measures. Several heuristic procedures [9–13] have also been proposed to solve some control problems with nonconvex constraints such as a controller with fixed or reduced order of the decentralized structure. In [10], a method is presented to solve some controller design problems when structure constraints are imposed. The procedure is based on a two-stage optimization process, each stage requires the solution of a convex optimization problem based on a kind of LMI expression, in which either the controller gain matrix or the Lyapunov function is considered as the optimization variable.

This paper proposes a way of designing a quadratically stabilizing LTI output feedback controller for LPV system where dependent parameters vary in a polytopic space. Different from gain-scheduled LPV controller design, besides rank constraints, another constraint condition in which the controller matrix should be the same one for each vertex plant of LPV system is added. This problem still remains a complex issue and not numerically tractable. Here, a heuristic ILMI approach is presented to solve an admissible solution for this control problem. And a method of setting an initial value for the Lyapunov matrix is also proposed to increase the possibility of obtaining a feasible solution to the ILMI approach. The proposed method is better than random assignment of the initial value. Even though this approach is not guaranteed to converge globally, it may provide a useful alternative design tool in practice.

2. Notation and definitions

Consider an LPV plant $P(\theta(t))$ described by state space equations as

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B_u u(t), \\ y(t) &= C_y u(t). \end{aligned} \tag{2.1}$$

Here, state-space matrices have compatible dimensions of time-varying dependent parameters $\theta(t) = [\theta_1(t)\theta_2(t)\cdots\theta_r(t)]^T \in \mathbb{R}^r$. Moreover, we have the following assumptions.

(1) The system state matrix $A(\theta)$ is a continuous and bounded function and depends affinely on $\theta(t)$.

(2) The immeasurable real parameters $\theta(t)$ exist in the LPV plant and vary in a polytope Θ as

$$\begin{aligned} \theta(t) \in \Theta &:= Co\{\omega_1, \omega_2, \ldots, \omega_N\} \\ &= \left\{ \sum_{i=1}^{N} \alpha_i(t)\omega_i : \alpha_i(t) \geq 0, \ \sum_{i=1}^{N} \alpha_i(t) = 1, \ N = 2^r \right\}. \end{aligned} \tag{2.2}$$

(3) The LPV plant is quadratically detectable and quadratically stabilizable.

With the above assumptions, the system state matrix $A(\theta)$ can be expressed as

$$A(\theta) = \sum_{i=1}^{N} \alpha_i(t) A_i \quad \text{with } \alpha_i \geq 0, \qquad \sum_{i=1}^{N} \alpha_i = 1. \tag{2.3}$$

Remark 2.1. It is assumed that the matrices B_u, C_y of the LPV plant are time invariant. When they are time varying, a simple way is to satisfy the requirement by filtering the control input and output through lowpass filters. These filters should have sufficiently large bandwidth. Then, the dependent parameters are shifted into the state matrix $A(\theta)$ in [3].

Definition 2.2 (quadratic stability [14]). Considering a LPV system, $\dot{x}(t) = A(\theta)x(t)$ is said to be quadratically stable if and only if there exists $P > 0$ such that

$$A^T(\theta)P + PA(\theta) < 0. \tag{2.4}$$

Remark 2.3. For polytopic LPV system, we have the equivalent conditions for (2.4) as

$$A_i^T P + PA_i < 0, \quad i = 1,\dots,N. \tag{2.5}$$

It should be noted that if LPV system is quadratically stable one, it is also exponentially stable.

3. Main results

In this section, a LTI output feedback controller is designed to achieve quadratic stability for LPV system where dependent parameters vary in a polytopic space.

We seek to design a controller ($A_K \in \mathbb{R}^{n_k \times n_k}$) of fixed order n_k as

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k y, \\ u &= C_k x_k + D_k y, \end{aligned} \tag{3.1}$$

where $x_K \in \mathbb{R}^{n_k}$ is the controller state. Substituting (3.1) into (2.1), the closed-loop state matrix A_{cl} has the following expression:

$$A_{\text{cl}}(\theta) = \begin{bmatrix} A(\theta) + B_u D_k C_y & B_u C_K \\ B_K C_y & A_K \end{bmatrix}. \tag{3.2}$$

