

*Research Article*

# **An Approximate Solution for Boundary Value Problems in Structural Engineering and Fluid Mechanics**

**A. Barari,<sup>1</sup> M. Omidvar,<sup>2</sup> D. D. Ganji,<sup>1</sup> and Abbas Tahmasebi Poor<sup>1</sup>**

<sup>1</sup>Departments of Civil Engineering and Mechanical Engineering, Mazandaran University of Technology, P.O. Box 484, Babol, Iran

<sup>2</sup>Technical and Engineering Faculty, Gorgan University of Agricultural Sciences and Natural Resources, Gorgan, Iran

Correspondence should be addressed to A. Barari, amin78404@yahoo.com

Received 10 January 2008; Accepted 19 May 2008

Recommended by David Chelidze

Variational iteration method (VIM) is applied to solve linear and nonlinear boundary value problems with particular significance in structural engineering and fluid mechanics. These problems are used as mathematical models in viscoelastic and inelastic flows, deformation of beams, and plate deflection theory. Comparison is made between the exact solutions and the results of the variational iteration method (VIM). The results reveal that this method is very effective and simple, and that it yields the exact solutions. It was shown that this method can be used effectively for solving linear and nonlinear boundary value problems.

Copyright © 2008 A. Barari et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## **1. Introduction**

This paper discusses the analytical approximate solution for fourth-order equations with nonlinear boundary conditions involving third-order derivatives. The general form of the equation for a fixed positive integer  $n$ ,  $n \geq 2$ , is a differential equation of order  $2n$ :

$$y^{(2n)} + f(x, y) = 0 \quad (1.1)$$

subject to the boundary conditions

$$y^{(2j)}(a) = A_{2j}, \quad y^{(2j)}(b) = B_{2j}, \quad j = 0(1)n - 1, \quad (1.2)$$

where  $-\infty < a \leq x \leq b < \infty$ ,  $A_{2j}$ ,  $B_{2j}$ ,  $j = 0(1)n - 1$  are finite constants.

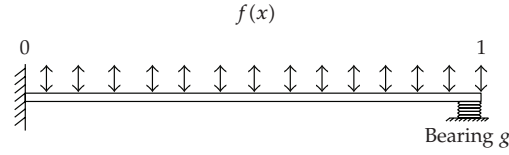


Figure 1: Beam on elastic bearing.

It is assumed that  $y$  is sufficiently differentiable and that a unique solution of (1.1) exists. Problems of this kind are commonly encountered in plate-deflection theory and in fluid mechanics for modeling viscoelastic and inelastic flows [1–3]. Usmani [1, 2] discussed sixth order methods for the linear differential equation  $y^{(4)} + P(x)y = q(x)$  subject to the boundary conditions  $y(a) = A_0, y''(a) = A_2, y(b) = B_0, y''(b) = B_2$ . The method described in [1] leads to five diagonal linear systems and involves  $p', p'', q', q''$  at  $a$  and  $b$ , while the method described in [2] leads to nine diagonal linear systems.

Ma and Silva [4] adopted iterative solutions for (1.1) representing beams on elastic foundations. Referring to the classical beam theory, they stated that if  $u = u(x)$  denotes the configuration of the deformed beam, then the bending moment satisfies the relation  $M = -EIu''$ , where  $E$  is the Young modulus of elasticity and  $I$  is the inertial moment. Considering the deformation caused by a load  $f = f(x)$ , they deduced, from a free-body diagram, that  $f = -v'$  and  $v = M' = -EIu'''$ , where  $v$  denotes the shear force. For  $u$  representing an elastic beam of length  $L = 1$ , which is clamped at its left side  $x = 0$ , and resting on an elastic bearing at its right side  $x = 1$ , and adding a load  $f$  along its length to cause deformations (Figure 1), Ma and Silva [4] arrived at the following boundary value problem assuming an  $EI = 1$ :

$$u^{(iv)}(x) = f(x, u(x)), \quad 0 < x < 1, \quad (1.3)$$

the boundary conditions were taken as

$$u(0) = u'(0) = 0, \quad (1.4)$$

$$u''(1) = 0, \quad u'''(1) = g(u(1)), \quad (1.5)$$

where  $f \in C([0, 1] \times \mathbb{R})$  and  $g \in C(\mathbb{R})$  are real functions. The physical interpretation of the boundary conditions is that  $u'''(1)$  is the shear force at  $x = 1$ , and the second condition in (1.5) means that the vertical force is equal to  $g(u(1))$ , which denotes a relation, possibly nonlinear, between the vertical force and the displacement  $u(1)$ . Furthermore, since  $u''(1) = 0$  indicates that there is no bending moment at  $x = 1$ , the beam is resting on the bearing  $g$ .

