Research Article

Stabilization and Observability of a Rotating Timoshenko Beam Model

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A control system describing the dynamics of a rotating Timoshenko beam is considered. We assume that the beam is driven by a control torque at one of its ends, and the other end carries a rigid body as a load. The model considered takes into account the longitudinal, vertical, and shear motions of the beam. For this distributed parameter system, we construct a family of Galerkin approximations based on solutions of the homogeneous Timoshenko beam equation. We derive sufficient conditions for stabilizability of such finite dimensional system. In addition, the equilibrium of the Galerkin approximation considered is proved to be stabilizable by an observer-based feedback law, and an explicit control design is proposed.

1. Introduction

Control issues for several models of flexible manipulators have been intensively studied by many authors. A particular list of references in this area can be found in monographs [1, 2]. There are two common approaches to represent the motion of such manipulators. The first approach deals with systems of rigid bodies [3], Galerkin approximations [4, 5], or finite element methods [6] to derive mathematical models with finite degrees of freedom. The second approach treats a manipulator as a distributed parameter system. The majority of publications in this distributed parameter approach are concentrated on the Euler-Bernoulli beam model (see [7], [8, Chapter 10.8], [1, Chapter 4], [2, 9, 10]).

A possible extension of the Euler-Bernoulli model was proposed by Timoshenko [11]. From the engineering viewpoint, the Timoshenko beam has an advantage of describing the effects of rotary inertia and the deflection due to shear. Control of Timoshenko beams was studied in [12–17], [1, Chapter 5.1.2]. The motion of a payload, usually attached to a

real manipulator, is neglected in all these publications. In [18], a clamped beam with an end mass is proved to be stabilizable by a feedback control applied to the tip. The author of [19] addresses the development of LQR techniques and computation algorithms for beams with controlling torques applied to the hub. A limitation of these results is that a knowledge of the full infinite dimensional space is required. In [5], a hybrid system of partial and ordinary differential equations, representing the oscillations of a flexible beam, has been studied for the case when the control is the acceleration at a point. We have considered a model for the vertical motion of a beam and estimated its physical parameters from measurements of modal frequencies in [20].

It should be emphasized that, in contrast to the above publications, we study here a rotating beam that carries a payload under the action of gravity, the control torque is applied at the hub, and the longitudinal motion is taken into account. The motivation for this study is to control the motion of a real flexible-link manipulator-turntable ladder. Such a turntable ladder has been described in [3], where a dynamical model with two rigid bodies (two degrees of freedom) has been used to represent the first mode of oscillations.

This paper is organized as follows. In Section 2, we derive the motion equations for a flexible beam with a load under the action of gravity and the control torque. Section 3 contains necessary details for computing the eigenvalues and eigenfunctions of the associated Sturm-Liouville problem. By using Galerkin's method, we approximate the dynamics by a system of ODEs in Section 4. In the derivation procedure, we exploit the variational form instead of taking the standard inner product in L^2. The order of approximation may be chosen arbitrarily. In Section 5, a state feedback control which stabilizes the equilibrium of the Galerkin approximation is obtained (Theorem 5.1). In order to justify a possibility of implementation of the controller proposed, we study the observability problem in Section 6. The closed-loop system is proved to be asymptotically stable, provided that the feedback is generated by a Luenberger-type observer (Theorem 6.2). The proof of Theorem 6.2 is based on the invariance principle. The main advantage of our approach is that the control design is done explicitly; all the parameters appearing in the feedback law and dynamical observer can be effectively computed through integral moments with respect to solutions of the Sturm-Liouville problem. It is also important that no derivatives of the input signals are needed for the state estimation.

2. Description of the model

Consider a flexible beam rotating in the vertical plane around the fixed point O (see Figure 2.1).

We assume that the beam is driven by a control torque M at one of its ends (the hub at O), and the other end (point C) carries a payload of mass m.

Let l be the length of the beam. We assume that the centerline of the beam in its undeformed reference configuration occupies the segment $[0,l]$ on the Ox-axis. Consider a particle P on the centerline and denote by x its coordinate in the reference configuration. At a given time t, let $(x + s(x,t), w(x,t))$ be the coordinates of the position vector for P in the Cartesian frame Oxy. We introduce the notation $\psi(x,t)$ for the rotation angle of the cross section area at P due to bending. By taking into account the longitudinal, vertical,
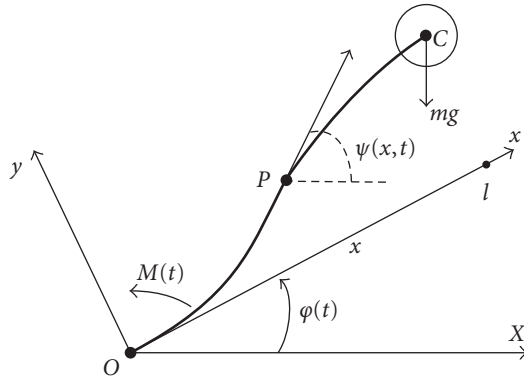
FIGURE 2.1. A rotating Timoshenko beam.

and shear motions, we derive the following expression for the kinetic energy of the system considered:

$$
2T = \int_0^l \left\{ \rho(x) \left[(\dot{w} + x\dot{\varphi})^2 + (w\dot{\varphi})^2 + \dot{s}^2 + 2\dot{\varphi}(\dot{w}s - \dot{s}w) + (\dot{\varphi})^2(s + 2x)s \right] + I_\rho(x)(\dot{\varphi} + \dot{\psi})^2 \right\} dx
$$

$$
+ J_0(\dot{\varphi})^2 + m \left\{ (\dot{w} + x\dot{\varphi})^2 + (w\dot{\varphi})^2 + \dot{s}^2 + 2\dot{\varphi}(\dot{w}s - \dot{s}w) + (\dot{\varphi})^2(s + 2x)s \right\} \big|_{x=l}
$$

$$
+ J_c \{ \dot{\varphi} + \dot{\psi} \}^2 |_{x=l},
$$

$$
(2.1)
$$

where $\varphi(t)$ is the angle between the moving axis Ox and the horizontal direction, $\rho(x)$ is the mass per unit length of the beam, $I_\rho(x)$ is the mass moment of inertia of the cross section, and J_0 is the hub moment of inertia. The mass distribution for the payload is characterized by the moment of inertia J_c with respect to its center of mass C.

In this paper, we use dots to denote derivatives with respect to time t, and primes to denote derivatives with respect to the space variable x.

Assuming that the beam is inextensible, we get the following relation on w and s:

$$
s' = -\frac{1}{2} w'^2 + o(w'^2).
$$

$$
(2.2)
$$

The integration of this relation, with the higher order terms being omitted, yields

$$
s(x,t) = -\frac{1}{2} \int_0^x w'^2(\xi, t) \, d\xi.
$$

$$
(2.3)
$$

We assume that the deformation of the beam is small and drop the terms of order higher than 2 relative to w when computing the Lagrangian of the system considered.