First, the following definitions are made as

$$J = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \qquad \overline{A}(\theta) = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \overline{B}_u = \begin{bmatrix} 0 & B_u \\ I & 0 \end{bmatrix}, \qquad \overline{C}_y = \begin{bmatrix} 0 & I \\ C_y & 0 \end{bmatrix}, \tag{3.3}$$

which are totally dependent on the state-space matrices of the controller and the LPV plant. Then, the closed-loop relation is parameterized in terms of the controller realization as

$$A_{\text{cl}}(\theta) = \overline{A}(\theta) + \overline{B}_u J \overline{C}_y. \tag{3.4}$$

Theorem 3.1. *Suppose LPV system is given in (3.4), and then the following are equivalent conditions.*

(1) *The closed-loop state matrix $A_{cl}(\theta)$ is quadratically stable.*

(2) *There exist a symmetric positive definite matrix P and matrix J such that*

$$\overline{A}(\theta)P + P\overline{A}^T(\theta) + \overline{B}_u J \overline{C}_y P + P\overline{C}_y^T J^T \overline{B}_u^T < \delta I \tag{3.5}$$

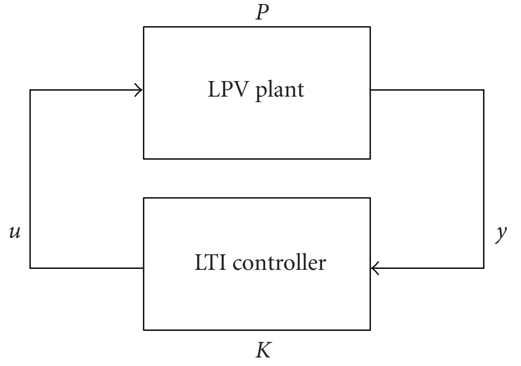

FIGURE 3.1. Relevant LPV control scheme.

or

$$\overline{A}_i P + P\overline{A}_i^T + \overline{B}_u J \overline{C}_y P + P\overline{C}_y^T J^T \overline{B}_u^T < \delta I, \tag{3.6}$$

$i = 1,\dots,N$, *for* δ *being a negative scalar value.*

Proof. According to Definition 2.2, the claims (3.5) or (3.6) can be established easily. □

From (3.6), system matrix J of the controller (3.1) should be the same one for each vertex plant of LPV system (3.2): it is also a nonconvex constraint and difficult to be solved. In the following section, necessary conditions for the existence of a constant matrix J for (3.6) are presented, then a heuristic ILMI algorithm is presented to supply a solution of J for (3.6). The choosing of an appropriate initial value to ILMI is very important to converge quickly to a feasible solution. Here, a method of setting an initial value to ILMI algorithm is also proposed.

THEOREM 3.2. *Given an LPV plant (2.1), if there exists a fixed order LTI controller of order* n_k *that makes the closed-loop LPV system as Figure 3.1 quadratically stable, then there exist* $n\times n$ *symmetric positive definite matrices* X, Y *such that*

$$N_o^T\left(A^T(\theta)X + XA(\theta)\right)N_o < 0, \qquad N_c^T\left(YA(\theta) + A^T(\theta)Y\right)N_c < 0. \tag{3.7}$$

Using the polytopic characteristic of the LPV plant, (3.7) can be equivalent to

$$N_o^T\left(A_i^T X + XA_i\right)N_o < 0, \qquad N_c^T\left(YA_i + A_i^T Y\right)N_c < 0, \tag{3.8}$$

$$\begin{gathered}\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \\ \mathit{rank}\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_k, \end{gathered} \tag{3.9}$$

where N_o *and* N_c *are full column rank matrices such that*

$$\operatorname{Im} N_o = \ker C_y, \qquad \operatorname{Im} N_c = \ker B_u^T. \tag{3.10}$$

The proof of the theorem can be easily taken from earlier results [3, 15].

Theorem 3.2 tells us necessary conditions of the existence of a stabilizing output feedback LTI controller for the LPV plant (2.1). Meanwhile, it also provides an efficient method for setting an initial value of the common Lyapunov matrix P, which is used to construct a stabilizing output feedback LTI controller.

Remark 3.3. Now, let us overview some results of LPV controller design for LPV plant. Consider the LPV plant (2.1), since this plant is assumed to be quadratically stabilizable and quadratically detectable, (3.8)-(3.9) are sufficient and necessary conditions for the existence of such a full-order gain-scheduled LPV controller that quadratically stabilizes LPV plant (2.1). In contrast to gain-scheduled LPV controller design [3], here only an LTI controller is designed to quadratically stabilize the LPV plant and conditions (3.8)-(3.9) become not sufficient but necessary just as Theorem 3.2.