Solving (1.3) by means of iterative procedures, Ma and Silva [4] obtained solutions and argued that the accuracy of results depends highly upon the integration method used in the iterative process.

With the rapid development of nonlinear science, many different methods were proposed to solve differential equations, including boundary value problems (BVPS). These two methods are the homotopy perturbation method (HPM) [5–7] and the variational iteration method (VIM) [8–17]. In this paper, it is aimed to apply the variational iteration method proposed by He [14] to different forms of (1.1) subject to boundary conditions of physical significance.

## 2. Basic idea of He's variational iteration method

To clarify the basic ideas of He's VIM, the following differential equation is considered:

$$L[u(t)] + N[u(t)] = g(t), \quad (2.1)$$

where  $L$  is a linear operator,  $N$  is a nonlinear operator, and  $g(t)$  is an inhomogeneous term. According to VIM, a correction functional could be written as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) (Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)) d\tau, \quad (2.2)$$

where  $\lambda$  is a general Lagrange multiplier which can be identified optimally via the variational theory. The subscript  $n$  indicates the  $n$ th approximation and  $\tilde{u}_n$  is considered as a restricted variation, that is,  $\delta\tilde{u}_n = 0$ .

For fourth-order boundary value problem with suitable boundary conditions, Lagrangian multiplier can be identified by substituting the problem into (2.2), upon making it stationary leads to the following:

$$\begin{aligned} \frac{d^4}{d\tau^4} \lambda &= 0, \\ -\lambda''' + 1|_{\tau=x} &= 0, \\ \lambda''|_{\tau=x} &= 0. \end{aligned} \quad (2.3)$$

Solving the system of (2.3) yields

$$\lambda = \frac{1}{6}(\tau - x)^3 \quad (2.4)$$

and the variational iteration formula is obtained in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3 (u_n^{(4)}(\tau) + f(\tau, u_n, u_n', u_n'', u_n''')) d\tau. \quad (2.5)$$

## 3. The applications of VIM method

In this section, the VIM is applied to different forms of the fourth-order boundary value problem introduced in through (1.1).

*Example 3.1.* Consider the following linear boundary value problem:

$$u^{(4)}(x) = 4e^x + u(x), \quad 0 < x < 1, \quad (3.1)$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 2, \quad u(1) = 2e, \quad u'(1) = 3e. \quad (3.2)$$

The exact solution for this problem is

$$u(x) = (1+x)e^x. \quad (3.3)$$

According to (2.5), the following iteration formulation is achieved:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau-x)^3(u_n^{(4)}(\tau) - u_n(\tau) - 4e^\tau) d\tau. \quad (3.4)$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d, \quad (3.5)$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are unknown constants to be further determined.

By the iteration formula (3.4), the following first-order approximation may be written:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau-x)^3(u_0^{(4)}(\tau) - u_0(\tau) - 4e^\tau) d\tau \\ &= ax^3 + bx^2 + cx + d + \int_0^x \frac{1}{6}(\tau-x)^3(-a\tau^3 - b\tau^2 - c\tau - d - 4e^\tau) d\tau \\ &= \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \frac{1}{120}cx^5 + \frac{1}{24}dx^4 + \left(-\frac{2}{3} + a\right)x^3 + (b-2)x^2 + (c-4)x + 4e^x + d - 4. \end{aligned} \quad (3.6)$$

Incorporating the boundary conditions (3.2), into  $u_1(x)$ , the following coefficients can be obtained:

$$a = -\frac{2289756}{301681} + \frac{916440}{301681}e, \quad b = \frac{4575063}{301681} - \frac{1516680}{301681}e, \quad c = 2, \quad d = 1. \quad (3.7)$$

Therefore, the following first-order approximate solution is derived:

$$\begin{aligned} u_1(x) &= \left(-\frac{27259}{3016810} + \frac{1091}{301681}e\right)x^7 + \left(\frac{1525021}{36201720} - \frac{4213}{301681}e\right)x^6 \\ &\quad + \frac{1}{60}x^5 + \frac{1}{24}x^4 + \left(-\frac{7472630}{905043} + \frac{916440}{301681}e\right)x^3 + \left(\frac{3971701}{301681} - \frac{1516680}{301681}e\right)x^2 - 2x - 3 + 4e^x. \end{aligned} \quad (3.8)$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 1, showing a remarkable agreement.