Following the Timoshenko beam model [11], [7, page 1142], and exploiting (2.1), (2.3), the Lagrangian takes the form

$$2L = \int_0^l \{\rho(x)((\dot{w}+x\dot{\varphi})^2 + \dot{\varphi}^2 w^2) - \rho_2(x)\dot{\varphi}^2 w'^2 + I_\rho(\dot{\varphi}+\dot{\psi})^2 - K(\psi-w')^2 - EI(\psi')^2\}dx$$

$$+ m\{\dot{\varphi}^2 w^2(l,t) + (l\dot{\varphi}+\dot{w}(l,t))^2\} + J_c\{\dot{\varphi}+\dot{\psi}(l,t)\}^2 + J_0\dot{\varphi}^2$$

$$- g\int_0^l \{(2\rho x - \rho_1 w'^2)\sin\varphi + 2\rho w\cos\varphi\}dx - 2mg\{l\sin\varphi + w(l,t)\cos\varphi\},$$

$$(2.4)$$

where

$$\rho_1(x) = \int_x^l \rho(\xi)\,d\xi + m, \qquad \rho_2(x) = \int_x^l \xi\rho(\xi)\,d\xi + ml. \tag{2.5}$$

Here E and I are Young's modulus and the moment of inertia of the cross section of the beam, respectively, g is the acceleration of gravity. The coefficient K is equal to kGA, where G is the modulus of elasticity in shear, A is the cross sectional area, and k is a constant depending on the shape of the cross section. We assume that ρ, I_ρ, EI, and K are all positive, differentiable functions of the space variable x.

If C^2-functions $(\varphi(t), w(x,t), \psi(x,t))$ define the motion of the system for the control torque $M(t)$ on a segment $t \in [t_1, t_2]$ then Hamilton's principle yields

$$\delta\left(\int_{t_1}^{t_2} L\,dt\right) + \int_{t_1}^{t_2} M(t)\delta\varphi(t)\,dt = 0, \tag{2.6}$$

for any admissible variations $(\delta\varphi(t), \delta w(x,t), \delta\psi(x,t))$ satisfying the boundary conditions

$$\delta\varphi|_{t=t_1} = \delta\varphi|_{t=t_2} = 0, \qquad \delta w|_{t=t_1} = \delta w|_{t=t_2} = 0, \qquad \delta\psi|_{t=t_1} = \delta\psi|_{t=t_2} = 0,$$
$$\delta w|_{x=0} = 0, \qquad \delta\psi|_{x=0} = 0. \tag{2.7}$$

By computing the first variation in (2.6) and integrating by parts, we get

$$\int_{t_1}^{t_2} \left\{\left(M + \frac{\partial L}{\partial\varphi} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}}\right)\delta\varphi(t) - \mu(\delta w(\cdot,t), \delta\psi(\cdot,t); \varphi, w, \psi)\right\}dt = 0, \tag{2.8}$$

where the functional μ is linear with respect to δw and $\delta\psi$:

$$\mu(\delta w(\cdot,t), \delta\psi(\cdot,t); \varphi, w, \psi)$$

$$= \int_0^l \delta w(x,t)\{(\ddot{w}+x\ddot{\varphi}-(\dot{\varphi})^2 w + g\cos\varphi)\rho + (K(\psi-w') + (g\rho_1\sin\varphi - (\dot{\varphi})^2\rho_2)w')'\}dx$$

$$+ \int_0^l \delta\psi(x,t)\{I_\rho(\ddot{\psi}+\ddot{\varphi}) + K(\psi-w') - (EI\psi')'\}dx + \delta\psi(l,t)\{J_c(\ddot{\varphi}+\ddot{\psi}) + EI\psi'\}|_{x=l}$$

$$+ \delta w(l,t)\{K(w'-\psi) + m(\ddot{w}+l\ddot{\varphi}-(\dot{\varphi})^2 w + g\cos\varphi) + m(l(\dot{\varphi})^2 - g\sin\varphi)w'\}|_{x=l}. \tag{2.9}$$

Thus, as (2.8) vanishes on each admissible variation satisfying (2.7), we get the following boundary value problem:

$$\ddot{w} + \frac{1}{\rho}\left(K(\psi - w')\right)' = -g\cos\varphi - x\ddot{\varphi} + \dot{\varphi}^2 w + \frac{1}{\rho}\left((\rho_2\dot{\varphi}^2 - g\rho_1\sin\varphi)w'\right)';$$

$$\ddot{\psi} + \frac{K}{I_\rho}(\psi - w') - \frac{1}{I_\rho}(EI\psi')' = -\ddot{\varphi}, \quad x \in (0,l);$$

$$w|_{x=0} = \psi|_{x=0} = 0;$$

$$K(\psi - w')|_{x=l} = m\left\{\ddot{w} + l\ddot{\varphi} - \dot{\varphi}^2 w + g\cos\varphi + (l\dot{\varphi}^2 - g\sin\varphi)w'\right\}|_{x=l};$$

$$-EI\psi'|_{x=l} = J_c(\ddot{\varphi} + \ddot{\psi}|_{x=l}),$$

$$M(t) = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = \left\{J_c + J_0 + m[l^2 + w^2(l,t)] + \int_0^l \left[I_\rho + (x^2 + w^2)\rho - \rho_2 w'^2\right]dx\right\}\ddot{\varphi}$$

$$+ \int_0^l \left(\rho x\ddot{w} + I_\rho\ddot{\psi} + 2\rho w\dot{\varphi}\dot{w} - 2\rho_2 w'\dot{\varphi}\dot{w}'\right)dx + m(l\ddot{w} + 2w\dot{\varphi}\dot{w})|_{x=l} + J_c\ddot{\psi}|_{x=l}$$

$$+ g\left\{\int_0^l \left(\rho x - \frac{1}{2}\rho_1 w'^2\right)dx + ml\right\}\cos\varphi - g\left\{\int_0^l \rho w\,dx + m w(l,t)\right\}\sin\varphi. \tag{2.10}$$

Straightforward computations show that the above control system admits an equilibrium

$$\varphi(t) = \varphi_0, \qquad w(x,t) = w_0(x), \qquad \psi(x,t) = \psi_0(x), \qquad M(t) = M_0 \tag{2.11}$$

if and only if the following conditions are satisfied:

$$\left(K(w_0'(x) - \psi_0(x))\right)' = g\{\rho\cos\varphi_0 + (\rho_1 w_0')'\sin\varphi_0\};$$

$$\left(EI\psi_0'(x)\right)' + K(w_0'(x) - \psi_0(x)) = 0, \quad x \in (0,l);$$

$$w_0(0) = \psi_0(0) = 0; \qquad \psi_0'(l) = 0; \qquad K(\psi_0(l) - w_0'(l)) = mg(\cos\varphi_0 - w_0'(l)\sin\varphi_0);$$

$$\frac{M_0}{g} = \left(\int_0^l \left(\rho x - \frac{1}{2}\rho_1 w_0'^2\right)dx + ml\right)\cos\varphi_0 - \left(\int_0^l \rho w_0\,dx + m w_0(l)\right)\sin\varphi_0. \tag{2.12}$$

Our goal is to control the system (2.10) around its steady state (2.12).

