Research Article

Homotopy Perturbation Method for Solving Fourth-Order Boundary Value Problems

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We apply the homotopy perturbation method for solving the fourth-order boundary value problems. The analytical results of the boundary value problems have been obtained in terms of convergent series with easily computable components. Several examples are given to illustrate the efficiency and implementation of the homotopy perturbation method. Comparisons are made to confirm the reliability of the method. Homotopy method can be considered an alternative method to Adomian decomposition method and its variant forms.

1. Introduction

In recent years, much attention has been given to develop some analytical methods for solving integral equations including the perturbation methods and decomposition method. It is well known that perturbation methods [1, 2] provide the most versatile tools available in nonlinear analysis of engineering problems. The major drawback in the traditional perturbation technique is the over dependence on the existence of small parameter. This condition is overstrict and greatly affects the applications of the perturbation techniques because most of the nonlinear problems (especially those having strong nonlinearity) do not even contain the so-called small parameter; moreover, the determination of the small parameter is a complicated process and requires special techniques. These facts have motivated to suggest alternative techniques such as the homotopy analysis method [3, 4], decomposition and the variational iteration method [5–8]. In order to overcome these drawbacks, combining the standard homotopy method and perturbation,

we obtain a modified method, which is called the homotopy perturbation method. This technique has been used by Noor and Mohyud-Din [9] for solving fifth-order boundary value problems. Using the idea of Noor and Mohyud-Din [9], we develop a homotopy perturbation method for solving a system of integral equations associated with fourth-order boundary value problems. It is shown that this method provides the solution in a rapid convergent series. We show that this method is easy to implement and it is more efficient than the Adomian method. We remark that to apply the Adomian method, one has to evaluate the derivative of the so-called Adomian polynomial, which is itself a complicated problem. On the other hand, homotopy perturbation is very simple to apply, which is the main characteristic of this method. Several examples are given to illustrate the performance of the method.

In this paper, we consider the general fifth-order boundary value problems of the type

$$u^{(v)}(x) = f(x, u, u', u'', u'''),\tag{1.1}$$

with boundary conditions

$$u(a) = \alpha_1, \qquad u'(a) = \alpha_2, \qquad u(b) = \beta_1, \qquad u'(b) = \beta_2,\tag{1.2}$$

where f is continuous function on $[a, b]$ and the parameters α_i and β_i, $i = 1, 2$, are real constants. Such type of boundary value problems arise in the mathematical modeling of the viscoelastic flows, deformation of beams, and plate deflection theory and other branches of mathematical, physical, and engineering sciences, see [10, 11, 7, 12, 8] and the references therein. Several numerical methods including finite difference, B-spline were developed for solving fourth-order boundary value problems, see [11]. Computational results have also been obtained in [12] for a special fourth-order boundary value problems with nonlinear boundary conditions of third-order. Noor and Mohydu-Din [8] used the variational iteration to solve the fourth-order boundary value problems. In this paper, we use the homotopy perturbation method coupled with the integral equations to solve the fourth-order boundary value problems. Several examples are given to illustrate the performance and efficiency of the method developed in this paper. Our experience shows that the homotopy perturbation technique can be considered as an alternative to decomposition and variational iteration techniques.

2. Homotopy perturbation method

Consider the following system of the integral equations:

$$F(t) = G(t) + \lambda \int_0^t K(t, s) F(s) ds,\tag{2.1}$$

where

$$F(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T,$$

$$G(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T,$$ (2.2)

$$K(t,s) = [k_{ij}(t,s)], \quad i = 1,2,3,\ldots,n : j = 1,2,3,\ldots,n.$$

To convey an idea of the homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0,$$ (2.3)

where L is an integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u),$$ (2.4)

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that

$$H(u, p) = 0,$$ (2.5)

from which we have $H(u, 0) = F(u)$ and $H(u, 1) = L(u)$.

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution $H(f, 1)$. The embedding parameter increases monotonically from zero to unit as the problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter can be considered as an expanding parameter [7]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [7] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots.$$ (2.6)

If $p \to 1$, then (2.5) corresponds to (2.3) and becomes the approximate solution of the form

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i.$$ (2.7)

It is well known that the series (2.7) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$, see [7]. We assume that problem (2.1) has a unique solution.