Similarly, the following second-order approximation is obtained:

$$\begin{aligned} u_2(x) &= u_1(x) + \int_0^x \frac{1}{6}(\tau-x)^3(u_1^{(4)}(\tau) - u_1(\tau) - 4e^\tau) d\tau \\ &= \frac{1}{6652800}ax^{11} + \frac{1}{1814400}bx^{10} + \frac{1}{362880}cx^9 + \frac{1}{40320}dx^8 + \left(\frac{1}{840}a - \frac{1}{1260}\right)x^7 \\ &\quad + \left(\frac{1}{360}b - \frac{1}{180}\right)x^6 + \left(-\frac{1}{30} + \frac{1}{120}c\right)x^5 + \left(-\frac{1}{6} + \frac{1}{24}d\right)x^4 \\ &\quad + \left(-\frac{4}{3} + a\right)x^3 + (b-4)x^2 + (c-8)x - 8 + 8e^x + d, \\ a &= -\frac{12706529114180}{681628862391} + \frac{85535681616000}{12042109902241}e, \quad c = 2, \\ b &= \frac{8416302814865}{227209620797} - \frac{157452726614400}{12042109902241}e, \quad d = 1. \end{aligned} \quad (3.9)$$

**Table 1:** Comparison of the first-order approximate solution with exact solution.

$x$	$U_E$	$U_1$	Error
0	1.000000000	1.000000000	0.0000E + 000
0.1	1.215688010	1.215681524	6.4860E - 006
0.2	1.465683310	1.465660890	2.2420E - 005
0.3	1.754816450	1.754773923	4.2527E - 005
0.4	2.088554577	2.088492979	6.1598E - 005
0.5	2.473081906	2.473007265	7.4641E - 005
0.6	2.915390080	2.915312734	7.7346E - 005
0.7	3.423379602	3.423312592	6.7010E - 005
0.8	4.005973670	4.005929404	4.4266E - 005
0.9	4.673245911	4.673229891	1.6020E - 005
1.0	$2e$	$2e$	0.0000E + 000

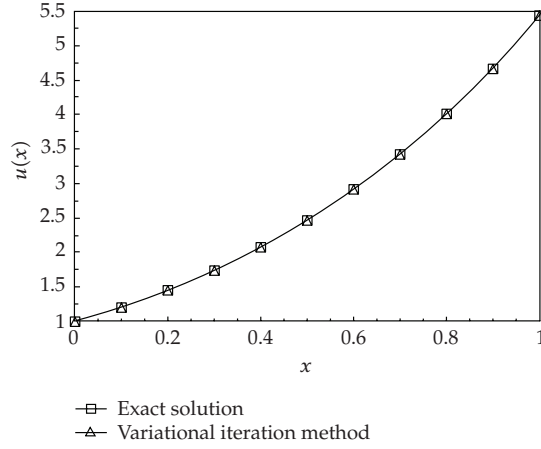
**Table 2:** Comparison of the second-order approximate solution with exact solution.

$x$	$U_E$	$U_2$	Error
0	1.000000000	1.000000000	0.0E + 000
0.1	1.215688010	1.215688008	2.0E - 009
0.2	1.465683310	1.465683305	5.0E - 009
0.3	1.754816450	1.754816444	6.0E - 009
0.4	2.088554577	2.088554566	1.1E - 008
0.5	2.473081906	2.473081902	4.0E - 009
0.6	2.915390080	2.915390064	1.6E - 008
0.7	3.423379602	3.423379600	2.0E - 009
0.8	4.005973670	4.005973650	2.0E - 008
0.9	4.673245911	4.673245930	1.9E - 008
1.0	$2e$	$2e$	0.0E + 000