3. Perturbed dynamics

Let (φ_0, w_0, ψ_0) be a solution of (2.12) with some M_0. Then plugging

$$\varphi = \varphi_0 + \tilde{\varphi}, \qquad w = w_0 + \tilde{w}, \qquad \psi = \psi_0 + \tilde{\psi}, \qquad M = M_0 + \widetilde{M} \tag{3.1}$$

into the dynamical equations (2.10) yields the following control system:

$$\ddot{\tilde{\varphi}} = v;$$

$$\ddot{\tilde{w}} + \frac{1}{\rho}\left(K(\tilde{\psi} - \tilde{w}')\right)' = -xv + g\tilde{\varphi}\sin\varphi_0 - \frac{g}{\rho}\left((\tilde{w}'\sin\varphi_0 + \tilde{\varphi}w_0'\cos\varphi_0)\rho_1\right)' + \cdots;$$

$$I_\rho\ddot{\tilde{\psi}} + K(\tilde{\psi} - \tilde{w}') - (EI\tilde{\psi}')' = -I_\rho v;$$

$$\tilde{w}|_{x=0} = \tilde{\psi}|_{x=0} = 0;$$

$$\left(\frac{K}{m}(\tilde{w}' - \tilde{\psi}) + \ddot{\tilde{w}} - g\left(\sin\varphi_0 + w_0'\cos\varphi_0\right)\tilde{\varphi} - g\tilde{w}'\sin\varphi_0 + \cdots\right)\Bigg|_{x=l} = -lv;$$

$$(EI\tilde{\psi}' + J_c\ddot{\tilde{\psi}})|_{x=l} = -J_c v,$$

(3.2)

where

$$v = \left(J_0 + \int_0^l \left[(w_0)^2\rho - (w_0')^2\rho_2\right]dx + mw_0^2(l)\right)^{-1}$$

$$\times \left\{\widetilde{M} + g\left(\int_0^l \left[(\rho w_0 - \rho_1 w_0')\cos\varphi_0 - \frac{1}{2}\rho_1(w_0')^2\sin\varphi_0\right]dx + mw_0(l)\cos\varphi_0\right)\tilde{\varphi}\right.$$

$$\left. + g\int_0^l \left[(w_0'\cos\varphi_0 - \sin\varphi_0)\rho_1\tilde{w}' + \rho\tilde{w}\sin\varphi_0\right]dx + mg\tilde{w}(l,t) - EI\tilde{\psi}'(0,t)\right\} + \cdots,$$

(3.3)

where the symbol "$\cdots$" denotes terms of order of smallness 2 or higher with respect to $\tilde{\varphi}$, $\tilde{w}$, $\tilde{\psi}$ and their derivatives.

As, for each state $(\tilde{\varphi}(t), \dot{\tilde{\varphi}}(t), \tilde{w}(\cdot,t), \dot{\tilde{w}}(\cdot,t), \tilde{\psi}(\cdot,t), \dot{\tilde{\psi}}(\cdot,t))$, there is a one-to-one correspondence between $\widetilde{M}$ and v, we may treat v as a control in (3.2) and assume that it may take any value in $\mathbb{R}$.

3.1. Separation of variables.

To derive a finite dimensional approximation, let us first study solutions of the control system (3.2) of a particular form

$$\tilde{\varphi}(t) \equiv 0, \qquad \tilde{w}(x,t) = \overline{w}(x)q(t), \qquad \tilde{\psi}(x,t) = \overline{\psi}(x)q(t). \tag{3.4}$$

By substituting the above relations into (3.2), we get $\ddot{q}(t) = -\lambda q(t)$ together with the following Sturm-Liouville problem:

$$\left(K(\overline{\psi} - \overline{w}') + g\rho_1\overline{w}'\sin\varphi_0\right)' - \lambda\rho\overline{w} = 0,$$

$$K(\overline{\psi} - \overline{w}') - (EI\overline{\psi}')' - \lambda I_\rho\overline{\psi} = 0, \quad x \in (0,l),$$

$$\overline{w}(0) = 0, \qquad \overline{\psi}(0) = 0, \tag{3.5}$$

$$K\left(\overline{w}'(l) - \overline{\psi}(l)\right) - mg\overline{w}'(l)\sin\varphi_0 - m\lambda\overline{w}(l) = 0,$$

$$EI\overline{\psi}'(l) - \lambda J_c\overline{\psi}(l) = 0,$$

where λ is a scalar parameter.

3.2. Eigenvalues of the Sturm-Liouville problem. Let

$$\mathcal{H} = \left\{ \begin{pmatrix} \overline{w} \\ \overline{\psi} \end{pmatrix} : \overline{w} \in H^1[0,l], \ \overline{\psi} \in H^1[0,l], \ \overline{w}(0) = \overline{\psi}(0) = 0 \right\}, \tag{3.6}$$

where $H^1[0,l]$ is the Sobolev space. Consider the following symmetric positive definite bilinear form on $\mathcal{H}$:

$$\left\langle \begin{matrix} w_1 \ w_2 \\ \psi_1 , \psi_2 \end{matrix} \right\rangle_{\mathcal{H}} = \int_0^l (\rho w_1 w_2 + I_\rho \psi_1 \psi_2) \, dx + m w_1(l) w_2(l) + J_c \psi_1(l) \psi_2(l). \tag{3.7}$$

A straightforward consequence of the above definition is the following.

LEMMA 3.1. *Let (λ_1, w_1, ψ_1) and (λ_2, w_2, ψ_2) be nontrivial solutions of (3.5). Then*

$$\left\langle \begin{matrix} w_1 \ w_2 \\ \psi_1 , \psi_2 \end{matrix} \right\rangle_{\mathcal{H}} = 0 \quad if \, \lambda_1 \neq \lambda_2. \tag{3.8}$$

Moreover, if $K(x) = $ const and

$$\frac{2(m + \int_0^l \rho \, dx) g \sin \varphi_0}{K} \leq 1, \qquad \frac{Kl^2}{EI} \leq 2 \tag{3.9}$$

then all eigenvalues λ of (3.5) are nonnegative real numbers.