Consider the ith equation of (2.1), take

$$f_1(t) = \sum_{i=0}^{\infty} p^i u_i,$$

$$f_2(t) = \sum_{i=0}^{\infty} p^i v_i,$$ (2.8)

$$f_3(t) = \sum_{i=0}^{\infty} p^i w_i,$$

$$\vdots$$

The comparison of like powers of p gives solution of various orders.

3. Applications

In this section, we first show that the fourth-order boundary value problems of the type (1.1) can be reformulated as a system of integral equations. We then use the homotopy perturbation method developed in Section 2 to solve the resultant system of integral equations. To illustrate the implementation of the homotopy method, we consider the following examples.

Example 3.1 [8]. Consider the following nonlinear initial boundary value problem:

$$u^{(iv)} = u^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \quad (3.1)$$

with boundary conditions

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = A, \qquad u'(1) = B. \quad (3.2)$$

The exact solution of this problem is

$$u(x) = x^5 - 2x^4 + 2x^2. \quad (3.3)$$

Using the transformation $du/dx = q(x)$, $dq/dx = f(x)$, $df/dx = z(x)$, we can rewrite the boundary value problem (3.1) and (3.2) as a system of differential equations:

$$\frac{du}{dx} = q(x),$$

$$\frac{dq}{dx} = f(x),$$

$$\frac{df}{dx} = z(x),$$ (3.4)

$$\frac{dz}{dx} = u^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$$

with $u(0) = 0$, $q(0) = 1$, $f(0) = A$, $z(0) = B$, which can be written as a system of integral equations:

$$u(x) = 0 + \int_0^x q^{(k)}(t)dt,$$

$$q(x) = 0 + \int_0^x f^{(k)}(t)dt,$$

$$f(x) = A + \int_0^x z^{(k)}(t)dt,$$

$$(3.5)$$

$$z(x) = B + \int_0^x \left(u^{(k)}(t) - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48\right)dt.$$

Using (2.4) and (2.6) for (3.5), we have

$$u_0 + pu_1 + p^2u_2 + \cdots = 0 + p\int_0^x \left(v_0 + pv_1 + p^2v_2 + \cdots\right)dx,$$

$$v_0 + pv_1 + p^2v_2 + \cdots = 0 + p\int_0^x \left(s_0 + ps_1 + s^2a_2 + \cdots\right)dx,$$

$$s_0 + ps_1 + p^2s_2 + \cdots = 0 + p\int_0^x \left(t_0 + pt_1 + p^2t_2 + \cdots\right)dx,$$

$$t_0 + pt_1 + p^2t_2 + \cdots = A + p\int_0^x \left(\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right.$$

$$\left. + \left(-x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48\right)\right)dx.$$

$$(3.6)$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)} : \begin{cases} u_0 = 0, \\ v_0 = 0, \\ s_0 = A, \\ t_0 = B, \end{cases} \qquad p^{(1)} : \begin{cases} u_1 = 0, \\ v_1 = Ax, \\ s_1 = Bx, \\ t_1 = -\dfrac{x^{11}}{11} + \dfrac{4x^{10}}{10} - \dfrac{4x^9}{9} - \dfrac{4x^8}{8} + 60x - 48x, \end{cases}$$

$$p^{(2)} : \begin{cases} u_2 = \dfrac{A}{2}x^2, \\ v_2 = \dfrac{Bx^2}{2}, \\ s_2 = -\dfrac{x^{12}}{132} + \dfrac{2x^{11}}{55} - \dfrac{2x^{10}}{45} - \dfrac{x^9}{18} + 20x^3 - 24x^2, \\ t_2 = 0, \end{cases}$$

$$p^{(3)}: \begin{cases} u_3 = \dfrac{Bx^3}{3!}, \\ v_3 = -\dfrac{x^{13}}{1716} + \dfrac{2x^{12}}{330} - \dfrac{2x^{11}}{495} - \dfrac{x^{10}}{180} + 5x^4 - 8x^3, \\ s_3 = 0, \\ t_3 = 0, \end{cases}$$

$$p^{(4)}: \begin{cases} u_4 = -\dfrac{x^{14}}{24024} + \dfrac{x^{13}}{4290} - \dfrac{x^{12}}{4290} - \dfrac{x^{11}}{1980} + x^5 - 2x^4, \\ v_4 = 0, \\ s_4 = 0, \\ t_4 = 0, \end{cases}$$