Therefore, the second-order approximate solution may be written as

$$\begin{aligned}
u_2(x) = & \left( -\frac{57756950519}{20612456798703840} + \frac{12857095}{12042109902241} e \right) x^{11} \\
& + \left( \frac{1683260562973}{82449827194815360} - \frac{86779501}{12042109902241} e \right) x^{10} + \frac{1}{181440} x^9 \\
& + \frac{1}{40320} x^8 + \left( -\frac{731163797543}{31809346911580} + \frac{101828192400}{12042109902241} e \right) x^7 \\
& + \left( \frac{7961883573271}{81795463486920} - \frac{437368685040}{12042109902241} e \right) x^6 - \frac{1}{60} x^5 \\
& - \frac{1}{8} x^4 + \left( -\frac{13615367597368}{681628862391} + \frac{85535681616000}{12042109902241} e \right) x^3 \\
& + \left( \frac{7507464331677}{227209620797} - \frac{157452726614400}{12042109902241} e \right) x^2 - 6x - 7 + 8e^x.
\end{aligned} \tag{3.10}$$

Again, the obtained solution is of distinguishing accuracy, as indicated in Table 2 and Figure 2.



**Figure 2:** Comparison between different solutions.

*Example 3.2.* Consider the following linear boundary value problem:

$$u^{(4)}(x) = u(x) + u''(x) + e^x(x-3), \quad 0 < x < 1, \quad (3.11)$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = -e. \quad (3.12)$$

The exact solution for this problem is

$$u(x) = (1-x)e^x. \quad (3.13)$$

According to (2.5), the iteration formulation may be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau-x)^3 \left( u_n^{(4)}(\tau) - u_n(\tau) - u_n''(\tau) - e^\tau(\tau-3) \right) d\tau. \quad (3.14)$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d. \quad (3.15)$$

Where  $a$ ,  $b$ ,  $c$ , and  $d$  are unknown constants to be further determined.

By the iteration formula (3.14), the following first-order approximation is developed:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau-x)^3 \left( u_0^{(4)}(\tau) - u_0(\tau) - u_0''(\tau) - e^\tau(\tau-3) \right) d\tau \\ &= ax^3 + bx^2 + cx + d + \int_0^x \frac{1}{6}(\tau-x)^3 \left( -a\tau^3 - b\tau^2 - (6a+c)\tau - 2b-d - e^\tau(\tau-3) \right) d\tau \\ &= \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \left( \frac{1}{20}a + \frac{1}{120}c \right)x^5 + \left( \frac{1}{12}b + \frac{1}{24}d \right)x^4 + \left( \frac{2}{3} + a \right)x^3 + \left( b + \frac{5}{2} \right)x^2 \\ &\quad + \left( e^x + 6 + c \right)x - 7e^x + 7 + d. \end{aligned} \quad (3.16)$$

**Table 3:** Comparison of the first-order approximate solution with exact solution.

$x$	$U_E$	$U_1$	Error
0	1.0000000000	1.0000000000	0.0000000E + 000
0.1	0.9946538262	0.9947931547	1.3932850E - 004
0.2	0.9771222064	0.9775949040	4.7269760E - 004
0.3	0.9449011656	0.9457776230	8.7645740E - 004
0.4	0.8950948188	0.8963297250	1.2349062E - 003
0.5	0.8243606355	0.8258087440	1.4481085E - 003
0.6	0.7288475200	0.7302919280	1.4444080E - 003
0.7	0.6041258121	0.6053240800	1.1982679E - 003
0.8	0.4451081856	0.4458625400	7.5435440E - 004
0.9	0.2459603111	0.2462193000	2.5898890E - 004
1.0	0.0000000000	0.0000000000	0.0000000E + 000

Incorporating the boundary conditions (3.12), into  $u_1(x)$ , it can be written as

$$a = \frac{7904470}{323149} - \frac{2950080}{323149}e, \quad b = -\frac{12770295}{323149} + \frac{4640400}{323149}e, \quad c = 0, \quad d = 1. \quad (3.17)$$