Proof. If (λ_1, w_1, ψ_1) is a solution of (3.5) then

$$\lambda_1 \left\langle \begin{matrix} w_1 \ w_2 \\ \psi_1 , \psi_2 \end{matrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{matrix} \lambda_1 w_1 \ w_2 \\ \lambda_1 \psi_1 , \psi_2 \end{matrix} \right\rangle_{\mathcal{H}}$$

$$= \int_0^l (K(\psi_1 - w_1') + g\rho_1 w_1' \sin \varphi_0)' w_2 \, dx + \int_0^l (K(\psi_1 - w_1') - (EI\psi_1')') \psi_2 \, dx$$

$$+ (K(w_1'(l) - \psi_1(l)) - mg w_1'(l) \sin \varphi_0) w_2(l) + EI\psi_1'(l) \psi_2(l). \tag{3.10}$$

Performing integration by parts in the above expression, we get

$$\lambda_1 \left\langle \begin{matrix} w_1 \ w_2 \\ \psi_1 , \psi_2 \end{matrix} \right\rangle_{\mathcal{H}} = \int_0^l \{ K(w_1' w_2' + \psi_1 \psi_2 - w_1' \psi_2 - w_2' \psi_1) + EI\psi_1' \psi_2' - g\rho_1 w_1' w_2' \sin \varphi_0 \} \, dx. \tag{3.11}$$

The permutation of arguments in (3.11) yields

$$\lambda_2 \left\langle \begin{matrix} w_1 \ w_2 \\ \psi_1 \ \psi_2 \end{matrix} \right\rangle_{\mathcal{H}} = \lambda_2 \left\langle \begin{matrix} w_2 \ w_1 \\ \psi_2 \ \psi_1 \end{matrix} \right\rangle_{\mathcal{H}} = \lambda_1 \left\langle \begin{matrix} w_1 \ w_2 \\ \psi_1 \ \psi_2 \end{matrix} \right\rangle_{\mathcal{H}}. \tag{3.12}$$

Hence, $\left\langle \begin{smallmatrix} w_1 \ w_2 \\ \psi_1 \ \psi_2 \end{smallmatrix} \right\rangle_{\mathcal{H}} = 0$ if $\lambda_1 \neq \lambda_2$. If $w_2 = w_1$ and $\psi_2 = \psi_1$ then (3.11) implies

$$\begin{aligned}
\lambda_1 \left\langle \begin{matrix} w_1 \ w_1 \\ \psi_1 \ \psi_1 \end{matrix} \right\rangle_{\mathcal{H}} &= \int_0^l \left(K(w_1' - \psi_1)^2 + EI\psi_1'^2 - g\rho_1 w_1'^2 \sin\varphi_0 \right) dx \\
&= \int_0^l \left(\frac{K}{2} w_1'^2 + EI\psi_1'^2 - K\psi_1^2 - g\rho_1 w_1'^2 \sin\varphi_0 \right) dx + \frac{1}{2}\int_0^l K(w_1' - 2\psi_1)^2 dx \\
&\geq \int_0^l \left(\left(\frac{K}{2} - g\rho_1 \sin\varphi_0 \right) w_1'^2 + EI\psi_1'^2 - K\psi_1^2 \right) dx.
\end{aligned} \tag{3.13}$$

The function $\psi_1(x)$ subject to the boundary condition $\psi_1(0) = 0$ satisfies Friedrichs' inequality of the following form (cf. [13, page 440]):

$$\int_0^l \psi_1^2(x)\,dx \leq \frac{l^2}{2}\int_0^l \psi_1'^2(x)\,dx. \tag{3.14}$$

Using this inequality in (3.13), we conclude that

$$\lambda_1 \left\langle \begin{matrix} w_1 \ w_1 \\ \psi_1 \ \psi_1 \end{matrix} \right\rangle_{\mathcal{H}} \geq \int_0^l \left(\left(\frac{K}{2} - g\rho_1 \sin\varphi_0 \right) w_1'^2 + \left(EI - \frac{Kl^2}{2} \right)\psi_1'^2 \right) dx \geq 0, \tag{3.15}$$

provided that the conditions (3.9) are satisfied. This proves that all eigenvalues λ are nonnegative. $\qquad\square$

For the rest of this section we assume that EI, I_ρ, K, and ρ are constants, and that $\sin\varphi_0 = 0$. The coefficients of the Sturm-Liouville problem are constant under this assumption, and, therefore, it is easy to find the general solution of the corresponding system of ODEs. This solution is needed for computing the coefficients of an approximate dynamical model in the sequel (formulae (4.5) define coefficients of the approximate system (4.4) through eigenvalues and eigenfunctions of (3.5)).

We introduce in (3.5) the following dimensionless functions:

$$\zeta\left(\frac{x}{l}\right) = \frac{\overline{w}(x)}{l}, \qquad \theta\left(\frac{x}{l}\right) = \overline{\psi}(x), \tag{3.16}$$

and parameters:

$$p_1 = \frac{\rho l^2}{K}, \qquad p_2 = \frac{Kl^2}{EI}, \qquad p_3 = \frac{I_\rho l^2}{EI}, \qquad p_4 = \frac{ml}{K}, \qquad p_5 = \frac{lJ_c}{EI}. \tag{3.17}$$

Then (3.5) is reduced to the following problem:

$$
\frac{d}{d\tau}
\begin{pmatrix}
\zeta(\tau) \\
\zeta_\tau(\tau) \\
\theta(\tau) \\
\theta_\tau(\tau)
\end{pmatrix}
=
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-\lambda p_1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & -p_2 & p_2 - \lambda p_3 & 0
\end{pmatrix}
\times
\begin{pmatrix}
\zeta(\tau) \\
\zeta_\tau(\tau) \\
\theta(\tau) \\
\theta_\tau(\tau)
\end{pmatrix},
\quad \tau = \frac{x}{l} \in (0,1); \quad (3.18)
$$

$$
\zeta_\tau(1) - \theta(1) = \lambda p_4 \zeta(1), \qquad \theta_\tau(1) = \lambda p_5 \theta(1), \qquad \zeta(0) = \theta(0) = 0, \qquad (3.19)
$$

where $\zeta_\tau(\tau)$ and $\theta_\tau(\tau)$ stand for derivatives with respect to τ. The eigenvalues and eigen-vectors of the matrix in (3.18) are, respectively, given by

$$
\mu_j = i\sigma_j, \quad v_j =
\begin{pmatrix}
4\mu_j (\lambda p_3 - \sigma_j{}^2) \\
\lambda c_3 + 4\lambda p_1 (\sigma_j{}^2 - \lambda p_3) \\
\lambda c_3 \\
\lambda c_3 \mu_j
\end{pmatrix},
\quad j = 1,2,3,4, \quad (3.20)
$$

where

$$
\sigma_1 = -\sigma_2 = \frac{\sqrt{2}}{2} \sqrt{c_1 \lambda - \sqrt{c_2{}^2 \lambda^2 + c_3 \lambda}}, \qquad \sigma_3 = -\sigma_4 = \frac{\sqrt{2}}{2} \sqrt{c_1 \lambda + \sqrt{c_2{}^2 \lambda^2 + c_3 \lambda}}, \quad (3.21)
$$

$$
c_1 = p_1 + p_3, \qquad c_2 = p_1 - p_3, \qquad c_3 = 4 p_1 p_2.
$$

The general solution of (3.18) therefore reads as

$$
(\zeta, \zeta_\tau, \theta, \theta_\tau)^T (\tau) = C_1 v_1 e^{i\sigma_1 \tau} + C_2 v_2 e^{-i\sigma_1 \tau} + C_3 v_3 e^{i\sigma_3 \tau} + C_4 v_4 e^{-i\sigma_3 \tau}. \quad (3.22)
$$