$$p^{(5)}: \begin{cases} u_5 = 0, \\ v_5 = 0, \\ s_5 = 0, \\ t_5 = \dfrac{A^2 x^5}{20}, \end{cases} \qquad p^{(6)}: \begin{cases} u_6 = 0, \\ v_6 = 0, \\ s_6 = \dfrac{A^2}{120}x^6, \\ t_6 = \dfrac{ABx^6}{36}, \end{cases}$$

$$p^{(7)}: \begin{cases} u_7 = 0, \\ v_7 = \dfrac{A^2 x^7}{840}, \\ s_7 = \dfrac{ABx^7}{252}, \\ t_7 = . \end{cases}$$

$$(3.7)$$

Combining all the terms, (3.7) gives

$$\begin{aligned} u(x) = {} & \frac{A}{2!}x^2 + \frac{B}{3!}x^3 - 24x^4 + x^5 - \frac{1}{420}x^8 + \frac{A^8 x^8}{6720} + \frac{A^2 B x^9}{18144} \\ & + \left(\frac{B^2 - 72A}{181440}\right)x^{10} - \frac{x^{11}}{1980} - \frac{x^{12}}{2970} - \frac{x^{14}}{24024} + \frac{Ax^{20}}{2793510786} \\ & + \frac{x^{13}}{4290} + \frac{Ax^{19}}{399072960} \cdots . \end{aligned} \qquad (3.8)$$

Using the boundary conditions at $x = 1$, we have

$$A = 3.0000000000008983, \qquad B = 3.193397011756958 \times 10^{13}. \qquad (3.9)$$

Table 3.1. Error estimates.

x	Exact solution	Series solution	*Errors
0.0	0.0000000	0.0000000	0.00000000
0.1	0.0198100000	0.0198099999	4.579E-16
0.2	0.0771200000	0.0771199999	1.5959E-15
0.3	0.1662300000	0.16622999999	3.1641E-15
0.4	0.2790400000	0.279039999	4.7739E-15
0.5	0.4062500000	0.4062499999	6.0507E-15
0.6	0.5385599999	0.538559999	6.6613E-15
0.7	0.6678700000	0.6678699999	6.6613E-15
0.8	0.7884800000	0.7884799999	5.2180E-15
0.9	0.8982900000	0.8982899999	2.5535E-15
1	1.00000000000	0.999999999	3.3306E-16

*Error = analytical solution − numerical solution.

The series solution is given by

$$u(x) = 1.9999999999999492x^2 + 5.32233 \times 10^{-14}x^3 - 2x^4 + x^5$$
$$- 1.21431 \times 10^{-16}x^8 + O(x^9), \tag{3.10}$$

which is exactly the same solution as obtained in [8] by using the variational iteration technique.

Table 3.1 exhibits a comparison between the exact solution and the series solution obtained by using the homotopy perturbation method. Higher accuracy can be obtained by evaluating more terms of $u(x)$.

Example 3.2 [8]. Consider the following nonlinear fifth-order boundary value problem:

$$u^{(iv)}(x) = u(x) + u''(x) + e^x(x - 3), \tag{3.11}$$

with boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = -e. \tag{3.12}$$

The exact solution is $u(x) = (1 - x)e^x$.

Using the transformation $dy/dx = q(x)$, $dq/dx = f(x)$, $df/dx = z(x)$, we can rewrite the fifth-order boundary value problem (3.11) and (3.12) as the system of differential equations

$$\frac{dy}{dx} = q(x), \quad \frac{dq}{dx} = f(x),$$
$$\frac{df}{dx} = z(x), \quad \frac{dz}{dx} = u(x) + f(x) + e^x(x - 3), \tag{3.13}$$

with $u(0) = 1$, $q(0) = 0$, $f(0) = A$, $z(0) = B$, which can be written as a system of integral equations:

$$u(x) = 1 + \int_0^x q(t)dt, \qquad q(x) = q(0) + \int_0^x f(t)dt,$$

$$f(x) = A + \int_0^x z(t)dt, \qquad z(x) = B + \int_0^x (u(t) + f(t) + e^t(t-3))dt.$$

(3.14)