Therefore, the following first-order approximate solution is obtained:

$$\begin{aligned} u_1(x) = & \left( \frac{112921}{3877788} - \frac{3512}{323149}e \right) x^7 + \left( -\frac{851353}{7755576} + \frac{12890}{323149}e \right) x^6 \\ & + \left( \frac{790447}{646298} - \frac{147504}{323149}e \right) x^5 + \left( -\frac{25217441}{7755576} + \frac{386700}{323149}e \right) x^4 \\ & + \left( \frac{24359708}{969447} - \frac{2950080}{323149}e \right) x^3 + \left( -\frac{23924845}{646298} + \frac{4640400}{323149}e \right) x^2 \\ & + (6 + e^x)x + 8 - 7e^x. \end{aligned} \quad (3.18)$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 3, again showing a clear agreement. Even higher accurate solutions could be obtained without any difficulty.

Similarly, the following second-order approximation can be written as

$$\begin{aligned} u_2(x) = & u_1(x) + \int_0^x \frac{1}{6}(\tau - x)^3 (u_1^{(4)}(\tau) - u_1(\tau) - u_1''(\tau) - e^\tau(\tau - 3)) d\tau \\ = & \frac{1}{6652800}ax^{11} + \frac{1}{1814400}bx^{10} + \left( \frac{1}{362880}c + \frac{1}{30240}a \right) x^9 + \left( \frac{1}{10080}b + \frac{1}{40320}d \right) x^8 \\ & + \left( \frac{1}{5040}c + \frac{1}{420}a + \frac{1}{1260} \right) x^7 + \left( \frac{1}{720}d + \frac{1}{144} + \frac{1}{180}b \right) x^6 + \left( \frac{1}{12} + \frac{1}{20}a + \frac{1}{120}c \right) x^5 \\ & + \left( \frac{1}{2} + \frac{1}{24}d + \frac{1}{12}b \right) x^4 + (3 + a)x^3 + \left( b + \frac{21}{2} \right) x^2 + (24 + 3e^x + c)x + 27 - 27e^x + d. \end{aligned} \quad (3.19)$$

Incorporating the boundary conditions, (3.12), into  $u_2(x)$ , yields

$$\begin{aligned} a &= \frac{381804789300110}{4289712004667} - \frac{140985028800000}{4289712004667}e, & c &= 0, \\ b &= -\frac{629495301082065}{4289712004667} + \frac{230790037363200}{4289712004667}e, & d &= 1. \end{aligned} \quad (3.20)$$

The following second-order approximate solution is then achieved in the following form:

$$\begin{aligned} u_2(x) &= \left( \frac{3470952630001}{259441782042260160} - \frac{63575500}{12869136014001}e \right) x^{11} \\ &+ \left( -\frac{41966353405471}{518883564084520320} + \frac{381597284}{12869136014001}e \right) x^{10} \\ &+ \left( \frac{38180478930011}{12972089102113008} - \frac{13986610000}{12869136014001}e \right) x^9 \\ &+ \left( -\frac{2513691492323593}{172961188028173440} + \frac{22895837040}{4289712004667}e \right) x^8 \\ &+ \left( \frac{1149704079904997}{5405037125880420} - \frac{335678640000}{4289712004667}e \right) x^7 \\ &+ \left( -\frac{415373822050043}{514765440560040} + \frac{1282166874240}{4289712004667}e \right) x^6 \\ &+ \left( \frac{233372585584733}{51476544056004} - \frac{7049251440000}{4289712004667}e \right) x^5 \\ &+ \left( -\frac{1203224346103459}{102953088112008} + \frac{19232503113600}{4289712004667}e \right) x^4 \\ &+ \left( \frac{394673925314111}{4289712004667} - \frac{140985028800000}{4289712004667}e \right) x^3 \\ &+ \left( -\frac{1168906650066123}{8579424009334} + \frac{230790037363200}{4289712004667}e \right) x^2 \\ &+ (3e^x + 24)x + 28 - 27e^x. \end{aligned} \quad (3.21)$$

The obtained solution is of evident accuracy, as shown in Table 4 and Figure 3.