Note that the matrix inequality (3.6) is a bilinear matrix problem with the constraint that controller gain matrix should be constant, and it is a nonconvex optimization problem. Here, a heuristic approach of alternately solving convex optimization problems is proposed based on LMI formulation. We minimize δ, over P and J, subject to (3.6). This problem is a convex optimization problem in J and δ for fixed P, and is convex in P and δ for fixed J. It also should be noted that this approach is guaranteed to converge, but not necessarily to the global optimum of the problem. The assignment of a proper initial value to P is the key to enhance probability of converging to the global optimum. Here, conditions (3.8)-(3.9) supply necessary conditions for the existence of such an LTI controller of order n_k. Therefore, conditions (3.8)-(3.9) of Theorem 3.2 also give us an effective method of setting an initial value to P.

Therefore, the ILMI algorithm proceeds as shown in Algorithm 3.1.

If, after the procedure is alternated several times, solution J is still infeasible, there are two cases: one is that a feasible J may still exit, for this procedure does not necessarily guarantee to the solution J; the other is that the LPV plant may not be quadratically stabilizable by only an LTI controller.

4. Numerical example

In this section, two numerical examples are considered to illustrate the proposed method. All LMI-related computations are performed with LMI toolbox of Matlab [4].

Example 4.1. We consider the problem of controlling the yaw angles of a satellite system that appears in [4]. The satellite system consisting of two rigid bodies joined by a flexible link has the state-space representation as

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \tag{4.1}$$

Step 1.

Set initial value $i = 0$, obtain $P_i = \begin{bmatrix} X & X_2 \\ X_2^T & I \end{bmatrix}$
subject to (3.8)-(3.9), where $X - Y^{-1} = X_2 X_2^T$.
Let δ_i be an arbitrary large positive real number.
$\delta_{\text{old}} = \delta_i$.

Step 2.

Repeat {
OP1: Solve eigenvalue problem, "minimize δ_{i1}, over J_i and δ_{i1}, subject to (3.6);"
$\delta_i = \delta_{i1\text{opt}}$, $J_i = J_{\text{opt}}$.
If $\delta_i < 0$, exit. J_i is an admissible solution.
OP2: Solve eigenvalue problem, "minimize δ_{i2}, over P_i and δ_{i2}, subject to (3.6) and $P_i > 0$";
$P_{i+1} = P_{i2\text{opt}}$. $\delta_i = \delta_{i2\text{opt}}$.
If $\delta_i < 0$, exit. J_i is an admissible solution.
If $\|\delta_i - \delta_{\text{old}}\| < \gamma$, a predetermined tolerance, exit.
Else $\delta_{\text{old}} = \delta_i$.
$i = i + 1$.
}

Algorithm 3.1

where k and f are torque constant and viscous damping, which vary in the following uncertainty ranges: $k \in [0.09 \quad 0.4]$ and $f \in [0.0038 \quad 0.04]$. A state-feedback controller $u = Kx$ is designed to achieve quadratic stability for all possible parameter trajectories in the polytopic space. The pre-determined tolerance γ is set to $1.0e - 4$. The following two cases are considered.

(1) Setting an arbitrary matrix to the initial P such as identity matrix. After 12 iterations, δ_{12} converges to -0.0857, therefore solution K is found as

$$K = [1061463.3 \quad -1061463.3 \quad -258208.45 \quad -7338.2]. \tag{4.2}$$

(2) Setting an initial matrix to P proposed in this paper. In this case, a state feedback is considered to construct, then an initial matrix of P satisfying (3.8) is chosen as

$$P_0 = \begin{bmatrix} 961.4 & 518.14 & -118.4 & 278.06 \\ 518.1 & 930.3 & -247.3 & -167.8 \\ -118.4 & -247.3 & 95.46 & -55.25 \\ 278.06 & -167.8 & -55.25 & 972.54 \end{bmatrix}. \tag{4.3}$$

Using the initial matrix P_0, after only 1 iteration, δ_1 converges to -9395817.73. An admissible K is found as

$$K = [10541311.8 \quad -24814284.7 \quad -60435459.05 \quad -15945712.7]. \tag{4.4}$$

Therefore, the proposed method has a quicker convergence to a feasible solution than the method of setting an arbitrary matrix as the initial matrix P.