Using (2.4) and (2.6) for (3.14), we have

$$a_0 + pa_1 + p^2 a_2 + \cdots = 0 + p \int_0^x (b_0 + bv_1 + p^2 b_2 + \cdots) dx,$$

$$b_0 + pb_1 + p^2 b_2 + \cdots = 0 + p \int_0^x (c_0 + pc_1 + p^2 c_2 + \cdots) dx,$$

$$c_0 + pc_1 + p^2 c_2 + \cdots = A + p \int_0^x (d_0 + pd_1 + p^2 d_2 + \cdots) dx,$$

$$d_0 + pd_1 + p^2 d_2 + \cdots = B + p \int_0^x \{(a_0 + pe_1 + p^2 a_2 + \cdots)$$

$$+ (c_0 + pc_1 + p^2 c_2 + \cdots) + e^t(t-3)\} dt.$$

(3.15)

Comparing the coefficient of like powers of p, we have

$$p^{(0)} : \begin{cases} a_0 = 1, \\ b_0 = 0, \\ c_0 = A, \\ d_0 = B, \end{cases}$$

$$p^{(1)} : \begin{cases} a_1 = 0, \\ b_1 = Ax, \\ c_1 = Bx, \\ d_1 = 4 + x + Ax - 4e^x + xe^x, \end{cases}$$

$$p^{(2)} : \begin{cases} a_2 = \dfrac{Ax^2}{2}, \\ b_2 = \dfrac{Bx^2}{2}, \\ c_2 = 5 + 4x + \dfrac{1}{2}x^2 + \dfrac{1}{2}Ax^2 - 5e^x + xe^x, \\ d_2 = \dfrac{1}{2}Bx^2, \end{cases}$$

$$p^{(3)} : \begin{cases} a_3 = \dfrac{Bx^3}{6}, \\ b_3 = 6 + 5x + 2x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{6}Ax^3 - 6e^x + xe^x, \\ c_3 = \dfrac{Bx^3}{6}, \\ d_3 = 6 + 5x + 2x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{6}Ax^3 - 6e^x + xe^x, \end{cases}$$

$$p^{(4)} : \begin{cases} a_4 = 7 + 6x + \dfrac{5}{2}x^2 + \dfrac{2}{3}x^3 + \dfrac{1}{24}x^4 + \dfrac{1}{24}Ax^4 - 7e^x + xe^x, \\ b_4 = \dfrac{1}{24}Bx^4, \\ c_4 = 7 + 6x + \dfrac{5}{2}x^2 + \dfrac{2}{3}x^3 + \dfrac{1}{24}x^4 + \dfrac{1}{24}Ax^4 - 7e^x + xe^x, \\ d_4 = \dfrac{1}{12}Bx^4, \end{cases}$$

$$p^{(5)} : \begin{cases} a_5 = \dfrac{1}{120}Bx^5, \\ b_5 = 8 + 7x + 3x^2 + \dfrac{5}{6}x^3 + \dfrac{1}{6}x^4 + \dfrac{1}{120}x^5 + \dfrac{1}{60}Ax^5 - 8e^x + xe^x, \\ c_5 = \dfrac{1}{60}Bx^5, \\ d_5 = 16 + 14x + 6x^2 + \dfrac{10}{6}x^3 + \dfrac{4}{120}x^5 + \dfrac{3}{120}Ax^5 - 16e^x + 2xe^x, \end{cases}$$

$$p^{(6)} : \begin{cases} a_6 = 9 + 8x + \dfrac{7}{2}x^2 + x^3 + \dfrac{5}{24}x^4 + \dfrac{1}{30}x^5 + \dfrac{1}{720}x^6 \\ \qquad + \dfrac{1}{300}x^4 - 9e^x + xe^x, \\ b_6 = \dfrac{1}{360}Bx^6, \\ c_6 = 18 + 16x + 7x^2 + 2x^3 + \dfrac{5}{12}x^4 + \dfrac{x^5}{5} + \dfrac{x^6}{360} + \dfrac{Ax^6}{240} - 18e^x + 2xe^x, \\ d_6 = \dfrac{1}{240}Bx^6. \end{cases}$$

$$(3.16)$$

Adding up all the terms, (3.16) gives

$$u(x) = 512 + 480x + \frac{1}{2}(499 + a)x^2 + \frac{1}{6}(418 + B)x^3 + \frac{1}{24}(385 + A)x^4 + \frac{1}{120}(354 + B)x^5$$

$$+ \frac{1}{360}(161 + A)x^6 + \left(\frac{146 + B}{2520}\right)x^7 + \left(\frac{259 + 3A}{40320}\right)x^8 + \left(\frac{230 + 3B}{362880}\right)x^9$$

$$+ \left(\frac{197 + 5A}{3628800}\right)x^{10} + O(x^{11}).$$

$$(3.17)$$

Table 3.2. Error estimates.