*Example 3.3.* Consider the following nonlinear boundary value problem:

$$u^{(4)}(x) = u^2(x) + g(x), \quad 0 < x < 1, \quad (3.22)$$

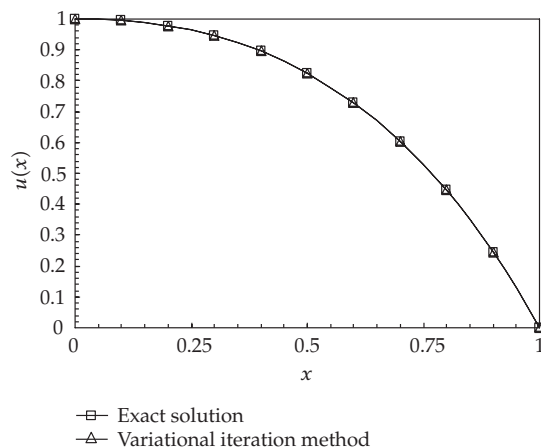
subject to the boundary conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 1, \quad (3.23)$$

where

$$g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48. \quad (3.24)$$





**Figure 3:** Comparison between different solutions.

**Table 4:** Comparison of the second-order approximate solution with exact solution.

$x$	$U_E$	$U_2$	Error
0	1.0000000000	1.0000000000	0.00000E + 000
0.1	0.9946538262	0.9946577580	3.93180E - 006
0.2	0.9771222064	0.9771357780	1.35716E - 005
0.3	0.9449011656	0.9449268900	2.57244E - 005
0.4	0.8950948188	0.8951321100	3.72912E - 005
0.5	0.8243606355	0.8244058800	4.52445E - 005
0.6	0.7288475200	0.7288945300	4.70100E - 005
0.7	0.6041258121	0.6041666500	4.08379E - 005
0.8	0.4451081856	0.4451352800	2.70944E - 005
0.9	0.2459603111	0.2459701300	9.81890E - 006
1.0	0.0000000000	0.0000000000	0.00000E + 000

The exact solution for this problem is

$$u(x) = x^5 - 2x^4 + 2x^2. \quad (3.25)$$

According to (2.5), the iteration formulation is written as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3 (u_n^{(4)}(\tau) - u_n^2(\tau) - g(\tau)) d\tau. \quad (3.26)$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d, \quad (3.27)$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are unknown constants to be further determined.

**Table 5:** Comparison of the first-order approximate solution with exact solution.

$x$	$U_E$	$U_1$	Error
0	0.0000000000	0.0000000000	0.0000000E + 000
0.1	0.0198100000	0.0198624243	5.2424300E - 005
0.2	0.0771200000	0.0773022107	1.8221070E - 004
0.3	0.1662300000	0.1665781379	3.4813790E - 004
0.4	0.2790400000	0.2795490972	5.0909720E - 004
0.5	0.4062500000	0.4068747265	6.2472650E - 004
0.6	0.5385600000	0.5392178270	6.5782700E - 004
0.7	0.6678700000	0.6684511385	5.8113850E - 004
0.8	0.7884800000	0.7888727023	3.9270230E - 004
0.9	0.8982900000	0.8984356964	1.4569640E - 004
1.0	1.0000000000	1.0000000000	0.0000000E + 000

By the iteration formula (3.26), the following first-order approximation is obtained:

$$\begin{aligned}
u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau - x)^3(u_0^{(4)}(\tau) - u_0^2(\tau) + \tau^{10} - 4\tau^9 + 4\tau^8 + 4\tau^7 - 8\tau^6 + 4\tau^4 - 120\tau + 48)d\tau \\
&= -\frac{1}{24024}x^{14} + \frac{1}{4290}x^{13} - \frac{1}{2970}x^{12} - \frac{1}{1980}x^{11} + \left(\frac{1}{5040}a^2 + \frac{1}{630}\right)x^{10} \\
&\quad + \frac{1}{1512}abx^9 + \left(-\frac{1}{420} + \frac{1}{1680}b^2 + \frac{1}{840}ac\right)x^8 + \left(\frac{1}{420}bc + \frac{1}{420}ad\right)x^7 \\
&\quad + \left(\frac{1}{180}bd + \frac{1}{360}c^2\right)x^6 + \left(1 + \frac{1}{60}cd\right)x^5 + \left(\frac{1}{24}d^2 - 2\right)x^4 + ax^3 + bx^2 + cx + d.
\end{aligned} \tag{3.28}$$