By substituting (3.22) into the boundary conditions (3.19), we get a system of linear algebraic equations with respect to (complex) variables C_1, C_2, C_3, C_4. That system has a nontrivial solution if

$$
\varkappa(\lambda) =
$$

$$
\begin{vmatrix}
e^{-i\sigma_1} & e^{i\sigma_1} & e^{-i\sigma_3} & e^{i\sigma_3} \\
\sigma_1 (\sigma_1^2 - \lambda p_3) e^{-i\sigma_1} & -\sigma_1 (\sigma_1^2 - \lambda p_3) e^{i\sigma_1} & \sigma_3 (\sigma_3^2 - \lambda p_3) e^{-i\sigma_3} & -\sigma_3 (\sigma_3^2 - \lambda p_3) e^{i\sigma_3} \\
(\sigma_1^2 - \lambda p_3)(p_1 + ip_4\sigma_1) & (\sigma_1^2 - \lambda p_3)(p_1 - ip_4\sigma_1) & (\sigma_3^2 - \lambda p_3)(p_1 + ip_4\sigma_3) & (\sigma_3^2 - \lambda p_3)(p_1 - ip_4\sigma_3) \\
i\sigma_1 - \lambda p_5 & i\sigma_1 + \lambda p_5 & i\sigma_3 - \lambda p_5 & i\sigma_3 + \lambda p_5
\end{vmatrix}
= 0.
$$

$$
(3.23)
$$

The roots of $\varkappa(\lambda) = 0$ define the eigenvalues λ for the Sturm-Liouville problem (3.5) when its coefficients are constant. It is clear that the function $\varkappa(\lambda)$, given by (3.23), is analytic in its domain of definition. Then the uniqueness theorem for analytic functions implies that either $\varkappa(\lambda) \equiv 0$ or the set of all eigenvalues for (3.5) is discrete. The for-mer is impossible for "typical" values of parameters (see, for example, [13], where the spectrum was estimated for a particular case $p_1 = p_2 = p_3 = 1$, $p_4 = p_5 = 0$). We do not estimate solutions of the characteristic equation (3.23) here. Such a study requires addi-tional assumptions on the mechanical parameters, based on real measurements, and is not of principal interest for this work.

4. The Galerkin approximation

To derive a Galerkin approximation (see, e.g., [6]), we consider a variational formulation of the boundary value problem as follows: if $(\widetilde{\varphi}(t), \widetilde{w}(x,t), \widetilde{\psi}(x,t))$ $(0 \le x \le l)$ is a solution of (3.2), corresponding to $M(t)$, on an interval $t \in \mathscr{I} \subset \mathbb{R}$ then

$$\ddot{\widetilde{\varphi}}(t) - v = 0,$$

$$\widetilde{\mu} = \int_0^l \delta\widetilde{w}(x,t)\big\{ (\ddot{\widetilde{w}} + xv - g\widetilde{\varphi}\sin\varphi_0)\rho$$

$$+ \big(K(\widetilde{\psi} - \widetilde{w}') + \rho_1 g(\widetilde{w}'\sin\varphi_0 + \widetilde{\varphi}w_0'\cos\varphi_0)\big)' + \cdots \big\}dx$$

$$+ \int_0^l \delta\widetilde{\psi}(x,t)\big\{ I_\rho\ddot{\widetilde{\psi}} + K(\widetilde{\psi} - \widetilde{w}') - (EI\widetilde{\psi}')' + I_\rho v\big\}dx + \delta\widetilde{\psi}(l,t)\big\{ J_c\ddot{\widetilde{\psi}} + EI\widetilde{\psi}' + J_c v\big\}|_{x=l}$$

$$+ \delta\widetilde{w}(l,t)\big\{ K(\widetilde{w}' - \widetilde{\psi}) + m(\ddot{\widetilde{w}} + lv - g(\widetilde{\varphi} + \widetilde{w}')\sin\varphi_0 - g\widetilde{\varphi}w_0'\cos\varphi_0) + \cdots \big\}|_{x=l} = 0,$$
$$\forall t \in \mathscr{I},$$
$$(4.1)$$

for each admissible variation $(\delta\widetilde{w}(x,t), \delta\widetilde{\psi}(x,t))$ satisfying the boundary conditions $\delta\widetilde{w}(0, t) = 0$ and $\delta\widetilde{\psi}(0,t) = 0$. (The derivation of $\widetilde{\mu}$ from (3.2) uses the standard technique: integration by parts, collecting terms, and so forth. The expression (4.1) may also be obtained by expanding (2.9) in a neighborhood of the equilibrium and neglecting the higher order terms.) Here v is given by the expression (3.3).

Let us fix an integer number $N \ge 1$ and consider nontrivial solutions (λ_j, w_j, ψ_j) of (3.5) for $j = 1, 2, \ldots, N$. We assume that all λ_j are different and substitute finite sums

$$\widetilde{w}(x,t) = \sum_{j=1}^N q_j(t)w_j(x), \quad \widetilde{\psi}(x,t) = \sum_{j=1}^N q_j(t)\psi_j(x) \tag{4.2}$$

into (3.3) and (4.1). We also restrict $\delta\widetilde{w}$ and $\delta\widetilde{\psi}$ to finite-dimensional subspaces:

$$\delta\widetilde{w}(\cdot,t) \in \mathrm{span}\{w_1(\cdot),\ldots,w_N(\cdot)\}, \qquad \delta\widetilde{\psi}(\cdot,t) \in \mathrm{span}\{\psi_1(\cdot),\ldots,\psi_N(\cdot)\}. \tag{4.3}$$

By assuming $\delta\widetilde{w}(x,t) = w_i(x)$ and $\delta\widetilde{\psi}(x,t) = \psi_i(x)$ in (4.1) for $i = 1,2,\ldots,N$ and exploiting Lemma 3.1, we obtain the following control system with respect to $\widetilde{\varphi}, q_1, q_2, \ldots, q_N$:

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 + B_1 u + R_1(z,u),$$
$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2 u + R_2(z,u), \quad z = (z_1^T, z_2^T)^T, \tag{4.4}$$

where z is the state, u is the control,

$$z_1 = (\widetilde{\varphi}, \dot{\widetilde{\varphi}})^T, \qquad z_2 = (q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_N, \dot{q}_N)^T,$$

$$u = \frac{\widetilde{M}}{J_0 + \int_0^l (w_0^2 \rho - w_0'^2 \rho_2)\,dx + m w_0^2(l)},$$

$$A_{11} = \begin{pmatrix} 0 & 1 \\ d_0 & 0 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ d_1 & 0 & d_2 & 0 & \cdots & d_N & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad B_2 = (0, -b_1, \ldots, 0, -b_N)^T,$$

$$A_{21} = \begin{pmatrix} 0 & 0 \\ a_1 - b_1 d_0 & 0 \\ 0 & 0 \\ a_2 - b_2 d_0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ a_N - b_N d_0 & 0 \end{pmatrix}, \qquad A_{22} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda_1 - b_1 d_1 & 0 & -b_1 d_2 & 0 & \cdots & -b_1 d_N & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ -b_2 d_1 & 0 & -\lambda_2 - b_2 d_2 & 0 & \cdots & -b_2 d_N & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -b_N d_1 & 0 & -b_N d_2 & 0 & \cdots & -\lambda_N - b_N d_N & 0 \end{pmatrix},$$