	Analytical solution	Series solution	*Errors
0.0	1.0000000000	1.0000000000	0.0000E+00
0.1	0.9946538262	0.9946538264	2.0038E-10
0.2	0.9771222065	0.9971222072	7.0052E-10
0.3	0.944011653	0.9449011666	13508E-9
0.4	0.8950948185	0.8950948205	2.0025E-9
0.5	0.8243606353	0.8243606378	2.5085E-9
0.6	0.7288475201	0.72884752228	2.722E-9
0.7	0.6041258122	0.6041258147	2.5125E-9
0.8	0.4451081856	0.4451081875	1.8035E-9
0.9	0.2459603111	0.2459603118	7.2528E-10
1	0.0000000000	-9.9475983006E-14	9.9476E-14

*Error = analytical solution − numerical solution.

Using the boundary conditions at $x = 1$, we have

$$A = -0.9999999547881531, \qquad B = -2.00000154679945. \qquad (3.18)$$

The series solution is

$$y(x) = 512 + 480x + 224.00000000226055x^2 + 69.33333308x^3 + 16.00000000029x^4$$
$$+ 02.933333320443x^5 + 0.44444444570033x^6 + 0.057142857081477x^7$$
$$+ 0.00634920635267x^8 + 0.000617284x^9 + O(x^{10}),$$
$$(3.19)$$

which is exactly the same solution as obtained in [8] by using the variational iteration technique.

Table 3.2 shows the comparison between exact solution and the series solution obtained using the proposed homotopy perturbation method. Higher accuracy can be obtained by evaluating some more terms of the solution $u(x)$.

Example 3.3 [8]. Consider the following nonlinear boundary value problem:

$$u^{(iv)}(x) = \sin x + \sin^2 x - \left(u''(x)\right)^2, \qquad (3.20)$$

with boundary conditions

$$u(0) = 0, \qquad u'(0) = 1, \qquad u(1) = \sin(1), \qquad u'(1) = \cos(1). \qquad (3.21)$$

Using the transformations $du/dx = q(x)$, $dq/dx = f(x)$, $df/dx = z(x)$, we can rewrite the above nonlinear boundary value problem as a system of differential equations:

$$\frac{du}{dx} = q(x), \qquad \frac{dq}{dx} = f(x),$$
$$\frac{df}{dx} = z(x), \qquad \frac{dz}{dx} = \sin x + \sin^2 x - (f(x))^2, \tag{3.22}$$

with $u(0) = 0$, $q(0) = 1$, $f(0) = A$, $z(0) = B$, which can be written as a system of integral equations:

$$u(x) = 0 + \int_0^x q^{(k)}(t)dt, \qquad q(x) = 1 + \int_0^x f^{(k)}(t)dt,$$
$$f(x) = A + \int_0^x z^{(k)}(t)dt, \qquad z(x) = B + \int_0^x \left(\sin x + \sin^2 x - (f(x))^2 \right) dt. \tag{3.23}$$

Using (2.4) and (2.6) for (3.23), we have

$$u_0 + pu_1 + p^2 u_2 + \cdots = 0 + p \int_0^x (v_0 + bv_1 + p^2 v_2 + \cdots) dx,$$

$$v_0 + pv_1 + p^2 v_2 + \cdots = 1 + p \int_0^x (s_0 + ps_1 + p^2 s_2 + \cdots) dx,$$

$$s_0 + ps_1 + p^2 s_2 + \cdots = A + p \int_0^x (t_0 + pt_1 + p^2 t_2 + \cdots) dx,$$

$$t_0 + pt_1 + p^2 t_2 + \cdots = B + p \int_0^x \left\{ \left(\sin x + \frac{1 - \cos 2x}{2} \right) - (s_0 + ps_1 + p^2 s_2 + \cdots)^2 \right\} dx. \tag{3.24}$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)} : \begin{cases} u_0 = 1, \\ v_0 = x, \\ s_0 = Ax, \\ t_0 = Bx, \end{cases} \qquad p^{(1)} : \begin{cases} u_1 = x, \\ v_1 = Ax, \\ s_1 = Bx, \\ t_1 = 1 + \dfrac{1}{2}x - Ax - \cos x - \dfrac{1}{4}\sin 2x, \end{cases}$$