Incorporating the boundary conditions (3.23), into  $u_1(x)$ , results in the following values:

$$a = -0.006871650809; \quad b = 2.005929593; \quad c = 0, \quad d = 0. \tag{3.29}$$

The following first-order approximate solution is then achieved:

$$\begin{aligned}
u_1(x) &= -4.162504162 \times 10^{-5}x^{14} + 2.331002331 \times 10^{-4}x^{13} \\
&\quad - 3.367003367 \times 10^{-4}x^{12} - 5.050505050 \times 10^{-4}x^{11} \\
&\quad + 1.587310956 \times 10^{-3}x^{10} - 9.116433669 \times 10^{-6}x^9 \\
&\quad + 1.4139007 \times 10^{-5}x^8 + x^5 - 2x^4 - 6.871650809 \times 10^{-3}x^3 \\
&\quad + 2.005929593x^2.
\end{aligned} \tag{3.30}$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 5, which once again shows an excellent agreement.

Similarly, the following second-order approximation may be written:

$$u_2(x) = u_1(x) + \int_0^x \frac{1}{6}(\tau - x)^3(u_1^{(4)}(\tau) - u_1^2(\tau) + \tau^{10} - 4\tau^9 + 4\tau^8 + 4\tau^7 - 8\tau^6 + 4\tau^4 - 120\tau + 48)d\tau. \tag{3.31}$$

**Table 6:** Comparison of the second-order approximate solution with exact solution.

$x$	$U_E$	$U_2$	Error
0	0.0000000000	0.0000000000	0.000E + 000
0.1	0.0198100000	0.0198100068	6.800E - 009
0.2	0.0771200000	0.0771200239	2.390E - 008
0.3	0.1662300000	0.1662300464	4.640E - 008
0.4	0.2790400000	0.2790400692	6.920E - 008
0.5	0.4062500000	0.4062500874	8.740E - 008
0.6	0.5385600000	0.5385600961	9.610E - 008
0.7	0.6678700000	0.6678700906	9.060E - 008
0.8	0.7884800000	0.7884800670	6.700E - 008
0.9	0.8982900000	0.8982900292	2.920E - 008
1.0	1.0000000000	1.0000000012	1.200E - 009

Incorporating the boundary conditions, (3.23), into  $u_2(x)$ , yields

$$a = -8.269548014E - 7; \quad b = 2.000000763; \quad c = 0, \quad d = 0. \quad (3.32)$$

The following second-order approximate solution is obtained:

$$\begin{aligned}
u_2(x) = & -1.093855974 \times 10^{-9}x^9 + 1.817 \times 10^{-9}x^8 - 2x^4 - 1.117934793 \times 10^{-8}x^{21} \\
& + 1.463705892 \times 10^{-9}x^{20} + 6.586694874 \times 10^{-8}x^{19} + 2.000000763x^2 \\
& - 8.269548014 \times 10^{-7}x^3 - 1.047931585 \times 10^{-7}x^{18} - 3.536760165 \times 10^{-8}x^{17} \\
& + 1.453571773 \times 10^{-7}x^{16} - 5.173598972 \times 10^{-13}x^{28} - 2.569735395 \times 10^{-14}x^{31} \\
& + 1.252296566 \times 10^{-13}x^{30} - 2.016131906 \times 10^{-13}x^{29} + 2.007605778 \times 10^{-15}x^{32} \\
& + 2.564345160 \times 10^{-12}x^{27} + 3.603899741 \times 10^{-9}x^{22} + 3.025 \times 10^{-13}x^{14} \\
& - 1.392179800 \times 10^{-10}x^{12} + 6.103539401 \times 10^{-10}x^{11} + x^5 + 9.879565106 \times 10^{-12}x^{24} \\
& - 2.268156651 \times 10^{-12}x^{26} - 5.281071651 \times 10^{-12}x^{25} - 3.917282540 \times 10^{-10}x^{23} \\
& - 1.335600908 \times 10^{-13}x^{15} - 6.059998643 \times 10^{-10}x^{10}.
\end{aligned} \quad (3.33)$$

The obtained solution is once again of remarkable accuracy, as shown in Table 6 and Figure 4.