$$a_j = g \frac{\left(\int_0^l \rho w_j\,dx + m w_j(l)\right)\sin\varphi_0 + \int_0^l \rho_1 w_0' w_j'\,dx \cos\varphi_0}{\int_0^l (\rho w_j^2 + I_\rho \psi_j^2)\,dx + m w_j^2(l) + J_c \psi_j^2(l)},$$

$$b_j = \frac{\int_0^l (\rho x w_j + I_\rho \psi_j)\,dx + m l w_j(l) + J_c \psi_j(l)}{\int_0^l (\rho w_j^2 + I_\rho \psi_j^2)\,dx + m w_j^2(l) + J_c \psi_j^2(l)},$$

$$d_0 = \frac{\int_0^l [(\rho w_0 - \rho_1 w_0')\cos\varphi_0 - (1/2)(w_0')^2 \rho_1 \sin\varphi_0]\,dx + m w_0(l)\cos\varphi_0}{J_0 + \int_0^l (w_0^2 \rho - w_0'^2 \rho_2)\,dx + m w_0^2(l)} g,$$

$$d_j = \frac{g \int_0^l [(w_0' \cos\varphi_0 - \sin\varphi_0)\rho_1 w_j' + \rho w_j \sin\varphi_0]\,dx + m g w_j(l)\sin\varphi_0 + EI \psi_j'(0)}{J_0 + \int_0^l (w_0^2 \rho - w_0'^2 \rho_2)\,dx + m w_0^2(l)},$$

(4.5)

the nonlinear term $R(z,u) = (R_1^T, R_2^T)^T$ satisfies the estimate

$$\|R(z,u)\| = O(\|z\|^2 + u^2) \tag{4.6}$$

around the equilibrium point $z = 0$, $u = 0$. The control system (4.4) is a finite dimensional approximation of (3.2) corresponding to the flexible coordinates of order up to N.

5. Stabilization in finite dimensions

In this section, an explicit procedure for stabilizing controller design is proposed.

THEOREM 5.1. *Assume that all eigenvalues $(\lambda_1,\dots,\lambda_N)$ are positive and different, and that $a_j + \lambda_j b_j \neq 0$ for each $j = 1,2,\dots,N$. Then system (4.4) is stabilizable by the following feedback control:*

$$u = Kz, \qquad K = (K_1, K_2), \qquad K_1 = \left(-d_0 - \frac{h_1 + \sum_{j=1}^{N} a_j(b_j + a_j/\lambda_j)}{h_2}, -\frac{h_0}{h_2} \right),$$

$$K_2 = \left(-d_1 + \frac{a_1 + \lambda_1 b_1}{h_2}, 0, -d_2 + \frac{a_2 + \lambda_2 b_2}{h_2}, 0, \dots, -d_N + \frac{a_N + \lambda_N b_N}{h_2}, 0 \right),$$

(5.1)

where h_0, h_1, and h_2 are arbitrary positive constants.

Proof. Consider a Lyapunov function candidate

$$2V(z) = \left(h_1 + \sum_{j=1}^{N} \frac{a_j^2}{\lambda_j} \right) \tilde{\varphi}^2 + \left(h_2 + \sum_{j=1}^{N} b_j^2 \right) \dot{\tilde{\varphi}}^2 + \sum_{j=1}^{N} \left(\lambda_j q_j^2 + \dot{q}_j^2 - 2a_j \tilde{\varphi} q_j + 2b_j \dot{\tilde{\varphi}} \dot{q}_j \right).$$

(5.2)

By applying the Cauchy-Schwartz inequality, we get

$$2V \geq G_1 \left(-|\tilde{\varphi}|, \left(\sum_{j=1}^{N} \lambda_j q_j^2 \right)^{1/2} \right) + G_2 \left(-|\dot{\tilde{\varphi}}|, \left(\sum_{j=1}^{N} \dot{q}_j^2 \right)^{1/2} \right),$$

(5.3)

where

$$G_1(\alpha,\beta) = \left(h_1 + \sum_{j=1}^{N} \frac{a_j^2}{\lambda_j} \right) \alpha^2 + 2 \left(\sum_{j=1}^{N} \frac{a_j^2}{\lambda_j} \right)^{1/2} \alpha\beta + \beta^2,$$

$$G_2(\alpha,\beta) = \left(h_2 + \sum_{j=1}^{N} b_j^2 \right) \alpha^2 + 2 \left(\sum_{j=1}^{N} b_j^2 \right)^{1/2} \alpha\beta + \beta^2.$$

(5.4)

Sylvester's criterion for quadratic forms G_1 and G_2 implies that both G_1 and G_2 are positive definite if $h_1 > 0$ and $h_2 > 0$. Then the quadratic form V is positive definite due to estimate (5.3).

The time-derivative of V along the trajectories of the linear part of (4.4) is

$$\dot{V} = h_2 \dot{\tilde{\varphi}} v + \left(h_1 + h_2 d_0 + \sum_{j=1}^{N} a_j \left(\frac{b_j + a_j}{\lambda_j} \right) \right) \tilde{\varphi} \dot{\tilde{\varphi}} + \dot{\tilde{\varphi}} \sum_{j=1}^{N} q_j (h_2 d_j - a_j - \lambda_j b_j).$$

(5.5)

We choose a constant $h_0 > 0$ arbitrarily and define the feedback control in order to have $\dot{V} = -h_0 \dot{\tilde{\varphi}}^2$. This yields expression (5.1).

Now we apply the Barbashin-Krasovskii theorem (or LaSalle's invariance principle, cf. [21]). For this purpose consider the set

$$Z_0 = \{(\widetilde{\varphi}, \dot{\widetilde{\varphi}}, q_1, \dot{q}_1, \ldots, q_N, \dot{q}_N) \in \mathbb{R}^{2N+2} : \dot{V} = 0\}. \tag{5.6}$$

Each positive semitrajectory of the linear approximation of (4.4) with (5.1) on Z_0 satisfies the following relations:

$$\ddot{q}_j = -\lambda_j q_j + a_j \widetilde{\varphi},$$

$$\sum_{j=1}^{N} \left[-a_j b_j \widetilde{\varphi} + (a_j + \lambda_j b_j) q_j \right] = \left(h_1 + \sum_{j=1}^{N} \frac{a_j^2}{\lambda_j} \right) \widetilde{\varphi} = \text{const}, \quad t \geq 0. \tag{5.7}$$