$$p^{(2)} : \begin{cases} u_2 = \dfrac{A}{2}x, \\ v_2 = \dfrac{B}{2}x^2, \\ s_2 = -\dfrac{1}{2} + x + \dfrac{1}{4}x^2 - \sin x + \dfrac{1}{8}\cos 2x, \\ t_2 = -\dfrac{B^2}{2}x^2, \end{cases}$$

$$p^{(3)}:\begin{cases} u_3 = \dfrac{B}{3!}x^3, \\[2mm] v_3 = -1 - \dfrac{1}{8}x + \dfrac{1}{2}x^2 + \dfrac{1}{3!}\left(\dfrac{1-2A}{2}\right)x^3 + \cos x + \dfrac{1}{16}\sin 2x, \\[2mm] s_3 = -\dfrac{B}{3!}x^3, \\[2mm] t_3 = +1 + \dfrac{1}{8}x - \dfrac{1}{2}x^2 - \dfrac{1}{3!}\left(\dfrac{1-2A}{2}\right)x^3 - \cos x - \dfrac{1}{16}\sin 2x, \end{cases}$$

$$p^{(4)}:\begin{cases} u_4 = \dfrac{1}{32} - x - \dfrac{1}{16}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{4!}\left(\dfrac{1-2A}{2}\right)x^4 + \sin x - \dfrac{1}{32}\cos 2x, \\[2mm] v_4 = -\dfrac{B}{32}x^4, \\[2mm] s_4 = -\dfrac{1}{32} + x + \dfrac{1}{16}x^2 - \dfrac{1}{6}x^3 - \dfrac{1}{4!}\left(\dfrac{1-2A}{2}\right)x^4 - \sin x - \dfrac{1}{32}\sin 2x, \\[2mm] t_4 = \dfrac{B}{32}x^4, \end{cases}$$

$$p^{(5)}:\begin{cases} u_5 = -\dfrac{B}{5!}x^5, \\[2mm] v_5 = \dfrac{31}{32} + \dfrac{1}{32}x + \dfrac{1}{2}x^2 - \dfrac{1}{48}x^3 - \dfrac{1}{4!}x^4 \\[2mm] \qquad - \dfrac{1}{5!}\left(\dfrac{1-2A}{2}\right)x^5 - \cos x + \dfrac{1}{32}\cos 2x, \\[2mm] s_5 = \dfrac{B}{5!}x^5, \\[2mm] t_5 = -\dfrac{31}{32} - \dfrac{1}{32}x - \dfrac{1}{2}x^2 + \dfrac{1}{48}x^3 + \dfrac{1}{4!}x^4 \\[2mm] \qquad + \dfrac{1}{5!}\left(\dfrac{1-2A}{2}\right)x^5 + \cos x - \dfrac{1}{32}\cos 2x, \end{cases}$$

$$p^{(6)}:\begin{cases} u_6 = \dfrac{31}{32}x + \dfrac{1}{64}x^2 + \dfrac{1}{3!}x^3 - \dfrac{1}{192}x^4 - \dfrac{1}{5!}x^5 \\[2mm] \qquad - \dfrac{1}{6!}\left(\dfrac{1-2A}{2}\right)x^6 - \sin x + \dfrac{1}{64}\sin 2x, \\[2mm] v_6 = \dfrac{B}{6!}x^6, \\[2mm] s_6 = -\dfrac{31}{32}x - \dfrac{1}{64}x^2 - \dfrac{1}{3!}x^3 + \dfrac{1}{192}x^4 + \dfrac{1}{5!}x^5 \\[2mm] \qquad + \dfrac{1}{6!}\left(\dfrac{1-2A}{2}\right)x^6 + \sin x - \dfrac{1}{64}\sin 2x, \\[2mm] t_6 = -\dfrac{B}{6!}x^6, \end{cases}$$