#### 4. Conclusion

This study showed that the variational iteration method is remarkably effective for solving boundary value problems. A fourth-order differential equation with particular engineering applications was solved using the VIM in order to prove its effectiveness. Different forms of the equation having boundary conditions of physical significance were considered. Comparison between the approximate and exact solutions showed that one iteration is enough to reach the exact solution. Therefore the VIM is able to solve partial differential equations using a minimum calculation process. This method is a very promoting method, which promises to find wide applications in engineering problems.

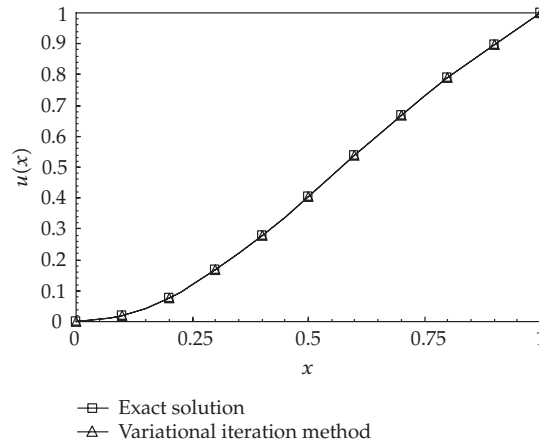


Figure 4: Comparison between different solutions.

## References

- [1] R. A. Usmani, "On the numerical integration of a boundary value problem involving a fourth order linear differential equation," *BIT*, vol. 17, no. 2, pp. 227–234, 1977.
- [2] R. A. Usmani, "An  $O(h^6)$  finite difference analogue for the solution of some differential equations occurring in plate deflection theory," *Journal of the Institute of Mathematics and Its Applications*, vol. 20, no. 3, pp. 331–333, 1977.
- [3] S. M. Momani, *Some problems in non-Newtonian fluid mechanics*, Ph.D. thesis, Wales University, Wales, UK, 1991.
- [4] T. F. Ma and J. da Silva, "Iterative solutions for a beam equation with nonlinear boundary conditions of third order," *Applied Mathematics and Computation*, vol. 159, no. 1, pp. 11–18, 2004.
- [5] J.-H. He, "New interpretation of homotopy perturbation method," *International Journal of Modern Physics B*, vol. 20, no. 18, pp. 2561–2568, 2006.
- [6] D. D. Ganji and A. Sadighi, "Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 4, pp. 411–418, 2006.
- [7] M. Rafei and D. D. Ganji, "Explicit solutions of Helmholtz equation and fifth-order Kdv equation using homotopy-perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 3, pp. 321–328, 2006.
- [8] N. H. Sweilam and M. M. Khader, "Variational iteration method for one dimensional nonlinear thermoelasticity," *Chaos, Solitons & Fractals*, vol. 32, no. 1, pp. 145–149, 2007.
- [9] S. M. Momani and Z. Odibat, "Numerical comparison of methods for solving linear differential equations of fractional order," *Chaos, Solitons & Fractals*, vol. 31, no. 5, pp. 1248–1255, 2007.
- [10] S. M. Momani and S. Abusad, "Application of He's variational iteration method to Helmholtz equation," *Chaos, Solitons & Fractals*, vol. 27, no. 5, pp. 1119–1123, 2006.
- [11] N. Bildik and A. Konuralp, "The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 65–70, 2006.
- [12] A. Barari, M. Omidvar, S. Gholitabar, and D. D. Ganji, "Variational iteration method and homotopy-perturbation method for solving second-order nonlinear wave equation," in *Proceedings of the International Conference of Numerical Analysis and Applied Mathematics (ICNAAM '07)*, vol. 936, pp. 81–85, Corfu, Greece, September 2007.
- [13] D. D. Ganji, M. Jannatabadi, and E. Mohseni, "Application of He's variational iteration method to nonlinear Jaulent-Miodek equations and comparing it with ADM," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 35–45, 2007.

- [14] J.-H. He, "Variational iteration method—a kind of nonlinear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.
- [15] Z. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 27–34, 2006.
- [16] H. Tari, D. D. Ganji, and M. Rostamian, "Approximate solutions of K (2.2), KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 2, pp. 203–210, 2007.
- [17] E. Yusufoglu, "Variational iteration method for construction of some compact and non-compact structures of Klein-Gordon equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 2, pp. 152–158, 2007.