The above relations imply

$$\sum_{j=1}^{N} (a_j + \lambda_j b_j) \left(r_{1j} \cos\left(\sqrt{\lambda_j} t\right) + r_{2j} \sin\left(\sqrt{\lambda_j} t\right) \right) = h_1 \widetilde{\varphi} \tag{5.8}$$

for some constants r_{1j}, r_{2j}, and $\widetilde{\varphi}$. Exploiting the fact that

$$\left\{ 1, \sin\left(\sqrt{\lambda_j} t\right), \cos\left(\sqrt{\lambda_j} t\right) \right\}_{j=1}^{N} \tag{5.9}$$

are linearly-independent functions on $[0, +\infty)$ (cf. [22]), we get that (5.8) is possible only if $\widetilde{\varphi} = 0$ and $r_{1j} = r_{2j} = 0$ for all $j = 1, 2, \ldots, N$. Thus, the only semitrajectory of the linearized closed-loop system on Z_0 is the trivial one, and the trivial solution of the linear part of (4.4), (5.1) is asymptotically stable by the Barbashin-Krasovskii theorem (LaSalle's invariance principle). Now local asymptotic stability of the nonlinear closed-loop system follows from Lyapunov's theorem on stability using linearization. □

Remark 5.2. As it follows from the representation $\dot{V} = -h_0 \dot{\widetilde{\varphi}}^2$, the choice of constant h_0 affects the decay rate of the Lyapunov function along trajectories of the closed-loop system. On the one hand, the more h_0 the faster convergence of solutions to the equilibrium could be achieved (for solutions with $\dot{\widetilde{\varphi}} \neq 0$). On the other hand, for large h_0, the gain $-h_0/h_2$, appearing in formula (5.1), may take large values if h_2 is small. This suggests us to choose h_0 as maximal as possible, and to select h_2 in such a way that the term $-(h_0/h_2)\dot{\widetilde{\varphi}}$, appearing in $u = Kz$, would not bring the control input u to its saturation bound (for typical disturbances $\dot{\widetilde{\varphi}}$). The constant h_1 should be then defined according to a desired geometry of the level surfaces for the quadratic form V. Indeed, constants h_1 and h_2 define a relation between semiaxes for the ellipsoids $V(z) = \text{const}$, and hence a desired ratio between overshoots for $\widetilde{\varphi}$ and $\dot{\widetilde{\varphi}}$ can be estimated in terms of h_1 and h_2. Certainly, this suggestion is based on the linearized system and does not give a rigorous characterization of the global behavior.

6. Observer design

In order to implement the feedback law (5.1) in practice, one should reconstruct the complete state vector of (4.4) from the outputs which can be measured. The values of $w(x, t)$

and $\psi(x,t)$ cannot be directly estimated in a real flexible manipulator. Instead, there is a set of strain gauges located at a point $x = l_0, 0 \leq l_0 \leq l$, which allows measurement of some components of the strain tensor. By using only the principal part of the strain at $x = l_0$, we get the output $\psi'(x,t)|_{x=l_0}$ for each $t \geq 0$. By subtracting from the signals $\varphi(t)$ and $\psi'(x,t)|_{x=l_0}$ their steady-state values and rescaling, we assume that the following outputs are available for the finite dimensional approximation (4.4):

$$y_1(t) = \tilde{\varphi}(t), \qquad y_2(t) = l^2 \tilde{\psi}'(x,t)|_{x=l_0} = \sum_{j=1}^{N} \chi_j q_j(t), \tag{6.1}$$

where $\chi_j = l^2 \psi_j'(l_0)$. We introduce the factor l^2 in order to get the dimension of length for the output y_2.

Let us rewrite the output (6.1) as follows:

$$y_1 = C_1 z_1, \quad y_2 = C_2 z_2, \quad C_1 = (1,0), \quad C_2 = (\chi_1, 0, \chi_2, 0, \dots, \chi_N, 0). \tag{6.2}$$

LEMMA 6.1. *The control system (4.4), (6.2) is locally observable at $z = 0$ if*

$$\begin{vmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1} & \pi_{N2} & \cdots & \pi_{NN} \end{vmatrix} \neq 0, \tag{6.3}$$

where $\pi_{1,j} = \chi_j$, $\pi_{k,j} = -\lambda_j \pi_{k-1,j} - d_j \sum_{i=1}^{N} \pi_{k-1,i} b_i$, $j = \overline{1,N}$, $k = \overline{2,N}$.
In particular, the condition (6.3) is equivalent to $\chi_1 \neq 1$ if $N = 1$ or

$$\chi_1 \chi_2 (\lambda_1 - \lambda_2 + b_1 d_1 - b_2 d_2) + b_2 \chi_2^2 d_1 - b_1 \chi_1^2 d_2 \neq 0 \quad \text{if } N = 2. \tag{6.4}$$

Proof. The linear part of (4.4), (6.2) can be written in terms of output y_1 as follows:

$$\begin{aligned} z_1 &= (y_1, \dot{y}_1)^T, \\ \dot{z}_2 &= A_{22} z_2 + B_2 u + (0, a_1 - b_1 d_0, 0, a_2 - b_2 d_0, \dots, 0, a_N - b_N d_0)^T y_1, \\ y_2 &= C_2 z_2. \end{aligned} \tag{6.5}$$

Hence, the above system is observable if the pair (A_{22}, C_2) satisfies the Kalman observability condition (cf. [23, Theorem 3.1, page 58]):

$$\text{rank} \begin{pmatrix} C_2 \\ C_2 A_{22} \\ \vdots \\ C_2 A_{22}^{2N-1} \end{pmatrix} = 2N. \tag{6.6}$$

Straightforward computations show that

$$\det \begin{pmatrix} C_2 \\ C_2 A_{22} \\ \vdots \\ C_2 A_{22}^{2N-1} \end{pmatrix} = \begin{vmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1} & \pi_{N2} & \cdots & \pi_{NN} \end{vmatrix}^2. \tag{6.7}$$

Therefore, (6.3) implies the observability rank condition for the linear part of (4.4), (6.2). It also means that (4.4), (6.2) is strongly locally observable at $z = 0$ by the Hermann-Krener theorem [24]. $\square$

The following theorem gives an explicit procedure for the Luenberger-type observer design.

THEOREM 6.2. *Suppose that the control system (4.4), (6.2) satisfies the observability condition (6.3), all λ_j are positive and different, $a_j + \lambda_j b_j \neq 0$, and $b_j d_j > 0$ for $j = \overline{1,N}$. Then the origin $z = 0$, $\bar{z} = 0$ of the extended system (4.4), (6.2) and*

$$\dot{\bar{z}}_1 = (A_{11} - F_1 C_1)\bar{z}_1 + A_{12}\bar{z}_2 + F_1 y_1 + B_1 u,$$
$$\dot{\bar{z}}_2 = (A_{22} - F_{22} C_2)\bar{z}_2 + F_{21} y_1 + F_{22} y_2 + B_2 u \tag{6.8}$$

with $u = K\bar{z}$ is locally asymptotically stable, where K is given by (5.1),

$$F_1 = (\phi_1, d_0 + \phi_2)^T, \qquad F_{21} = (0, a_1 - b_1 d_0, 0, a_2 - b_2 d_0, \ldots, 0, a_N - b_N d_0)^T,$$
$$F_{22} = (f_1, 0, f_2, 0, \ldots, f_N, 0)^T, \qquad (f_1, f_2, \ldots, f_N)^T = \gamma Q^{-1}(\chi_1, \chi_2, \ldots, \chi_N)^T,$$

$$Q = \begin{pmatrix} \dfrac{\lambda_1 d_1}{b_1} + d_1^2 & d_1 d_2 & \cdots & d_1 d_N \\ d_2 d_1 & \dfrac{\lambda_2 d_2}{b_2} + d_2^2 & \cdots & d_2 d_N \\ \vdots & \vdots & \ddots & \vdots \\ d_N d_1 & d_N d_2 & \cdots & \dfrac{\lambda_N d_N}{b_N} + d_N^2 \end{pmatrix}. \tag{6.9}$$

Here ϕ_1, ϕ_2, and γ are any positive constants.