$$p^{(7)}: \begin{cases} u_7 = \dfrac{B}{7!}x^7, \\[2mm] v_7 = \dfrac{127}{128} - \dfrac{31}{62}x^2 - \dfrac{1}{192}x^3 - \dfrac{1}{4!}x^4 + \dfrac{1}{960}x^5 + \dfrac{1}{6!}x^6 \\[2mm] \qquad + \dfrac{1}{7!}(1-2A)x^7 - \cos x + \dfrac{1}{128}\cos 2x, \\[2mm] s_7 = -\dfrac{B}{7!}x^7, \\[2mm] t_7 = -\dfrac{127}{128} + \dfrac{31}{62}x^2 + \dfrac{1}{192}x^3 + \dfrac{1}{4!}x^4 - \dfrac{1}{960}x^5 - \dfrac{1}{6!}x^6 \\[2mm] \qquad - \dfrac{1}{7!}(1-2A)x^7 + \cos x - \dfrac{1}{128}\cos 2x, \end{cases}$$

$$p^{(8)}: \begin{cases} u_8 = \dfrac{127}{128} - \dfrac{31}{186}x^3 - \dfrac{1}{768}x^4 - \dfrac{1}{5!}x^5 + \dfrac{1}{5760}x^6 + \dfrac{1}{6!}x^6 \\[2mm] \qquad + \dfrac{1}{8!}(1-2A)x^8 - \sin x + \dfrac{1}{256}\sin 2x, \\[2mm] v_8 = -\dfrac{B}{8!}x^8, \\[2mm] s_8 = \dfrac{127}{128} + \dfrac{31}{62}x^3 + \dfrac{1}{768}x^4 + \dfrac{1}{5!}x^5 - \dfrac{1}{5760}x^6 - \dfrac{1}{6!}x^6 \\[2mm] \qquad - \dfrac{1}{8!}(1-2A)x^8 + \sin x - \dfrac{1}{256}\sin 2x, \\[2mm] t_8 = \dfrac{B}{8!}x^8. \end{cases}$$

$$(3.25)$$

Adding all the terms (3.25) gives the solution

$$u(x) = 0.03125 + x + \frac{A}{2}x^2 + \frac{B}{6}x^3 + \frac{x}{48}\left(-48 - 3x + 8x^2 + x^3\right) + 0.03125\cos 2x$$
$$+ \sin x - \frac{A}{24}x^4 + \frac{B}{120}x^5 + \cdots .$$

$$(3.26)$$

Using the boundary conditions at $x = 1$, we have

$$A = 0.00017529213456789, \qquad B = -1.000057468112352. \qquad (3.27)$$

Thus the series solution of the boundary value problem is given as

$$u(x) = 11.3472 - 262.827x - 3.40768x^3 + 0.142081x^4 - 0.0027763x^5$$
$$- 0.00161889x^6 + 0.00079363x^7 + 0.0000414643x^8$$
$$+ 7.28477 \times 10^{-10}x^9 + O(x^{10}),$$

$$(3.28)$$

which is exactly the same as in [8] by using the variational iteration method.

Table 3.3. Error estimates.

x	Exact solution	Series solution	*Errors
0.0	0.0000000	9.592369E-14	9.592369E-14
0.1	0.0998334166	0.0998334945	7.7856E-8
0.2	0.1986693307	0.1986696031	2.723E-7
0.3	0.2955202066	0.2955207315	5.2489E-7
0.4	0.3894183423	0.3894191196	7.7730E-7
0.5	0.4794255386	0.4794265100	9.7145E-7
0.6	0.5646424733	0.564635236	1.0502E-6
0.7	0.6442176872	0.6442186501	9.6286E-7
0.8	0.7173560908	0.7173567749	6.8407E-7
0.9	0.7833269096	0.7833271803	2.7069E-7
1.0	0.8414709848	0.8414709848	1.5676E-13

*Error = analytical solution − numerical solution.

Table 3.3 shows the comparison between exact solution and the series solution obtained using the proposed homotopy perturbation method. Higher accuracy can be obtained by evaluating some more terms of the solution $u(x)$.

4. Conclusion

In this paper, we have shown that the homotopy perturbation method can be used successfully for finding the solution of linear and nonlinear boundary value problems of fourth-order by reformulating it as a system of integral equations. It may be concluded that this technique is very powerful and efficient in finding the analytical solutions for a large class of integral and differential equations. This technique provides more realistic series solutions as compared with the Adomian decomposition and variational iteration techniques.

Acknowledgments

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