Example 4.2. A classical example of parameter-varying unstable plant that can be viewed as a mass-spring-damper system with time-varying spring stiffness is considered [16]. The state-space equation of this unstable LPV plant is as follows:

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -0.5 - 0.5\theta & -0.2 \end{bmatrix}, \qquad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_y = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_{uy} = 0. \tag{4.5}$$

Here, the scope of time-varying parameter $\theta(t)$ is assumed in the polytope space $\Theta := Co\{-1, 1\}$. An LTI output feedback controller is designed to achieve quadratic stability for all possible parameter trajectories in the polytopic space. The predetermined tolerance γ is set to $1.0e - 4$.

Just like Example 4.1, the following two cases are considered.

(1) Setting an arbitrary matrix to the initial P, such as identity matrix. After 5 iterations, δ_5 converges to 0.163, which is larger than zero. Therefore solution J is found infeasible.

(2) Setting an initial matrix to P proposed in this paper. In this case, a full-order output feedback controller is considered to construct, then an initial matrix of P satisfying (3.8)-(3.9) is as follows:

$$P_0 = \begin{bmatrix} 17.62 & -11.99 & 4.01 & -1.22 \\ -11.99 & 35.23 & -1.22 & 5.80 \\ 4.01 & -1.22 & 1 & 0 \\ -1.22 & 5.80 & 0 & 1 \end{bmatrix}. \tag{4.6}$$

Using the initial matrix P_0, after only 1 iteration, δ_1 converges to -3.998. An admissible J is solved as

$$J = 1.0e8 * \begin{bmatrix} -4.00 & -0.53 & -7.8e-7 \\ -0.53 & -5.56 & 2.94e-7 \\ 3.76e-8 & 6.0e-10 & 2.0e-7 \end{bmatrix}. \tag{4.7}$$

Therefore, an LTI output feedback controller to satisfy quadratic stability of closed-loop LPV system is constructed as

$$A_k = 1.0e8 * \begin{bmatrix} -4.00 & -0.53 \\ -0.53 & -5.56 \end{bmatrix}, \qquad B_K = \begin{bmatrix} -78 \\ 29.4 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 3.76 & 0.06 \end{bmatrix}, \qquad D_k = 20. \tag{4.8}$$

When the trajectory of dependent parameter is assumed as $\theta(t) = 0.63 + 0.1 \cdot e^{-t}$, the trajectory of the output of this plant can be drawn for the initial values $x(0) = [-0.25 \quad 0.15]^T$ as shown in Figure 4.1.

Comparing these two cases above, numerical examples demonstrate that the proposed method of setting the initial value to ILMI algorithm is more efficient than the method of setting an arbitrary matrix as the initial value.

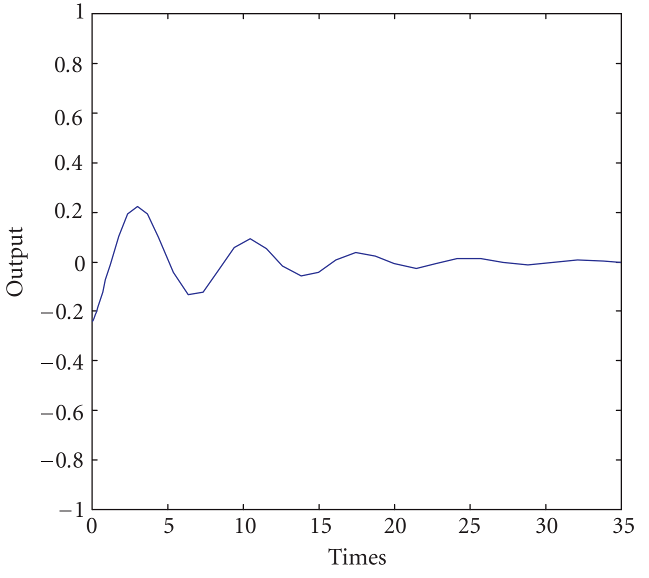

FIGURE 4.1. Trajectory of the output of this plant with initial values $x(0) = [-0.25 \quad 0.15]^T$.

5. Conclusions

In this paper, an LTI output feedback controller has been designed for LPV system to ensure that the closed-loop system achieves quadratic stability for all possible dependent parameters in a polytopic space. A heuristic iterative algorithm to solve such a controller has been presented in terms of LMI formulation. It also should be noted that the procedure is heuristic and the choice of initial value is important to ensure convergence to an acceptable solution. Finally, some numerical examples have been presented to illustrate the design method.

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