Proof. Consider the observation errors $e_1 = z_1 - \bar{z}_1$, $e_2 = z_2 - \bar{z}_2$. Then subtracting (6.8) from (4.4) yields the following dynamics:

$$\dot{e}_1 = H_1 e_1 + A_{12} e_2 + R_1(z,u), \qquad \dot{e}_2 = H_2 e_2 + R_2(z,u), \tag{6.10}$$

here $H_1 = A_{11} - F_1 C_1$ and $H_2 = A_{22} - F_{22} C_2$. We see that the roots of the polynomial

$$\det(H_1 - \mu I) = \begin{vmatrix} -\phi_1 - \mu & 1 \\ -\phi_2 & -\mu \end{vmatrix} = \mu^2 + \phi_1 \mu + \phi_2 \tag{6.11}$$

have negative real parts if and only if $\phi_1 > 0$ and $\phi_2 > 0$. Our goal is to show that the real parts of all eigenvalues of

$$
H_2 = \begin{pmatrix}
-f_1\chi_1 & 1 & -f_1\chi_2 & 0 & \cdots & -f_1\chi_N & 0 \\
-\lambda_1 - b_1 d_1 & 0 & -b_1 d_2 & 0 & \cdots & -b_1 d_N & 0 \\
-f_2\chi_1 & 0 & -f_2\chi_2 & 1 & \cdots & -f_2\chi_N & 0 \\
-b_2 d_1 & 0 & -\lambda_2 - b_2 d_2 & 0 & \cdots & -b_2 d_N & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-f_N\chi_1 & 0 & -f_N\chi_2 & 0 & \cdots & -f_N\chi_N & 1 \\
-b_N d_1 & 0 & -b_N d_2 & 0 & \cdots & -\lambda_N - b_N d_N & 0
\end{pmatrix} \tag{6.12}
$$

are also negative if the conditions of Theorem 6.2 are satisfied. Let us denote $e_2 = (\xi_1, \eta_1, \ldots, \xi_N, \eta_N)^T$ and consider the following quadratic form:

$$
2W(e_2) = \sum_{j=1}^{N} \frac{d_j \eta_j^2}{b_j} + (\xi_1, \xi_2, \ldots, \xi_N) Q (\xi_1, \xi_2, \ldots, \xi_N)^T. \tag{6.13}
$$

This form is positive definite as $\lambda_j > 0$ and $b_j d_j > 0$. Indeed, all principal minors Δ_j of Q are positive:

$$
\Delta_j = \frac{(\lambda_1 d_1)(\lambda_2 d_2) \cdots (\lambda_j d_j)}{b_1 b_2 \cdots b_j} \left(1 + \sum_{i=1}^{j} \frac{b_i d_i}{\lambda_i} \right) > 0, \quad j = \overline{1, N}. \tag{6.14}
$$

Then Sylvester's criterion implies that W is positive definite. The inequality $\det(Q) = \Delta_N > 0$ also proves invertibility of Q in (6.9). By computing the time derivative of W along the trajectories of the linear system $\dot{e}_2 = H_2 e_2$, we get

$$
\dot{W}(e_2) = -\gamma (C_2 e_2)^2 \leq 0, \tag{6.15}
$$

provided that F_{22} is defined by (6.9). As the time derivative of W is negative semi-definite and vanishes on $\ker C_2 = \{e_2 \in \mathbb{R}^{2N} : C_2 e_2 = 0\}$, we check whether the linear system $\dot{e}_2 = H_2 e_2$ admits a nontrivial semitrajectory on $\ker C_2$. Let $C_2 e_2(t) \equiv 0$, $t \geq 0$, then

$$
\frac{d^k}{dt^k} C_2 e_2(t) = C_2 (A_{22} - F_{22} C_2)^k e_2(t) = C_2 A_{22}^k e_2(t) = 0, \quad t \geq 0, \; k \geq 0. \tag{6.16}
$$

This implies that, for each $t \geq 0$, $e_2(t)$ is a solution of the following system of linear algebraic equations:

$$
C_2 A_{22}^k e_2(t) = 0, \quad k = \overline{0, 2N - 1}. \tag{6.17}
$$

The above system has only the trivial solution $e_2(t) = 0$ because of the observability rank condition (6.3). This proves asymptotic stability of the linear system $\dot{e}_2 = H_2 e_2$ by the Barbashin-Krasovskii theorem (LaSalle's invariance principle).

We have shown that the matrices H_1 and H_2 are Hurwitz. The nonlinear closed-loop system (4.4), (6.2), (6.8) with $u = K\bar{z}$ can be written in variables (z, e) as follows:

$$\begin{pmatrix} \dot{z} \\ \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} H_0 & -BK_1 & -BK_2 \\ 0 & H_1 & A_{12} \\ 0 & 0 & H_2 \end{pmatrix} \begin{pmatrix} z \\ e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} R(z, K(z - e)) \\ R_1(z, K(z - e)) \\ R_2(z, K(z - e)) \end{pmatrix}, \qquad (6.18)$$

where

$$H_0 = \begin{pmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{pmatrix}, \quad B = (B_1^T, B_2^T)^T. \qquad (6.19)$$

As $a_j + \lambda_j b_j \neq 0$ then the conditions of Theorem 5.1 are satisfied and H_0 is Hurwitz. Hence, the trivial solution of (6.18) is asymptotically stable by linear approximation as the spectrum of its matrix is the union of spectra of the Hurwitz matrices H_0, H_1, and H_2. $\qquad\qquad\square$

7. Conclusions

We have proposed a feedback controller that stabilizes the equilibrium of a Galerkin approximation for a rotating Timoshenko beam, provided that measurements of the raising angle and the strain at a point are available. The feedback law and coefficients of the dynamical observer are computed explicitly for any number of modal coordinates. A potential field of application of these results is the control design for fire-rescue turntable ladders. An advantage of our approach is that the identification procedure can be reduced significantly in comparison with a multibody model. In addition, the higher modes can be calculated explicitly, which is important for the design of an oscillation damping control of a turntable ladder. For a possible implementation of the controller, it is necessary to integrate a system of ordinary differential equations in real time. We do not consider here such issues as spillover analysis, convergence of Galerkin approximations, computational complexity, or limitation of the sampling rate with respect to the calculation time leaving these problems for future work.

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